HERTZ POTENTIALS AND DIFFERENTIAL GEOMETRY

A Thesis

by

JEFFREY DAVID BOUAS

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2011

Major Subject: Mathematics

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ABSTRACT

Hertz Potentials and Differential Geometry. (May 2011)

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I review the construction of Hertz potentials in vector calculus starting from Maxwell's equations. From here, I lay the minimal foundations of differential geometry to construct Hertz potentials for a general (spatially compact) Lorentzian manifold with or without boundary. In this general framework, I discuss "scalar" Hertz potentials as they apply to the vector calculus situation, and I consider their possible generalization, showing which procedures used by previous authors fail to generalize and which succeed, if any. I give specific examples, including the standard flat coordinate systems and an example of a non-flat metric, specifically a spherically symmetric black hole. Additionally, I generalize the introduction of gauge terms, and I present techniques for introducing gauge terms of arbitrary order. Finally, I give a treatment of one application of Hertz potentials, namely calculating electromagnetic Casimir interactions for a couple of systems.

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CHAPTER I

INTRODUCTION

The theory of Hertzian potentials is arguably as well developed as the theory of usual electromagnetic potentials, dating back to the turn of the 20th century where they were introduced possibly for the first time by Whittaker [1]. However, most treatments of electromagnetism give Hertz potentials only a brief mention, and many of the popular modern textbooks leave out the topic entirely.

Due to this lack of common treatment, generalizations of Hertz potentials starting from the work of Nisbet [2] have been re-derived numerous times [3] [4] [5] in a variety of contexts with varied notation. This means that despite their prevalent use, readers can easily become confused when referencing multiple sources.

The goal of this thesis is to provide a cumulative overview of Hertz potentials, starting with their classical formulation and moving into their modern geometrical interpretation, developing on the construction provided in [6]. This provides a unified treatment that spans most of the groundwork required by current research. The expression of this goal is given in three parts.

First, in order to integrate Hertz potentials seamlessly into modern electromagnetic theory, one must find their place in classical electromagnetic theory. I present a brief review of the classical theory of electromagnetism, along with the construction of Hertz potentials. Since the intended audience of this thesis has a cursory, qualitative understanding of the history of electromagnetism, I present the material quickly

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This thesis follows the style of *Physical Review A*.

and in full, focusing more on the geometry and the mathematics rather than the physics. At numerous places in this chapter I note seemingly-arbitrary symmetries and consequences whose geometrical nature is developed further in later chapters.

Second, as with the comprehension of the general theory of relativity, an understanding of modern differential geometry reveals the geometrical nature of electromagnetism. Here I present a full discussion of modern differential geometry with a focus on the topics required to describe electromagnetic theory in a "nice" but not necessarily flat spacetime. What exactly constitutes "nice" will be treated rigorously.

Third, I develop and explore Hertz potentials in the differential geometric framework, namely using the formalism and notation of differential forms built in the previous chapter, and I give a number of new calculations of what can and cannot be done with these potentials. I develop the scalar framework explored initially in a flat spacetime setting [2] and show what can and cannot generalize. After this, I present the application of scalar Hertz potentials as it applies to quantum vacuum energy, which was the original motivation for this research.

In the final chapter, I qualitatively discuss what has been accomplished and what work remains.

CHAPTER II

ELECTROMAGNETISM USING VECTOR CALCULUS

A. Maxwell's Equations

Electromagnetism begins with the concept of the electric and magnetic vector fields $\vec{E}$ and $\vec{B}$, respectively, that each takes as input a position in space and time and each outputs a vector in $\mathbb{R}^3$ (i.e. $\vec{E}, \vec{B} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$). These satisfy Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{2.1}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{\jmath} \tag{2.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.3}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.4}$$

where the function ρ takes as input a position in space and time and outputs a real number, and $\vec{\jmath}$ takes as input a position in space and time and outputs a vector in $\mathbb{R}^3$ (i.e. $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ and $\vec{\jmath} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$).

Along with the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}), \tag{2.5}$$

this system of partial differential equations completely describes the classical motion of charged matter in classical space and time.

Note that immediately from Maxwell's equations comes a charge conservation relation

$$\vec{\nabla} \cdot \vec{\jmath} + \frac{\partial \rho}{\partial t} = 0, \qquad (2.6)$$

which restricts the nature of the functions ρ and $\vec{\jmath}$.

The classical theory of electricity and magnetism is developed in order to solve various boundary-value/initial-value problems associated with the positions of charge densities and current densities enclosed in (or partially enclosed in or unenclosed by) boundaries of varying conductivity and permeability.

The special case, called the "source-free" case, is when $\rho \equiv 0$ and $\vec{\jmath} \equiv \vec{0}$. For various reasons, beginning with the case's simplicity and for geometrical and algebraic significance explored in later chapters, this case will be studied frequently throughout this text. This case describes all possible forms the vector-valued electric and magnetic functions of space and time may take when the space and time considered is removed from actual physical charges. This includes regions of space arbitrarily close to a source, but not containing the source, that are topologically equivalent to $\mathbb{R}^3$.

B. Potential Functions

Even though Maxwell's equations are first-order differential equations, within them many components of the electric and magnetic fields are mixed together (called coupling), which often makes finding solutions difficult. Luckily, for compact domains or electromagnetic fields that decay sufficiently rapidly, the Helmholtz Decomposition Theorem provides functions called potentials that are commonly much easier to work with, despite the fact that they turn Maxwell's equations into second-order differential equations.

Equation (2.3) guarantees that there exists a vector field $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{2.7}$$

This, along with equation (2.4) yields

$$\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0, \tag{2.8}$$

which guarantees there exists a scalar field Φ, taking input from space and time and outputting a real number, such that

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}. \tag{2.9}$$

There exists an additional freedom in the determination of the potential functions Φ and $\vec{A}$ that does not exist in the fields $\vec{E}$ and $\vec{B}$. This freedom is called a gauge invariance.

Let Φ and $\vec{A}$ be potential functions for a given physical system. Let χ be any scalar function such that $(\frac{\partial}{\partial t}\vec{\nabla} - \vec{\nabla}\frac{\partial}{\partial t})\chi = 0$. Define new potential functions Φ' and $\vec{A}'$ by

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \tag{2.10}$$

$$\Phi' = \Phi - \frac{\partial \chi}{\partial t}. \tag{2.11}$$

Then this gives new electric and magnetic fields $\vec{E}'$ and $\vec{B}'$ where

$$\vec{E}' = -\vec{\nabla}\Phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}(\Phi - \frac{\partial \chi}{\partial t}) - \frac{\partial(\vec{A} + \vec{\nabla}\chi)}{\partial t} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} = \vec{E}$$
$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\chi) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}\chi) = \vec{\nabla} \times \vec{A} = \vec{B}$$

since $\vec{\nabla} \times (\vec{\nabla}\chi) = 0$ for any function χ.

Given this freedom, further conditions may be applied to Φ and $\vec{A}$. One of these is the Lorenz gauge condition

$$\vec{\nabla} \cdot \vec{A} + \frac{\partial \Phi}{\partial t} = 0, \tag{2.12}$$

which can be formed of any potentials Φ and $\vec{A}$ by solving the equation

$$(\frac{\partial^2}{\partial t^2} - \nabla^2)\chi = \vec{\nabla} \cdot \vec{A} + \frac{\partial \Phi}{\partial t} \tag{2.13}$$

for χ. The geometrical significance of this choice of gauge condition will come to light when deriving the Hertz potentials in both vector calculus and differential geometry. This reduces Maxwell's equations on $\vec{E}$ and $\vec{B}$ to the homogeneous wave equations on Φ and $\vec{A}$:

$$\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = 0 \tag{2.14}$$
$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 0. \tag{2.15}$$

C. Hertz Potentials

The process of deriving potentials can be repeated. Since the scalar potential $-\Phi$ is a differentiable function defined on the vector space $\mathbb{R}^3$, there exists a vector field $\vec{\Pi}_e$

such that

$$\vec{\nabla} \cdot \vec{\Pi}_e = -\Phi. \tag{2.16}$$

Using this, the Lorenz gauge condition (2.12) yields

$$\vec{\nabla} \cdot \vec{A} + \frac{\partial}{\partial t}(-\vec{\nabla} \cdot \vec{\Pi}_e) = \vec{\nabla} \cdot (\vec{A} - \frac{\partial \vec{\Pi}_e}{\partial t}) = 0$$

and thus there exists $\vec{\Pi}_m$ such that

$$\vec{\nabla} \times \vec{\Pi}_m = \vec{A} - \frac{\partial \vec{\Pi}_e}{\partial t}. \tag{2.17}$$

Using the relationships between the electric and magnetic fields and the electromagnetic potentials yields the following relationships between these new Hertz potentials and the desired fields:

$$\vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{\Pi}_e) - \frac{\partial^2 \vec{\Pi}_e}{\partial t^2} - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{\Pi}_m \tag{2.18}$$

$$\vec{B} = \vec{\nabla} \times \frac{\partial \vec{\Pi}_e}{\partial t} + \vec{\nabla} \times (\vec{\nabla} \times \vec{\Pi}_m). \tag{2.19}$$

These field relations automatically satisfy equations (2.3) and (2.4). Applying the other two Maxwell equations yield

$$\begin{aligned}\rho &= \vec{\nabla}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}_e) - \vec{\nabla}\cdot\frac{\partial^2\vec{\Pi}_e}{\partial t^2}\\ &= \nabla^2(\vec{\nabla}\cdot\vec{\Pi}_e) - \frac{\partial^2\vec{\nabla}\cdot\vec{\Pi}_e}{\partial t^2}\\ &= \vec{\nabla}\cdot(\nabla^2\vec{\Pi}_e - \frac{\partial^2\vec{\Pi}_e}{\partial t^2}) = -\vec{\nabla}\cdot(\Box\vec{\Pi}_e) \qquad (2.20)\\ \vec{\jmath} &= \vec{\nabla}\times(\vec{\nabla}\times\frac{\partial\vec{\Pi}_e}{\partial t}) - \vec{\nabla}(\vec{\nabla}\cdot\frac{\partial\vec{\Pi}_e}{\partial t}) + \frac{\partial^3\vec{\Pi}_e}{\partial t^3} + \vec{\nabla}\times(\vec{\nabla}\times(\vec{\nabla}\times\vec{\Pi}_m)) + \vec{\nabla}\times\frac{\partial^2\vec{\Pi}_m}{\partial t^2}\\ &= -\frac{\partial}{\partial t}\nabla^2\vec{\Pi}_e + \frac{\partial}{\partial t}\frac{\partial^2\vec{\Pi}_e}{\partial t^2} + \vec{\nabla}\times(\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}_m) - \nabla^2\vec{\Pi}_m) + \vec{\nabla}\times\frac{\partial^2\vec{\Pi}_m}{\partial t^2}\\ &= \frac{\partial}{\partial t}(\Box\vec{\Pi}_e) + \vec{\nabla}\times(\Box\vec{\Pi}_m). \qquad (2.21)\end{aligned}$$

To greatly simplify the notation, I define the operator $\Box$ as

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}. \qquad (2.22)$$

Let us now examine the source-free case, where these relations become

$$0 = \vec{\nabla}\cdot(\Box\vec{\Pi}_e) \qquad (2.23)$$

$$0 = \frac{\partial}{\partial t}(\Box\vec{\Pi}_e) + \vec{\nabla}\times(\Box\vec{\Pi}_m). \qquad (2.24)$$

Note that this is very reminicent of the initial relations for $\vec{E}$ and $\vec{B}$ that allowed for the creation of the electromagnetic potentials. In that vein, define $\vec{W} : \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$ such that

$$\Box\vec{\Pi}_e = \vec{\nabla}\times\vec{W}, \qquad (2.25)$$

and, as above, we see

$$\vec{\nabla}\times(\frac{\partial\vec{W}}{\partial t} + \Box\vec{\Pi}_m) = 0, \qquad (2.26)$$

so define $w : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ such that

$$\Box \vec{\Pi}_m = -\frac{\partial \vec{W}}{\partial t} - \vec{\nabla} w. \tag{2.27}$$

As we note here an apparent asymmetry between the two Hertz potentials, let us back up and remember how $\vec{\Pi}_e$ and $\vec{\Pi}_m$ came to be. We specified a particular gauge for $\vec{A}$ and Φ by equation (2.12), but suppose we relax this condition by introducing a new vector function $\vec{G}$ and new scalar function g such that

$$\vec{\nabla} \cdot \vec{A} + \frac{\partial \Phi}{\partial t} = -\vec{\nabla} \cdot \vec{G} - \frac{\partial g}{\partial t}. \tag{2.28}$$

Then

$$\vec{\nabla} \cdot (\vec{A} + \vec{G}) + \frac{\partial (\Phi + g)}{\partial t} = 0, \tag{2.29}$$

which gives new definitions of $\vec{\Pi}_e$ and $\vec{\Pi}_m$, namely

$$\vec{\nabla} \cdot \vec{\Pi}_e = -\Phi - g \tag{2.30}$$

$$\vec{\nabla} \times \vec{\Pi}_m = \vec{A} + \vec{G} - \frac{\partial \vec{\Pi}_e}{\partial t}. \tag{2.31}$$

Repeating the calculations above with our new $\vec{\Pi}_e$ and $\vec{\Pi}_m$ yield

$$\Box \vec{\Pi}_e = \vec{\nabla} \times \vec{W} + \vec{\nabla} g + \frac{\partial \vec{G}}{\partial t} \tag{2.32}$$

$$\Box \vec{\Pi}_m = -\frac{\partial \vec{W}}{\partial t} - \vec{\nabla} w + \vec{\nabla} \times \vec{G}, \tag{2.33}$$

restoring the symmetry.

CHAPTER III

DIFFERENTIAL GEOMETRY

The language and construction of Differential Geometry affords a multitude of benefits in the generalization and refinement of classical electrodynamics. But before electromagnetism can be studied, first we must study the mathematics.

This chapter begins with a spartan formulation of topology; just enough modern definitions and examples are given to provide the scaffolding necessary for constructing the desired differentiable structures. After the topology, the modern formulation of Differential Geometry is given, paying special attention to vector fields and differential forms, and objects built from these, as these are the extensions of classical vector fields that are of importance to this topic.

A. Topology

A topological space is the pairing of a set with a particular collection of subsets of that set. These subsets are called **open**. The formal definition axiomatically constructs the term "open set" to generalize certain aspects of open intervals of the real line, where these open intervals are defined in terms of the Euclidean distance function or the natural ordering on $\mathbb{R}$. An open interval (a, b) is a subset of the real line $\mathbb{R}$ defined by

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

and a closed interval $[a, b]$ is defined by

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

Notice first that for any number $x \in (a, b)$ there exists a smaller open interval centered at x completely contained within (a, b). For example, consider the interval $(1, 2)$, and let $\epsilon > 0$ be any very small number (such as 10^{-10}). Then $2-\epsilon$ is contained in this interval, and the interval $(2 - \frac{3\epsilon}{2}, 2 - \frac{\epsilon}{2})$ is centered on $2 - \epsilon$ and is contained completely in $(1, 2)$.

This is demonstrably not true with a closed interval. For instance, take now $[1, 2]$ and consider the endpoint $2 \in [1, 2]$. No matter what open interval we choose, it must always contain some number $2 + \epsilon$ for some $\epsilon > 0$. This is not in the interval $[1, 2]$.

Next, let $A_i = (0, 2 - \frac{1}{i})$ and $B_i = (0, 1 + \frac{1}{i})$. Then $\bigcup_{i=1}^{\infty} A_i = (0, 2)$. Since this infinite union produces an open set, so should the union of infinitely many open sets be open. But $\bigcap_{i=1}^{\infty} B_i = (0, 1]$, so only the intersection of finitely many open sets must be open.

Note also $\bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}$ and $(-1, 0) \cap (0, 1) = \emptyset$, so the entirety of a set and the empty set must be considered open as well. From these three properties we give the formal definition.

Definition III.1. *A* **topological space** *is a set X together with a collection of subsets $\mathcal{T}$ called a* **topology** *such that*

1. *the empty set and X are elements of $\mathcal{T}$,*

2. *any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$,*

3. any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

If $U \in \mathcal{T}$, then $U \subset X$ and U is called **open**.

Example III.2. *Real Numbers with the Standard Topology*

As the source of the intuition, the real numbers $\mathbb{R}$ with the topology

$$\mathcal{T} = \left\{ \bigcup_{i \in I} (a_i, b_i) : a_i, b_i \in \mathbb{R}, a_i < b_i \right\}$$

satisfy this definition.

However, this is not the only topology one can place on the real numbers.

Example III.3. *Real Numbers with the Finite Complement Topology*

Consider the set $\mathbb{R}$ of real numbers with the topology

$$\mathcal{T} = \{U \subset \mathbb{R} : \mathbb{R} \setminus U = \{u_1, u_2, \ldots, u_n\}, u_1, \ldots, u_n \in \mathbb{R}, n \in \mathbb{Z}^+\}.$$

This is called the **finite complement** topology. Conceptually, in the finite complement topology, a set is open of it consists of the real line with only finitely many numbers punched out. This means no "strips" (intervals) are missing, just singularities, and only so many of them.

To show that it is a topology, note that taking a union of sets can only decrease the number of missing numbers. Also, finitely many intersections increases the number of missing elements by only finite amounts, so such sets will still be open.

Coming up with an explicit formulation of every open set in a topological space can be difficult at times, and so we would find useful a way to reduce our effort, but still to provide rigorously everything one would need to reconstruct our topology. A

basis for a topology does just this and mimics somewhat the concept of a basis from linear algebra.

Definition III.4. *A subset* $\mathcal{B} \subset \mathcal{T}$ *is called a* **basis** *if every set* $U \in \mathcal{T}$ *can be written as a union* $U = \bigcup_{B_i \in \mathcal{B}} B_i$ *of some* $B_i \in \mathcal{B}$*.*

Equivalently, a subset $\mathcal{B} \subset \mathcal{T}$ is a **basis** for the topology $\mathcal{T}$ if for every $U, V \in \mathcal{B}$ and $x \in U \cap V$ there exists $W \in \mathcal{B}$ such that $x \in W$, $W \subset U \cap V$, and $X = \bigcup_{B_i \in \mathcal{B}} B_i$.

Example III.5. *Open Intervals with Rational Endpoints*

A set $A \subset X$ is dense in X if for every $x \in X$ and every open set $U \subset X$ with $x \in U$, $A \cap U \neq \emptyset$. Since the rational numbers ($\mathbb{Q}$) are dense in the real numbers ($\mathbb{R}$), take for a basis the set of all open intervals with rational endpoints. That is, for the real numbers $\mathbb{R}$ with the standard topology, let

$$\mathcal{B} = \{(a, b)|a, b \in \mathbb{Q}, a < b\}.$$

For any real numbers $x, y \in \mathbb{R}$ with $x < y$, let $\{x_i\}_{i=0}^{\infty}$, with $x_i < y$, be a sequence converging to x from above and let $\{y_i\}_{i=0}^{\infty}$, with $y_i > x$, be a sequence converging to y from below. Then $(x, y) = \bigcup_{i=0}^{\infty}(x_i, y_i)$.

This example along with the next will be critical soon.

Example III.6. *Open Balls with Rational Centers and Radii*

The example above can be extended to $\mathbb{R}^n$ by taking

$$\mathcal{B} = \{B_q(r)|r \in \mathbb{Q}, q \in \mathbb{Q}^+\}$$

where $B_q(r) = \{x \in \mathbb{R}|d(x, r) \leq q\}$ with $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the usual Euclidean distance function.

The following two definitions will later restrict pathological topological spaces from complicating our discussion of differential geometry. I include them here simply for completeness and note their influence on physics is arguably minimal.

Definition III.7. *A topological space is called* **second countable** *if there exists a basis for the topology $\mathcal{B}$ such that the cardinality of $\mathcal{B}$ is countable (i.e. the set $\mathcal{B}$ contains countably many elements).*

Definition III.8. *A topological space X is called* **Hausdorff** *or* $\mathbf{T_2}$ *if for every $x, y \in X$ with $x \neq y$ there exist $U, V \subset X$ with U and V open such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

We are now zeroing in on exactly how we want to construct our generalized spacetime with regards to topology, however, to relate topological spaces to each other, we need a purely topological definition of an analytical construct from calculus, namely the continuous function. For spaces similar to Euclidean space ("similar" will be defined more rigorously shortly) this definition is equivalent to the usual epsilon-delta limit definition.

Definition III.9. *A function $f : X \to Y$ between topological spaces is called* **continuous** *if for every $V \subset Y$ such that V is open in Y, $f^{-1}(V)$ is open in X.*

Next, we would like to characterise when two spaces are topologically indistinguishable, and for this we use the following definition.

Definition III.10. *A function $f : X \to Y$ between topological spaces is called a* **homeomorphism** *if*

1. *f is a 1-1 and onto (i.e. f is* **bijective***),*
2. *f is continuous,*
3. *and f^{-1} is continuous.*

We can now define what it means to be "similar" to Euclidean space.

Definition III.11. *A topological space X is* **locally Euclidean** *or* **locally homeomorphic to Euclidean space** *if there exists an n such that for every $x \in X$ there* *exists a neighborhood $N \subset X$, with $x \in N$, where N is homeomorphic to $\mathbb{R}^n$. For this n, we can also say X is* **locally homeomorphic to** $\mathbb{R}^n$.

Finally, we finish this section with the definitions most pertinent to our geometric investigations.

Definition III.12. *A* **topological manifold** *is a second countable Hausdorff space that is locally Euclidean.*

Definition III.13. *A* **topological manifold with boundary** *is a second countable Hausdorff space that is locally homeomorphic to* $\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^1 \geq 0\}$.

B. Differentiable Manifolds

A topological manifold provides homeomorphisms between itself and Euclidean space. In the discussion of differentiable manifolds, these homeomorphisms are called **charts**. Consider two such maps, $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ with $U \cap V \neq \emptyset$. Since each is a homeomorphism, each is bijective, and so we can consider the map

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V).$$

This provides a homeomorphism from a subset of $\mathbb{R}^n$ to another subset of $\mathbb{R}^n$. Since calculus is defined on such domains already, it makes sense to say whether or not $\phi \circ \psi^{-1}$ is differentiable.

Definition III.14. *Let M be a topological manifold. Let $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ with $U \cap V \neq \emptyset$ and $U, V \subset M$. Then ϕ and ψ are called* **compatible** *if the two* *maps*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$
$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are differentiable as functions on Euclidean space.

But this is only a local consideration. To describe the entire manifold, we need a collection of charts where every point on the manifold is in the domain of some chart.

Definition III.15. *An* **atlas** *is a collection of compatible charts that covers the manifold. That is, let M be a topological manifold, let $\mathcal{A}$ be a set of charts (U_i, x_i), and let $p \in M$. Then $\mathcal{A}$ is an atlas if the charts (U_i, x_i) are compatible and $p \in U_j$ for some j.*

Definition III.16. *Two atlases are* **compatible** *if all charts with overlapping domains are compatible.*

Definition III.17. *We say two atlases $\mathcal{A}, \mathcal{A}'$ are equivalent if they can be combined to form another atlas $\mathcal{A}'' = \mathcal{A} \cup \mathcal{A}'$. A* **differentiable manifold** *is an equivalence class of compatible atlases under this equivalence relation.*

Definition III.18. *A manifold is called a* **$\mathbf{C}^k$ manifold** *if all transition maps are k-times continuously differentiable, and a manifold is called a* **$\mathbf{C}^\infty$ manifold** *or* **smooth manifold** *if all transition maps have derivatives of all orders.*

Unless otherwise stated, we assume all manifolds to be smooth, since present experiment does not seem to imply that spacetime (up to quantum corrections) is not smooth.

Definition III.19. *Let M and N be manifolds. A function $f : M \to N$ is* **k-times differentiable** *or* $\mathbf{C}^k$ *if for all coordinate charts (U, x) in M and (V, y) in N, $x^{-1} \circ f \circ y$ is C^k where defined as a function between subsets of Euclidean space.*

With all of these definitions, let us take a moment to explore some examples.

Example III.20. *Euclidean Space* ($\mathbb{R}^n$)

As a trivial example, we can consider the manifold consisting of Euclidean space. To show that this is a manifold, we need an atlas, and the manifold is then the equivalence class of atlases for which our atlas is a representative. For simplicity, we can consider a single chart consisting of the whole space under the identity map $i : \mathbb{R}^n \to \mathbb{R}^n$. Symbolically, this means $U = \mathbb{R}^n$ and $x(p) = i(p) = p$, where $p = (p_1, \ldots, p_n)$. Thus $\mathbb{R}^n$ is a manifold.

Example III.21. *2-Sphere* ($S^2 \subset \mathbb{R}^3$)

The sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ is a manifold of dimension 2. To define this manifold, again we need to specify an atlas. In this case, there is no one chart that can cover the whole sphere (as the sphere is compact, and compact sets in $\mathbb{R}^3$ are closed). Instead, we can choose the sphere less the point $(0, 0, 1)$ under the map of stereographic projection, and likewise take the sphere less the point $(0, 0, -1)$ again under stereographic projection.

Let $p = (p_x, p_y, p_z) \in S^2$ be a point on the sphere. We define the first stereographic projection by the function $\phi : S^2 \to \mathbb{R}^2$,

$$(p_x, p_y, p_z) \mapsto (\frac{p_x}{1-p_z}, \frac{p_y}{1-p_z}),$$

and the second stereographic projection by the function $\psi : S^2 \to \mathbb{R}^2$,

$$(p_x, p_y, p_z) \mapsto (\frac{p_x}{1+p_z}, \frac{p_y}{1+p_z}).$$

C. Vectors and Covectors

In vector calculus, the concept of a vector field is the pairing of each point in Euclidean space with a vector based at that point. For Euclidean space, this is well-defined since the space containing the point is also a vector space. However, for a differentiable manifold, this is not the case in general, and so a more careful definition of vectors and vector fields must be chosen.

First, we need to determine what it means to have a vector based at a point on a manifold. With this understanding, we then need to decide what a vector field looks like.

1. Vectors and Covectors

Instead of viewing a vector as an independent entity, we can think of it as the rate of change of a curve. If we imagine a curve as the trajectory of a particle, the instantaneous rate of change of the position of that particle tells us the direction the particle is moving and the magnitude of its velocity. We can abstract this view to a curve on an arbitrary differentiable manifold via the following definition.

Definition III.22. *Let M be a smooth manifold with $p \in M$, let (U, x) be a chart with $p \in U$, and let $\gamma_0 : (-\epsilon, \epsilon) \to U$ with $\epsilon > 0$ be a function such that $\gamma_0(0) = p$ and $(x \circ \gamma_0)(t)$ is differentiable at $t = 0$.*

The class of all functions $\gamma : (-\epsilon, \epsilon) \to U$ such that $\gamma(0) = p$ and $\frac{d}{dt}[x \circ \gamma]_{t=0} = \frac{d}{dt}[x \circ \gamma_0]_{t=0}$ forms an equivalence class, and this class is called a **tangent vector***.*

It is a standard exercise to show that the space of all tangent vectors at a point $p \in M$ forms a vector space of the same dimension as M. This vector space is denoted T_pM and is called the **tangent space of M at p**.

This definition affords another view of the tangent vector: that of an operator on differentiable functions. To illustrate this, let M be a smooth manifold, and let $f : M \to \mathbb{R}$ be a C^1 function. We can then make the following definition: for $v \in T_xM$ with representative $\gamma : (-\epsilon, \epsilon) \to M$,

$$v(f) = \frac{d}{dt}[f \circ \gamma]_{t=0}.$$

This is called the **derivative of f in the direction of v**, and is a generalization of the directional derivative from vector calculus.

So far the tangent space seems like an arbitrary, abstract construction we have simply attached to a point on M. However we can build a basis for the tangent space based on the coordinate chart around p, and when this is accomplished, the nature of the tangent space becomes clear.

Before we do this, however, we need to explore the dual vector space to the tangent space, called the **cotangent space**.

Definition III.23. *Let M be a smooth manifold, let $p \in M$ and let (U, x) be a* *coordinate chart with $p \in U$.*

Given the tangent space T_pM, we define the **cotangent space** *to be $T_p^*M = (T_pM)^*$, the dual space of T_pM. Elements of T_p^*M are called* **covectors**.

This means that elements of the cotangent space are linear functionals on the tangent space, and that the two vector spaces have the same dimension. We can use the coordinate chart (U, x) to define a natural basis on the cotangent space using the concept of the differential of a function.

Definition III.24. *Let $f : M \to N$ be a differentiable function between smooth manifolds M, N. The* **differential of** *f (or the* **pushforward** *of f) is a linear map $df_p = f_{*p} : T_pM \to T_{f(p)}N$ defined as follows. For any $v = [\gamma] \in T_pM$,*

$$df_p(v) = [f \circ \gamma] \in T_{f(p)}N.$$

We now note that the coordinate map, $x : U \to \mathbb{R}^n$ consists of n functions, namely $x = (x^1, \ldots, x^n)$ with $x^j : M \to \mathbb{R}$. The differentials of each of the these n maps at $p \in M$ forms a basis for T_p^*M, namely $\{dx^i\}_{i=1}^n$.

The dual to this dual space basis becomes the basis for the tangent space, and it is denoted $\{\partial_i\}_{i=1}^n$ in order to realize the following notation.

$$dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial x^i}{\partial x^j} = \delta_j^i,$$

where δ_j^i denotes the Kronecker delta ($\delta_i^i = 1$ and $\delta_j^i = 0$ for $i \neq j$).

Likewise, we observe the following about the direction derivative. Let $v = v^i\partial_i \in T_pM$ and let $f : M \to \mathbb{R}^n$ be a C^k function. Then the derivative of f in the direction of v is

$$v(f) = (v^i \partial_i)(f) = v^i(\partial_i f),$$

which matches the usual definition *and* notation from vector calculus.

Example III.25. *Vectors and Covectors in Euclidean 3-Space* ($\mathbb{R}^3$)

Let us consider the usual Euclidean three-dimensional space. For our coordinate chart we can consider the whole space and the usual Cartesian coordinates. Under this association, familiar vectors in 3-space based at particular points (such as $v = 2\hat{x} + 3\hat{z}$ at the point $p = (0, 1, 0)$) become vectors in the tangent spaces of those points (likewise, $v = 2\partial_x + 3\partial_z$ at the same point $p = (0, 1, 0)$).

Using this notation, transforming vectors under a coordinate transformation becomes straightforward as a pushforward between manifolds. Consider spherical coordinates. The manifold mapping from $\mathbb{R}^+ \times (0, \pi) \times (-\pi, \pi)$ to $\mathbb{R}^3$ is represented by the function

$$x(r, \theta, \phi) = (r \sin\theta \sin\phi, r \sin\theta \cos\phi, r \cos\theta).$$

This function maps an open half-infinite box in $\mathbb{R}^3$ to all of $\mathbb{R}^3$ with the negative-x part of the xz-plane removed. The inverse map is

$$x^{-1}(x, y, z) = \left(\sqrt{x^2 + y^2 + z^2}, \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right), \operatorname{Arg}(x + iy) \right)$$

where the argument function is expressable as a piecewise-defined function of arctan that is nonetheless differentiable.

For an arbitrary point (x, y, z), $d(x^{-1})$ takes the form of the matrix

$$d(x^{-1}) = \begin{pmatrix} \frac{\partial (x^{-1})^1}{\partial x} & \frac{\partial (x^{-1})^1}{\partial y} & \frac{\partial (x^{-1})^1}{\partial z} \\ \frac{\partial (x^{-1})^2}{\partial x} & \frac{\partial (x^{-1})^2}{\partial y} & \frac{\partial (x^{-1})^2}{\partial z} \\ \frac{\partial (x^{-1})^3}{\partial x} & \frac{\partial (x^{-1})^3}{\partial y} & \frac{\partial (x^{-1})^3}{\partial z} \end{pmatrix},$$

which for $p = (0, 1, 0)$ takes the form

$$d(x^{-1})|_p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence, our vector of $2\partial_x + 3\partial_z$ transforms to $-3\partial_\theta - 2\partial\phi$ at the point $(0, 1, 0)$.

2. Vector Fields and Differential Forms

The disjoint union of all vector spaces of M,

$$TM = \bigcup_{p \in M} \{p\} \times T_p M,$$

is also a differentiable manifold of dimension twice that of M, and it is called the **tangent bundle** of M. The reason we want to look at such a union is because, unlike for Euclidean space, on a manifold there is no canonical transformation that takes us from one tangent space to another, so if we want to talk about a vector field on a manifold, it is not enough to simply supply a position and a vector in an arbitrary n-dimensional vector space. We must specify the point $p \in M$ and the vector *in that point's tangent space* T_pM. This brings us to our next definitions.

Definition III.26. *A* **vector field** *is a function* $X : M \to TM$ *that sends a point*

$p \in M$ to a vector in that point's tangent space. That is, $p \mapsto X_p \in T_pM$.

A **covector field** *(or* **1-form***) is a function $\alpha : M \to T^*M$ that sends a point $p \in M$ to a covector in that point's cotangent space. That is, $p \mapsto \alpha_p \in T_p^*M$.*

Definition III.27. *A vector field $X : M \to TM$ is C^k or* **k-times differentiable** *if $X(f) : M \to \mathbb{R}$, $p \mapsto X_p(f)$ is a C^k function for every C^m function f with $m > k$. The vector field $X : M \to TM$ is* **smooth** *if this is true for all k.*

A covector field $\alpha : M \to T^*M$ is C^k or **k-times differentiable** *if $\alpha(X) : M \to \mathbb{R}$, $p \mapsto \alpha_p(X_p)$ is a C^k function for every C^m vector field X with $m \geq k$. The covector field is* **smooth** *if this is true for all k.*

We denote the space of all C^k 1-forms on M as $\Omega_k(M)$ or $\Omega_k^1(M)$. The space of all smooth 1-forms on M is denoted $\Omega(M)$ or $\Omega^1(M)$.

D. Multivectors and k-Forms

To extend the concept of the vector field and 1-form into higher-dimensional analogues, we first need a coordinate-independent way to express such objects.

Definition III.28. *A k-linear functional $f : (T_pM)^k \to \mathbb{R}$ is* **alternating** *if for every $v \in T_pM$,*

$$f(\ldots, v, \ldots, v, \ldots) = 0$$

The space of all alternating k-linear functionals on T_pM is denoted $\bigwedge^k T_p^*M$.

*The space of k-***vectors** *$\bigwedge^k T_pM$ (which are* **multivectors***) is defined as the dual to the space of alternating k-linear functionals on T_pM, namely*

$$\bigwedge^k T_pM = (\bigwedge^k T_p^*M)^*.$$

The space $\bigwedge^k T_p^* M$ is a vector space of dimension $\binom{n}{k}$, where n is the dimension of M. As in the extensions of vectors and covectors to vector fields and 1-forms, we can extend these point-wise definitions to form C^k fields over our manifold M.

Definition III.29. *A* k**-form** *is a function* $\alpha : M \to \bigwedge^k T^* M$ *such that each point* $p \in M$ *is sent to an element of* $\bigwedge^k T_p^* M$.

A k**-vector field** *is a function* $X : M \to \bigwedge^k TM$ *such that each point* $p \in M$ *is sent to an element of* $\bigwedge^k T_p M$.

A k*-form* α *is* C^l *if for every set of* C^m *vector fields* $X_1, \ldots, X_k : M \to TM$ *with* $m \geq k$*, the function* $\alpha(X_1, \ldots, X_k) : M \to \mathbb{R}$, $p \mapsto \alpha_p((X_1)_p, \ldots, (X_k)_p)$ *is* C^l.

A k*-vector field* X *is* C^l *if for every set of* C^m *functions* $f_1, \ldots, f_k : M \to \mathbb{R}$ *with* $m > k$*, the function* $X(f_1, \ldots, f_k) : M \to \mathbb{R}$ *is* C^l.

Definition III.30. *Let* α *be a* k*-form and* β *be an* l*-form. Let* $X_1, \ldots, X_{k+l}$ *be vector fields, and let* S_{k+l} *be the set of permutations on* $k + l$ *letters. The* **wedge product** $\alpha \wedge \beta$ *is defined as*

$$\alpha \wedge \beta(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma) \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}).$$

Let X *be a* k*-vector and* Y *be an* l*-vector. Let* $f_1, \ldots, f_{k+l}$ *be functions on* M*, and let* S_{k+l} *as above. The* **wedge product** $X \wedge Y$ *is defined as*

$$X \wedge Y(f_1, \ldots, f_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) X(f_{\sigma(1)}, \ldots, f_{\sigma(k)}) Y(f_{\sigma(k+1)}, f_{\sigma(k+l)}).$$

E. The Exterior Derivative and De Rham Cohomology

The exterior derivative is a generalization of the gradient, divergence, and curl operations from vector calculus. It is the unique operator that is defined by the following three conditions, but reduces in local coordinates to a simple to use form.

Definition III.31. *Let M be a C^k manifold of dimension n, and let $m \leq k$. Then $d : \Omega^l(M) \to \Omega^{l+1}(M)$ is called an* **exterior derivative** *if*

1. *df is the differential of f for $f \in C^m(M)$,*
2. *$d(df) = 0$ for any $f \in C^m(M)$,*
3. *$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$.*

Let α be a C^m p-form on a C^k manifold M of dimension n, and let (U, x) be a coordinate chart. Then in U we can express α in terms of $\binom{n}{k}$ functions $\alpha_{i_1 \cdots i_p}$ by

$$\alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

The exterior derivative of α is then

$$d\alpha = \frac{1}{p!} \frac{\partial \alpha_{i_1 \cdots i_p}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Theorem III.32. *Poincaré Lemma*

Let $U \subset \mathbb{R}^n$ be open and simply connected, and let α be a p-form such that $d\alpha = 0$. Then there exists a $p - 1$-form β such that

$$\alpha = d\beta.$$

We denote by Z_p the set of p-forms with vanishing exterior derivative; these are the **closed** **p-forms**. That is, if $d\alpha = 0$, then $\alpha \in Z_p$. We denote by B_p the set of p-forms that are the derivatives of $p-1$-forms; these are the **exact** **p-forms**. That is, if $\alpha = d\beta$ for some form $p-1$-form β, then $\alpha \in B_p$. Since $d^2\alpha = 0$ for any α, we see that $B_p \subset Z_p$.

Definition III.33. *The* **de Rham cohomology space of degree** k *is*

$$H^k(M) := \frac{Z_k}{B_k}.$$

Even though the cohomology space is defined by this quotient of spaces, it is best calculated by other means and then applied to forms. For example, if $B_k = Z_k$, then $H^k(M)$ becomes trivial, and we know that every closed k-form is also exact, as with the Poincaré Lemma.

F. The Metric

All of our work so far has been independent of any **metric**, or notion of length of a vector and distance. Now we incorporate this concept into our study.

Definition III.34. *A* **metric** *is a function* $g_p : T_pM \otimes T_pM \rightarrow \mathbb{R}$ *such that for* $u, v, w \in T_p(M); a, b \in \mathbb{R}$,

$$\begin{aligned} g(au + bv, w) &= ag(u, w) + bg(v, w) && \textit{(g is linear)} \\ g(u, w) &= g(w, u) && \textit{(g is symmetric)} \\ g(u, X) = 0 \textit{ for all } X \in T_pM &\iff u = 0 && \textit{(g is non-degenerate)} \end{aligned}$$

In a traditional treatment of the metric in Differential Geometry, instead of non-degeneracy, one would impose that the metric is positive definite, which means $g(v, v) \geq 0$ for all $v \in T_pM$ and vanishes if and only if $v = 0$. This requirement turns each tangent and cotangent space into a Hilbert space, allowing the utilization of a great many theorems related to Hilbert spaces. However, since the goal is to apply all work here to spacetime or compact subsects of spacetime, we are not afforded this luxury.

Our definition of a metric is, so far, limited to individual tangent spaces; however we can extend this in the following way.

Definition III.35. *A* **metric tensor** *is a function $g : M \times TM \otimes TM \to \mathbb{R}$ such that for all $p \in M$, $g(p, \cdot, \cdot) = g_p(\cdot, \cdot)$ is a metric. We say the metric tensor g is C^k if for all C^k vector fields $v, w \in TM$, $g(v, w) : M \to \mathbb{R}$ is a C^k function on M.*

Such a metric tensor can be generated by, say, the Einstein field equations.

Theorem III.36. *Let $p \in U \subset M$, and let $x : U \to \mathbb{R}^n$ be a coordinate chart around p. A metric g can be uniquely represented by an $n \times n$ symmetric matrix with elements denoted $g_{\mu\nu}$.*

Knowing this now about our metric allows us to utilize concepts and theorems about matrices from linear algebra. One of important note is that of **signature**. The definition of signature varies widely by author, but the purpose of signature is to carry information about the sign of the eigenvalues of the metric.

Definition III.37. *Let g be a metric on an n-dimensional manifold M with eigen-*

values $e_1, \ldots, e_n$. The **signature** *of g is the sum*

$$s = \sum_{i=1}^{n} sgn(e_i) = \sum_{i=1}^{n} \frac{e_i}{|e_i|}$$

We note that if g is a C^k metric tensor with $k \geq 1$ on a connected manifold M, its signature is constant on all of M.

Definition III.38. *Let M be an n-dimensional C^k manifold with $k \geq 2$. Let g be a C^k metric tensor on M with $k \geq 2$. Then we say (M, g) is a* **Lorentzian Manifold** *if the signature of g is $s = n - 2$ or $s = 2 - n$.*

Since g is non-degenerate, it can have only non-vanishing eigenvalues. Thus, this definition of a Lorentzian manifold means that the metric tensor has precisely 1 negative and $n - 1$ positive eigenvalues or 1 positive and $n - 1$ negative eigenvalues.

Theorem III.39. *Let g be a metric. Then g forms an isomorphism $g : T_pM \to T_p^*M$, $v \mapsto g(v, \cdot)$.*

Let (U, x) be a coordinate chart with $p \in U \subset M$ and $v \in T_pM$. Theorems III.36 and III.39 allow us to write the associated covector to v as $v_\mu = g_{\mu\nu}v^\nu$ where $\{\frac{\partial}{\partial x^\mu}\}$ forms the basis for T_pM and $\{dx^\mu\}$ forms the basis for T_p^*M.

Theorem III.40. *Let g be a C^k metric tensor, and let $v \in TM$ be a C^j vector field. Then g forms an isomorphism $g : TM \to T^*M$, and $v \mapsto g(v, \cdot)$ preserves differentiability of v for $j \leq k$.*

G. The Hodge Dual and Coderivative

A given metric $\langle \cdot, \cdot \rangle_p = g_p(\cdot, \cdot) : T_p^* M \otimes T_p^* M \rightarrow \mathbb{R}$ defines a unique unit volume n-form $\nu \in \Omega^n(M)$. We define $*_k : \Omega^k(M) \rightarrow \Omega^{n-k}(M)$, called the **Hodge star** by requiring that for every $\alpha, \beta \in \Omega^k(M)$,

$$\int_M \alpha \wedge *_k \beta = \int_M \langle \alpha, \beta \rangle \nu, \tag{3.1}$$

where we note that for given α, β, $\langle \alpha, \beta \rangle : M \rightarrow \mathbb{R}$ is a real-valued function on the manifold. From this point forward we drop the subscript on the Hodge star to improve readability and rely on the degree of its operand to remove ambiguity.

We can use the metric and the volume form to define a metric on the space of k-forms as follows.

Definition III.41. *For any $\alpha, \beta \in \Omega^k(M)$,*

$$<\alpha, \beta> := \int_M \langle \alpha, \beta \rangle \nu \tag{3.2}$$

Definition III.42. *The* **coderivative** $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ *is given by the expression*

$$\delta = (-1)^{k+1} *^{-1} d \, * \, .$$

We define the coderivative in this way to make the following observation. Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$ (such that α, β vanish on ∂M if $\partial M \neq \emptyset$). Then

$\alpha\wedge*\beta \in \Omega^{n-1}(M)$, so

$$\begin{aligned} d(\alpha\wedge*\beta) &= (d\alpha)\wedge*\beta + (-1)^k\alpha\wedge d*\beta \\ &= (d\alpha)\wedge*\beta) - \alpha\wedge*[(-1)^k *^{-1} d*\beta] \\ &= (\langle d\alpha,\beta\rangle - \langle\alpha,\delta\beta\rangle)\nu, \end{aligned}$$

so

$$\begin{aligned} 0 &= \int_M d(\alpha\wedge\beta) \\ &= \int_M \langle d\alpha,\beta\rangle\nu - \int_M \langle\alpha,\delta\beta\rangle\nu, \end{aligned}$$

by Stokes' theorem, which means $<d\alpha,\beta> = <\alpha,\delta\beta>$, so that δ is the adjoint operator to d under the $<\cdot,\cdot>$ metric.

H. Hodge Decomposition

For a compact, Riemannian manifold M, there exists the following decomposition:

$$\Omega^k(M) = B_k \times c_k \times H_k,$$

where B_k is the set of exact k-forms, c_k the set of co-exact k-forms, and H_k the space of harmonic k-forms, that is the space of k-forms α such that $\Delta\alpha = (d\delta + \delta d)\alpha = 0$. This means that every k-form ω can be written as

$$\omega = d\alpha + \delta\beta + \gamma,$$

where α is a $k-1$-form, β a $k+1$-form, and γ a harmonic k-form.

For a compact, pseudo-Riemannian manifold M, the Hodge decomposition follows similarly, though with a additional subtlety. We define a set Λ_k as follows:

$$\Lambda_k = B_k \times c_k \times (Z_k \cap C_k),$$

where Z_k is the set of closed k-forms, and C_k is the set of co-exact k-forms. Then one of the following is true:

1. M is *Strongly de Rham* (in the sense of [7]) if $\Lambda_k = \Omega^k(M)$.

2. M is *Weakly de Rham* (in the sense of [7]) if Λ_k is dense in $\Omega^k(M)$.

For the purposes of this thesis, we assume every manifold is strongly de Rham. Thus for every k-form ω there exist α, β, γ such that

$$\omega = d\alpha + \delta\beta + \gamma,$$

where α is a $k-1$-form, β a $k+1$-form, and γ a k-form such that $d\gamma = 0$ and $\delta\gamma = 0$.

CHAPTER IV

ELECTROMAGNETISM AND DIFFERENTIAL GEOMETRY

A. Electromagnetic 2-Forms

With a firm grasp of the relevant parts of differential geometry under our belt, we can now pursue a differential geometric formulation of electromagnetic theory.

On a Lorentzian manifold, a 2-form is called an electromagnetic field if it satisfies two conditions,

$$dF = 0 \tag{4.1}$$

$$\delta F = J, \tag{4.2}$$

where J is a 1-form that satisfies the charge conservation condition $\delta J = 0$. From a geometrical perspective, this means that all closed 2-forms are electromagnetic fields for some charge distribution. In the following example, we see how this is a generalization of the vector calculus Maxwell equations (2.1)–(2.4).

Example IV.1. *Cartesian Flat Space*

For the case of a 3+1-dimensional flat space using Cartesian coordinates, the metric takes values $1 = -g^{00} = g^{11} = g^{22} = g^{33}$ and $0 = g^{\alpha\beta}, \alpha \neq \beta$, which gives for values of the volume form $\eta^{0123} = \eta_{0123} = -1$. The electromagnetic field then takes the form $F = -E_i dx^0 \wedge dx^i + B_i * (dx^0 \wedge dx^i)$, and the current density 1-form is $J = -\rho dx^0 + j_i dx^i$ where $\vec{\jmath} = j^i \vec{e}_i$ is the current vector seen in Chapter II (indices are raised and lowered using the metric as usual: $J^i = g^{i\mu} J_\mu$).

Equations (4.1) and (4.2) then become

$$\begin{aligned} dF &= (-\partial_k E_j + \eta^{0i}{}_{jk}\partial_0 B_i)dx^0\wedge dx^j\wedge dx^k + (\eta^{0i}{}_{jk}\partial_i B_i)dx^i\wedge dx^j\wedge dx^k \\ &= \sum_{j<k}(\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t})\cdot\vec{e}_i\eta^{0i}{}_{jk}dx^0\wedge dx^j\wedge dx^k + (\vec{\nabla}\cdot\vec{B})dx^1\wedge dx^2\wedge dx^3 \qquad (4.3) \\ \delta F &= *[(-\partial_k B_j - \eta^{0i}{}_{jk}\partial_0 E_i)dx^0\wedge dx^j\wedge dx^k - (\eta^{0i}{}_{jk}\partial_i E_i)dx^i\wedge dx^j\wedge dx^k] \\ &= -\vec{\nabla}\cdot\vec{E}dx^0 + (\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t})\cdot\vec{e}_i dx^i \qquad (4.4) \end{aligned}$$

Compare the results of equation (4.4) with equations (2.1) and (2.2), and equation (4.3) with equations (2.3) and (2.4). We see that equations (4.1) and (4.2) form a faithful, coordinate-independent generalization of Maxwell's equations.

B. Potential 1-Forms

In the case of a simply-connected manifold, (4.1) means F is closed, and so by the Poincaré lemma F is also exact. Thus, there is a 1-form A such that

$$F = dA. \qquad (4.5)$$

Moreover, this form is not unique. Let χ be any function. Then for $A' = A + d\chi$ we find

$$dA' = d(A + d\chi) = dA = F. \qquad (4.6)$$

Example IV.2. *Cartesian Flat Space Revisited*

We again wish to see what this looks like in Cartesian coordinates. For $A =$

$-\Phi dx^0 + A_i dx^i$, we find

$$dA = (\partial_0 A_i - \partial_i A_0) dx^0 \wedge dx^i + \partial_j A_k dx^j \wedge dx^k \tag{4.7}$$

$$= -((-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}) \cdot \vec{e}_i) dx^0 \wedge dx^i + (\vec{\nabla} \times \vec{A}) \cdot \vec{e}_i * (dx^0 \wedge dx^i), \tag{4.8}$$

which precisely matches the results in the second chapter.

C. Non-Trivial Topology

For this section, we assume our Lorentzian manifold and metric g are "strongly de Rham" in the sense of [7]. Then, equation (4.1) implies there exist A, a 1-form, and f, a 2-form, such that

$$F = dA + f \tag{4.9}$$

$$df = 0, \ \delta f = 0. \tag{4.10}$$

Note that this means F is the consequence of an electromagnetic potential A superimposed on a solution to the vacuum ($J \equiv 0$) Maxwell equations *generated by no such potential.*

Considerations of these topological complications can be seen as in the treatment in [8], wherein the singularity caused by a black hole produces additional freedoms in field quantities. Principally, a non-vanishing magnetic flux can exist over a closed surface surrounding the singularity, making the black hole appear to have a magnetic charge.

CHAPTER V

HERTZ POTENTIALS AND DIFFERENTIAL GEOMETRY

This chapter is a deeper treatment of the form representation of Hertz potentials used in [6], along with a construction for the primary purpose of treating the use of Hertz potentials in [9] and related work.

A. Hertz Potential 2-Forms

As we saw in Chapter IV, the theorems from vector calculus in Chapter II that bore the electromagnetic potentials were really a specific expression of the Poincaré lemma. This process can be repeated utilizing the gauge freedom of the electromagnetic potential 1-form.

For any particular choice of A, take χ such that $\delta d\chi = -\delta A$. Then taking $A' = A + d\chi$ yields

$$\delta A' = 0. \tag{5.1}$$

This is the Lorenz guage, and this makes A' coclosed, so, in this simply connected space, coexact. (From here, we drop the prime from A'.)

Since A is coexact, there exists a 2-form Π such that $\delta\Pi = A$. This is the Hertz potential 2-form.

Now the electromagnetic 2-form takes the form

$$F = dA = d\delta\Pi. \tag{5.2}$$

Example V.1. *Cartesian Coordinates*

Let's look at the Hertz potential 2-form under specific coordinates to see how it is a generalization of our derivation from vector calculus. The simplest expression is under Cartesian coordinates. In this frame, the metric takes the form $1 = -g^{00} = g^{11} = g^{22} = g^{33}$ and $0 = g^{\alpha\beta}, \alpha \neq \beta$. As we saw in the previous chapter, this makes $F = -E_i dx^0 \wedge dx^i + B_i * (dx^0 \wedge dx^i)$, and $A = -\phi dx^0 + A_i dx^i$. If we make the definition $\Pi = (\Pi_e)_i dx^0 \wedge dx^i + (\Pi_m)_i * (dx^0 \wedge dx^i)$, we then see that

$$\delta\Pi = (\vec{\nabla} \cdot \vec{\Pi}_e) dx^0 + ((\vec{\nabla} \times \vec{\Pi}_m)_i + (\frac{\partial \vec{\Pi}_e}{\partial t})_i) dx^i.$$

This matches our definitions (2.16) and (2.17) from Chapter II.

B. Gauge Invariance

Applying (4.2) to (5.2) gives

$$J = \delta F = \delta dA = \delta d\delta\Pi = \delta(d\delta + \delta d)\Pi = \delta\Box\Pi. \tag{5.3}$$

Since J is coclosed, it is coexact. Therefore let Q be a 2-form such that $\delta Q = J$. Then $\delta\Box\Pi = \delta Q$, and so

$$\Box\Pi = Q + \delta C$$

for some 3-form C. Since Q is already arbitrary, we can assume that $\delta C = 0$, yielding

$$\Box\Pi = Q. \tag{5.4}$$

Again let us look at the source-free case, as that is the condition for vacuum. This means $J = 0$, which leads to $\delta Q = 0$, making Q coclosed/coexact. So let W be a 3-form such that $\delta W = Q$. Then we obtain

$$\Box \Pi = \delta W. \tag{5.5}$$

This is one of the two gauge freedom four-vectors seen in Chapter II. The other arises from observing the following.

We noted before that the condition (5.1) gives us the ability to define Π, but this condition can be relaxed if we introduce a new 1-form G via the relation

$$\delta A = -\delta G. \tag{5.6}$$

This changes the definition of Π to

$$\delta \Pi = A + G \tag{5.7}$$

yielding the new field relationship as

$$F = d(\delta \Pi - G) = d\delta \Pi - dG. \tag{5.8}$$

This means we must update equation (5.3) to

$$\delta d \delta \Pi - \delta d G = J \tag{5.9}$$

and so finally, through the same manipulations as before, we obtain

$$\Box \Pi = dG + \delta W. \tag{5.10}$$

This expresses the full gauge freedom, up to second order, of the Hertz potentials, without changing A (even by its own gauge transformation). In effect, (5.10) determines the maximum freedom inherent in choosing Π.

C. Scalar Hertz Potentials

Throughout the history of electromagnetic theory, scalar potentials whose derivatives yield the standard electromagnetic potentials have occasionally been used to greatly simplify various calculations. [1] [10] These potentials always come in pairs and are actually Hertz potentials of two non-vanishing components.

In more modern quantum field theoretical applications, quantization of scalar fields is more well-known and direct than the quantization of vector fields. Therefore, in certain circumstances, it becomes advantageous to find a scalar field or collection of scalar fields that relate to the desired vector fields through known, fixed relationships. Hertz potentials under the formulation above are apt for some of these circumstances.

Obviously, by the above formulation, the Hertz potentials are not truly scalar fields. But, by breaking the background independence inherent in the differential geometry, we may choose the 2-form Π to attain non-vanishing values in only two dimensions of the vector space of 2-forms. These components of Π comprise the scalar fields we seek.

In the following examples, we bring together the works of several past authors. As each author has their preferred notation, the notation so far developed is used instead so as to illustrate the unification of ideas under the differential geometry formalism.

Examples of Scalar Hertz Potentials

Example V.2. *Cartesian Scalar Hertz Potentials*

We consider first Cartesian coordinates, where $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. The earliest such treatment of Hertz potentials is found in [1]; however a more general vector-based treatment is given in [2], with [3] converting it to relativistic covariance. Finally, the results are used in [9] in a manner we will mirror in the next chapter. Let ϕ, ψ be defined such that $\Pi = \phi dx^{01} + \psi dx^{23}$, with the notation $dx^{\alpha_1 \cdots \alpha_n} = dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_n}$. Then

$$\Box\Pi = \Box\phi dx^{01} + \Box\psi * (dx^{01}), \tag{5.11}$$

and so to satisfy $\Box\phi = 0 = \Box\psi$, we take simply $\Box\Pi = 0$, $G = 0$, and $W = 0$.

This yields the electromagnetic 2-form F in terms of the Hertz potential Π as

$$\begin{aligned} F = & -E_z dt\wedge dz - E_x dt\wedge dx - E_y dt\wedge dy + B_z*(dt\wedge dz) + B_x*(dt\wedge dx) + B_y*(dt\wedge dy) \\ = & (\partial_t^2\phi - \partial_z^2\phi)dt\wedge dz + (\partial_t\partial_y\psi - \partial_x\partial_z\phi)dt\wedge dx + (-\partial_t\partial_x\psi - \partial_y\partial_z\phi)dt\wedge dy \qquad (5.12) \\ & + (-\partial_x^2\psi - \partial_y^2\psi)*(dt\wedge dz) + (\partial_t\partial_y\phi + \partial_z\partial_x\psi)*(dt\wedge dx) + (\partial_y\partial_z\psi - \partial_t\partial_x\phi)*(dt\wedge dy) \end{aligned}$$

Note that here we needed no gauge functions. The significance of this construction is that the wave equation yields all solutions to the vacuum states of the electromagnetic field in Cartesian coordinates through the relationship between Hertz potentials and electromagnetic fields.

Example V.3. *Axial Scalar Hertz Potentials in Cylindrical Coordinates*

Consider now cylindrical coordinates. To use a similar form as the above derivation, we take our coordinates to be ordered non-typically, namely let $(x^0, x^1, x^2, x^3) =$

(t, z, ρ, θ). Note, however, that this merely rearranges the metric terms; the volume form η remains unchanged. So let ϕ, ψ be defined such that $\Pi = \phi dx^{01} + \psi{*}dx^{01}$. Again, this system has been treated by [2], and [5] and [11] use its results in applications related to Casimir interactions.

Essentially identically to the Cartesian case, we obtain

$$\Box\Pi = \Box\phi dx^{01} + \Box\psi{*}(dx^{01}). \tag{5.13}$$

Again note that this simple scalar relation on ϕ and ψ appears without requiring the gauge forms G and W to be nonvanishing.

The electromagnetic field 2-form F takes the form

$$\begin{aligned} F = & -E_z dt{\wedge}dz - E_\rho dt{\wedge}d\rho - E_\varphi dt{\wedge}d\varphi \\ & + B_z{*}(dt{\wedge}dz) + B_\rho{*}(dt{\wedge}d\rho) + B_\varphi{*}(dt{\wedge}d\varphi) \\ = & (\partial_t^2\phi - \partial_z^2\phi)dt{\wedge}dz + (\frac{1}{\rho}\partial_t\partial_\varphi\psi - \partial_z\partial_\rho\phi)dt{\wedge}d\rho \\ & + (-\rho\partial_t\partial_\rho\psi - \partial_\varphi\partial_z\phi)dt{\wedge}d\varphi + (-\frac{1}{\rho}\partial_\rho\rho\partial_\rho\psi - \frac{1}{\rho^2}\partial_\varphi^2\psi){*}(dt{\wedge}dz) \\ & + (\frac{1}{\rho}\partial_t\partial_\varphi\phi + \partial_z\partial_\rho\psi){*}(dt{\wedge}d\rho) + (\partial_\varphi\partial_z\psi - \rho\partial_t\partial_\rho\phi){*}(dt{\wedge}d\varphi) \end{aligned} \tag{5.14}$$

Example V.4. *Spherical Scalar Hertz Potentials*

The spherical case makes use of the gauge forms G and W in a straightforward way. It was first treated by [12] and again independently by [10]. A basic discussion is given by [2], upon which the following treatment builds. This time, we define ϕ and ψ such that

$$\Pi = \phi dx^{01} + \psi({*}dx^{01}) = \phi dx^{01} + \psi({\eta^{01}}_{23}dx^{23}) = \phi dx^{01} + \psi r^2 \sin\theta dx^{23}. \tag{5.15}$$

Here I would like to define a new operator, $\hat{\Box}$, defined in spherical coordinates as

$$\hat{\Box} f = \Box f + \frac{2}{r}\partial_r f = \partial_t^2 f - \partial_r^2 f - \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\partial_\theta f) - \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f. \tag{5.16}$$

Note that $\hat{\Box}\phi = 0$ still denotes a homogeneous wave equation for ϕ, but $\hat{\Box}$ is not the D'Alembertian operator for spherical coordinates.

After laborious calculations, we find

$$\begin{aligned}
\Box\Pi &= (\hat{\Box}\phi - \frac{2}{r}\partial_r\phi + \frac{2}{r^2}\phi)dx^{01} + (-\frac{2}{r}\partial_\theta\phi)dx^{02} + (-\frac{2}{r}\partial_\varphi\phi)dx^{03} \\
&\quad + (\hat{\Box}\psi - \frac{2}{r}\partial_r\psi + \frac{2}{r^2}\psi)(r^2\sin\theta dx^{23}) + (-\frac{2}{r}\partial_\theta\psi)(-\sin\theta dx^{13}) \\
&\quad + (-\frac{2}{r}\partial_\varphi\psi)(\frac{1}{\sin\theta}dx^{12}) \qquad (5.17)\\
&= (\hat{\Box}\phi - \frac{2}{r}\partial_r\phi + \frac{2}{r^2}\phi)dx^{01} + (-\frac{2}{r}\partial_\theta\phi)dx^{02} + (-\frac{2}{r}\partial_\varphi\phi)dx^{03} \\
&\quad + (\hat{\Box}\psi - \frac{2}{r}\partial_r\psi + \frac{2}{r^2}\psi){*dx^{01}} + (-\frac{2}{r}\partial_\theta\psi){*dx^{02}} + (-\frac{2}{r}\partial_\varphi\psi){*dx^{03}}. \qquad (5.18)
\end{aligned}$$

Let us pause here for a moment to look at the 2-form just calculated. If we were to forget that we have G and W at our disposal and could only take $\Box\Pi = 0$, this would put some odd constraints on ϕ and ψ. The dx^{02} and dx^{03} (and their conjugate dx^{13} and dx^{12}) terms imply that our scalar functions are independent of solid angle by both the θ and φ coordinates. Further, if we still attempt to impose the conditions $\hat{\Box}\phi = 0 = \hat{\Box}\psi$, then the dx^{01} and dx^{23} terms show that our functions are independent of the radial coordinate as well! So it becomes imperative, rather than just interesting, that these gauge freedoms exist.

Turning back now to calculations, the relations $\hat{\Box}\phi = 0 = \hat{\Box}\psi$ and $\Box\Pi =$

$dG + \delta W$ show

$$dG + \delta W = (\frac{2}{r^2}\phi - \frac{2}{r}\partial_r\phi)dx^{01} + (-\frac{2}{r}\partial_\theta\phi)dx^{02} + (-\frac{2}{r}\partial_\varphi\phi)dx^{03} + (\frac{2}{r^2}\psi - \frac{2}{r}\partial_r\psi)*dx^{01} + (-\frac{2}{r}\partial_\theta\psi)*dx^{02} + (-\frac{2}{r}\partial_\varphi\psi)*dx^{03}. \quad (5.19)$$

Taking $G = 2\frac{\phi}{r}dx^0$ and $*W = 2\frac{\psi}{r}dx^0$ satisfies this requirement, thus leaving us again with scalar fields that satisfy a homogeneous wave equation and provide the general electromagnetic fields in vacuum.

This yields an expression for the electromagnetic field F as

$$\begin{aligned} F =&(\partial_t^2\phi - \partial_r^2\phi)dt\wedge dr + (\frac{1}{\sin\theta}\partial_t\partial_\varphi\psi - \partial_r\partial_\theta\phi)dt\wedge d\theta \\ &+ (-\sin\theta\partial_t\partial_\theta\psi - \partial_\varphi\partial_r\phi)dt\wedge d\varphi + (-\frac{1}{r^2\sin\theta}\partial_\theta\sin\theta\partial_\theta\psi - \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\psi)*(dt\wedge dr) \\ &+ (\frac{1}{\sin\theta}\partial_t\partial_\varphi\phi + \partial_r\partial_\theta\psi)*(dt\wedge d\theta) + (\partial_\varphi\partial_r\psi - \sin\theta\partial_t\partial_\theta\phi)*(dt\wedge d\varphi). \quad (5.20)\end{aligned}$$

Example V.5. *Spherical Schwarzschild Metric*

One of the benefits of this formulation is perfect agreement with General Relativity. Consequently, metrics other than flat space can be taken and yield consistent results. One such result is a case for scalar Hertz potentials that satisfy a wave-like equation near an uncharged, non-rotating, spherically symmetric massive body.

For this construction we take the spherical Schwarzschild metric

$$ds^2 = (1 - \frac{r_s}{r})dt^2 + (\frac{1}{1 - \frac{r_s}{r}})dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

To simplify the notation, set $\zeta = 1 - \frac{r_s}{r}$. This gives

$$g^{00} = \frac{1}{\zeta}, g^{11} = \zeta, g^{22} = \frac{1}{r^2}, g^{33} = \frac{1}{r^2\sin^2\theta},$$

and again we take

$$\Pi = \phi dx^{01} + \psi(*dx^{01}) = \phi dx^{01} + \psi r^2 \sin\theta dx^{23}. \tag{5.21}$$

Note that, for this coordinate system, the operator $\Box$ takes the form

$$\Box = \frac{1}{\zeta}\partial_t^2 - \frac{1}{r^2}\partial_r \zeta r^2 \partial_r - \frac{1}{r^2}\partial_\theta^2 - \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 \tag{5.22}$$

$$\Rightarrow \Box f = \frac{1}{\zeta}\partial_t^2 f - \frac{1}{r^2}\partial_r(\zeta r^2(\partial_r f)) - \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) - \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f. \tag{5.23}$$

So once more, define a new operator $\tilde{\Box}$ by

$$\tilde{\Box} f = \Box f + \frac{2}{r}\zeta\partial_r f = \frac{1}{\zeta}\partial_t^2 f - \partial_r(\zeta\partial_r f) - \frac{1}{r^2}\partial_\theta^2 f - \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 f. \tag{5.24}$$

Take $\tilde{\Box}\phi = 0 = \tilde{\Box}\psi$ like in the standard spherical case, and (5.10) yields

$$\begin{aligned} dG + \delta W = (&-\frac{2\zeta}{r}\partial_r\phi - (\frac{2\zeta'}{r} - \frac{2\zeta}{r^2})\phi)dx^{01} + (-\frac{2\zeta}{r}\partial_\theta\phi)dx^{02} \\ &+ (-\frac{2\zeta}{r}\partial_\varphi\phi)dx^{03} + (-\frac{2\zeta}{r}\partial_r\psi - (\frac{2\zeta'}{r} - \frac{2\zeta}{r^2})\psi)*dx^{01} \\ &+ (-\frac{2\zeta}{r}\partial_\theta\psi)*dx^{02} + (-\frac{2\zeta}{r}\partial_\varphi\psi)*dx^{03}. \end{aligned} \tag{5.25}$$

Therefore, we find natural the choices $G = \frac{2\zeta}{r}\phi dx^0$ and $*W = \frac{2\zeta}{r}\psi dx^0$.

Since the only fact used about ζ was its dependence strictly on the first coordinate, these results can be immediately generalized to any metric taking the form

$$ds^2 = \frac{dt^2}{\zeta} - \zeta dr^2 - \frac{d\theta^2}{r^2} - \frac{d\varphi^2}{r^2\sin^2\theta}, \tag{5.26}$$

which includes the Reissner-Nørdström metric of a charged, non-rotating black hole.

Example V.6. *Radial Scalar Hertz Potentials in Cylindrical Coordinates*

We return now to cylindrical coordinates, taking the traditional arrangement of $(x^0, x^1, x^2, x^3) = (t, \rho, \varphi, z)$. The Hertz 2-form takes the form

$$\begin{aligned}\Pi &= \phi\ dx^{01} + \psi\ {*dx^{01}} \\ &= \phi\ dt{\wedge}d\rho + \psi\ \rho d\varphi{\wedge}dz.\end{aligned}$$

This choice of non-vanishing Hertz potential components does not appear in the literature, and in fact it is this choice that inspired the search for the general formulation.

We now wish to find the equations of motion for this Hertz potential, and through calculations find

$$\Box\Pi = (\Box\phi + \frac{\phi}{\rho^2})dx^{01} - \frac{2}{\rho}\partial_\varphi\phi dx^{02} + (\Box\psi + \frac{\psi}{\rho^2}) * dx^{01} - \frac{2}{\rho}\partial_\varphi\psi * dx^{02}. \qquad (5.27)$$

And at this point, we pause. If we treat this as the Cartesian scenario, with vanishing gauge terms, then we obtain a similar problem to that in the spherical case. Namely, vanishing gauge terms imply, from the second term above, that the scalar Hertz potentials do not depend on the polar coordinate. We know from the scalar Hertz potentials in the axial direction above that this is not the case. However, if we treat this as the spherical case, with gauge terms G and W dependent upon first derivatives of ϕ and ψ, respectively, then one finds that the scalar potentials become independent of the *axial* direction, which is again contradicted by the above case. In fact, assuming G independent of ψ and W independent of ϕ (while each dependent on the other function, respectively, and possibly explicitly on the coordinate variables) leads to the scalar potentials being independent of at least one coordinate, which is unacceptable.

At this point, the author admits an impasse with the case and an inability to find general solutions to ϕ and ψ that construct all possible electromagnetic fields F via the relation $d\delta(\phi dx^{01} + \psi{*}dx^{01}) = F$.

For reference, the expression for the electromagnetic 2-form without including a G term is

$$
\begin{aligned}
F =&(\partial_t^2\phi - \partial_\rho \frac{1}{\rho}\partial_\rho \rho\phi) dt{\wedge}d\rho \\
&+ (\rho\partial_t\partial_z\psi - \frac{1}{\rho}\partial_\rho\rho\partial_\varphi\phi) dt{\wedge}d\varphi \\
&+ (-\frac{1}{\rho}\partial_t\partial_\varphi\psi - \partial_z\frac{1}{\rho}\partial_\rho\rho\phi) dt{\wedge}dz \\
&+ (-\frac{1}{\rho^2}\partial_\varphi^2\psi - \partial_z^2\psi){*}(dt{\wedge}d\rho) \\
&+ (\rho\partial_t\partial_z\phi + \rho\partial_\rho\frac{1}{\rho}\partial_\varphi\psi){*}(dt{\wedge}d\varphi) \\
&+ (\partial_z\frac{1}{\rho}\partial_\rho\rho\psi - \partial_t\partial_\varphi\psi){*}(dt{\wedge}dz).
\end{aligned} \tag{5.28}
$$

As a final note on the radial cylindrical scalar Hertz potentials, we would like to highlight the following. For our choice of $\Pi = \phi \; dt{\wedge}d\rho + \psi \; {*}(dt{\wedge}d\rho)$, consider

$$G = \delta\Pi - \delta\Pi_A$$

$$W = d\Pi - d\Pi_A$$

where $\Pi_A = \phi_A \; dt{\wedge}dz + \psi_A \; {*}(dt{\wedge}dz)$, which is the axial cylindrical Hertz potential.

We remember that $\Box\Pi = d\delta\Pi + \delta d\Pi$, so we may rewrite $\Box\Pi = dG + \delta W$ as

$$\begin{aligned} 0 &= d\delta\Pi - dG + \delta d\Pi - \delta W \qquad (5.29) \\ &= d(\delta\Pi - G) + \delta(d\Pi - W) \\ &= d(\delta\Pi - [\delta\Pi - \delta\Pi_A]) + \delta(d\Pi - [d\Pi - d\Pi_A]) \\ &= d\delta\Pi_A + \delta d\Pi_A. \qquad (5.30) \end{aligned}$$

What we have done here is use the gauge terms G and W to turn the radial construction into the axial construction, effectly "decoupling" the choice of scalar Hertz potential from coordinate system. This, however, is of little practical use, as the form of the electromagnetic field F will be exactly that of the axial cylindrical case, and the equations of motion for the scalar potentials are, again, those from the axial case.

D. Higher-Order Gauge Transformations

We have seen that we can take a weak restriction on the gauge of A to find a d'Alembertian equation, (5.10), for the Hertz potential 2-form Π with two gauge terms. All derivatives in these expressions remain at or below second order. Now we take a look at higher-order gauge transformations.

Let Π be a Hertz potential that generates some source-free ($J \equiv 0$) electromagnetic field satisfying (5.10) with gauge terms G and W. Then we define a new 2-form

Π', along with G' and W' by the relations

$$\Pi' = \Pi - d\Gamma - \delta\Lambda \tag{5.31}$$

$$G' = G - \delta d\Gamma \tag{5.32}$$

$$W' = W - d\delta\Lambda \tag{5.33}$$

where Γ is an arbitrary 1-form and Λ an arbitrary 3-form. Note that

$$\begin{aligned} \Box\Pi' &= (d\delta + \delta d)\Pi' \\ &= d\delta(\Pi - d\Gamma) + \delta d(\Pi - \delta\Lambda) \\ &= \Box\Pi - d\delta d\Gamma - \delta d\delta\Lambda \\ &= d(G - \delta d\Gamma) + \delta(W - d\delta\Lambda) \\ &= dG' + \delta W', \end{aligned}$$

and

$$\begin{aligned} \delta\Pi' &= \delta(\Pi - d\Gamma - \delta\Lambda) \\ &= \delta\Pi - \delta d\Gamma \\ &= A + (G - \delta d\Gamma) \\ &= A + G'. \end{aligned}$$

This means Π' satisfies a similar type of equation as Π and generates exactly the same potential A, which means it generates the same electromagnetic field F.

Due to the identities $d^2 = 0, \delta^2 = 0$, we can, in fact, add even more terms,

$$\Pi' = \Pi - d(\Gamma_0 + \delta(\Gamma_1 + d(\Gamma_2 + \cdots))) - \delta(\Lambda_0 + d(\Lambda_1 + \delta(\Lambda_2 + \cdots))) \tag{5.34}$$

$$G' = G - \delta d(\Gamma_0 + \delta(\Gamma_1 + d(\Gamma_2 + \cdots))) \tag{5.35}$$

$$W' = W - d\delta(\Lambda_0 + d(\Lambda_1 + \delta(\Lambda_2 + \cdots))), \tag{5.36}$$

depending on the symmetries and demands of the system investigated.

The apparent wealth of available gauge terms of this form might be exploited to achieve constructions of scalar Hertz potentials as sought in the previous section. An investigation of this was not done during this project, but it merits future research.

E. Non-Trivial Topology

All constructions to this point have been done in a trivial topology. This means the space contains no singularities or significant contortions. Now, utilizing the power of de Rham Cohomology and Hodge theory on Lorentzian manifolds [7], we drop the constraint of trivial topology and reformulate Hertz potentials in spacetimes with more interesting geometry.

As in the section of the same name from Chapter IV, let F be a 2-form that satisfies Maxwell's equations. Then there exist A, a 1-form, and f, a 2-form, such that $F = dA + f$ and $df = 0, \delta f = 0$. Now, suppose A satisfies the Lorenz condition $\delta A = 0$, then there exist Π, a 2-form, and a, a 1-form, such that

$$A = \delta\Pi + a \tag{5.37}$$

$$da = 0, \delta a = 0. \tag{5.38}$$

This means that

$$\begin{aligned} F &= dA + f \\ &= d(\delta\Pi + a) + f \\ &= d\delta\Pi + f \end{aligned}$$

where the last relation is due to equation (5.38). From this we can deduce, as before,

$$\begin{aligned} J &= \delta F = \delta(dA + f) \\ &= \delta d(\delta\Pi + a) = \delta(d\delta\Pi + \delta d\Pi) \\ &= \delta\Box\Pi. \end{aligned}$$

Let us once again examine the source-free case. Then $\delta\Box\Pi = 0$ implies there exist W, a 3-form, and β, a 2-form, such that

$$\Box\Pi = \delta W + \beta$$

where β is a solution to the vacuum Maxwell equations. We can repeat the procedure as in the trivial topology case and again insert a dG term, yielding

$$\Box\Pi = dG + \delta W + \beta, \tag{5.39}$$

where, again, $d\beta = 0, \delta\beta = 0$, and is thus a solution to the vacuum Maxwell equations.

The existence of an electromagnetic field with no corresponding Hertz potential corresponds to the so-called TEM modes of classical electromagnetism. A specific example of this will be treated at the end of the next chapter.

CHAPTER VI

APPLICATIONS – QUANTUM VACUUM ENERGY

A more recent use of scalar Hertz potentials has been in the calculation of the Casimir force through vacuum energy calculations. By treating the scalar Hertz potentials as a pair of non-interacting scalar fields, we can more easily quantize the electromagnetic field in a given compact geometry and produce vacuum expectation values for the energy of the field. This is accomplished by calculating the normal modes of the scalar potential functions, quantizing them with a special normalizing factor, and finding the expectation value for the energy in a vacuum state.

A. Perfectly Conducting Rectangular Cavity

This section is a summary and sharpening of material in [9]. Consider a hollow box with side lengths a, b, and c running in the x, y, and z directions, respectively, with a corner on the origin. We take the vector Hertz potentials to be in the $\hat{z}$ direction, making the Hertz 2-form take the form

$$\Pi = \phi dt \wedge dz + \psi * (dt \wedge dz). \tag{6.1}$$

Perfectly conducting boundary conditions on the sides lead to Neumann boundary conditions on the z-dimensional boundaries and Dirichlet conditions on the remaining boundaries for the ϕ function, and the reverse for the ψ function (Dirichlet on the $z = 0$ and $z = c$ planes, and Neumann on the remaining boundaries). This yields classical solutions of the form

$$\phi(t,x,y,z) = \sum_{l,m,n=0}^{\infty} \frac{1}{k_l^2 + k_m^2} \sin(k_l x)\sin(k_m y)\cos(k_n z) \times (E_{lmn}^{+} e^{i\omega_{lmn}t} + E_{lmn}^{-} e^{-i\omega_{lmn}t}) \tag{6.2}$$

$$\psi(t,x,y,z) = \sum_{l,m,n=0}^{\infty} \frac{B_{lmn}}{k_l^2 + k_m^2} \cos(k_l x)\cos(k_m y)\sin(k_n z) \times (B_{lmn}^{+} e^{i\omega_{lmn}t} + B_{lmn}^{-} e^{-i\omega_{lmn}t}), \tag{6.3}$$

where $k_l = \frac{\pi l}{a}, k_m = \frac{\pi m}{b}, k_n = \frac{\pi n}{c}$, and $\omega_{lmn}^2 = k_l^2 + k_m^2 + k_n^2$.

Moving into the quantum realm, we want the resulting electromagnetic field operators to satisfy the commutation relations

$$[E_i(\vec{x}), B_j(\vec{y})] = i\epsilon_{ij}{}^{k} \frac{\partial}{\partial x^k}\delta(\vec{x}-\vec{y}), \tag{6.4}$$

$$[E_i(\vec{x}), E_j(\vec{y})] = 0, \tag{6.5}$$

$$[B_i(\vec{x}), B_j(\vec{y})] = 0, \tag{6.6}$$

which take t in both arguments to be the same, so this requires a non-standard quantization of the above fields. The resulting fields are then

$$\phi(t,\vec{x}) = \sum_{l,m,n=0}^{\infty} \frac{N_{lmn}}{\sqrt{2k_\perp^2 \omega_{lmn}}} \sin(k_l x)\sin(k_m y)\cos(k_n z) \times (a_{lmn} e^{-i\omega_{lmn} t} + a^\dagger_{lmn} e^{i\omega_{lmn} t}) \tag{6.7}$$

$$\psi(t,\vec{x}) = \sum_{l,m,n=0}^{\infty} \frac{N_{lmn}}{\sqrt{2k_\perp^2 \omega_{lmn}}} \cos(k_l x)\cos(k_m y)\sin(k_n z) \times (b_{lmn} e^{-i\omega_{lmn} t} + b^\dagger_{lmn} e^{i\omega_{lmn} t}), \tag{6.8}$$

where $k_\perp^2 = k_l^2 + k_m^2$ and $N_{lmn} = 2\sqrt{2}/\sqrt{abc}$, and $a, a^\dagger, b, b^\dagger$ satisfy the commutation relations

$$[a_{lmn}, b_{l'm'n'}] = 0 = [a^\dagger_{lmn}, b_{l'm'n'}] = [a_{lmn}, b^\dagger_{l'm'n'}] = [a^\dagger_{lmn}, b^\dagger_{l'm'n'}] \tag{6.9}$$

$$[a_{lmn}, a^\dagger_{l'm'n'}] = \delta_{ll'}\delta_{mm'}\delta_{nn'} = [b_{lmn}, b^\dagger_{l'm'n'}]. \tag{6.10}$$

1. Commutation Relations

Now we must check that our choice of ϕ and ψ satisfy the commutation relations given above. Beginning with (6.5),

$$\begin{aligned}
[E_x(\vec{x}), E_y(\vec{y})] =& [\partial_z\partial_x\phi(t,x,y,z) - \partial_t\partial_y\psi(t,x,y,z), \partial_{y'}\partial_{z'}\phi(t,x',y',z') + \partial_t\partial_{x'}\psi(t,x',y',z')] \\
=& [\partial_z\partial_x\phi(t,x,y,z), \partial_{y'}\partial_{z'}\phi(t,x',y',z')] - [\partial_t\partial_y\psi(t,x,y,z), \partial_t\partial_{x'}\psi(t,x',y',z')] \\
=& \sum_{lmn}\sum_{l'm'n'} C_{lmn,l'm'n'} \cos(k_l x)\sin(k_{l'}x')\sin(k_m y)\cos(k_{m'}y')\sin(k_n z)\sin(k_{n'}z') \\
& \times ([a_{lmn}, a^\dagger_{l'm'n'}] + [a^\dagger_{lmn}, a_{l'm'n'}]) \\
& + i\sum_{lmn}\sum_{l'm'n'} D_{lmn,l'm'n'} \cos(k_l x)\sin(k_{l'}x')\sin(k_m y)\cos(k_{m'}y')\sin(k_n z)\sin(k_{n'}z') \\
& \times ([b_{lmn}, -b^\dagger_{l'm'n'}] + [-b^\dagger_{lmn}, b_{l'm'n'}]) \\
=& 0
\end{aligned}$$

by the relation $[a_{lmn}, a^\dagger_{l'm'n'}] = \delta_{ll'}\delta_{mm'}\delta_{nn'} = [b_{lmn}, b^\dagger_{l'm'n'}]$, where

$$C_{lmn,l'm'n'} = \frac{N_{lmn}N_{l'm'n'}k_l k_{m'} k_n k_{n'}}{2\sqrt{k_\perp^2 k_{\perp'}^2 \omega_{lmn}\omega_{l'm'n'}}} \tag{6.11}$$

and

$$D_{lmn,l'm'n'} = \frac{N_{lmn}N_{l'm'n'}k_{l'} k_m \omega_{lmn}\omega_{l'm'n'}}{2\sqrt{k_\perp^2 k_{\perp'}^2 \omega_{lmn}\omega_{l'm'n'}}}. \tag{6.12}$$

Similar relations follow for any commutations that appear symmetric in time derivatives, that is, with the same number of time derivatives in the first argument of the commutator as the second. This occurs for every iteration of (6.5) and (6.6), so all that remains is to check (6.4). We begin with E_z and B_x to obtain

$$
\begin{aligned}
[E_z(\vec{x}), B_x(\vec{y})] &= [-\nabla_\perp \phi(t,x,y,z), \partial_t \partial_{y'} \phi(t,x',y',z') + \partial_{z'}\partial_{x'}\psi(t,x',y',z')] \\
&= -\partial_{y'}[\nabla_\perp \phi(t,x,y,z), \partial_t \phi(t,x',y',z')] \qquad (6.13) \\
&= -i\partial_{y'} \sum_{l,m,n} \sum_{l',m',n'} \frac{N_{lmn} N_{l'm'n'} |k_\perp| \sqrt{\omega_{lmn}}}{2|k_{\perp'}|\sqrt{\omega_{l'm'n'}}} \\
&\quad \times \sin(k_l x)\sin(k_{l'} x')\sin(k_m y)\sin(k_{m'} y')\cos(k_n z)\cos(k_{n'} z') \\
&\quad \times ([a_{lmn}, a^\dagger_{l'm'n'}] + [a^\dagger_{lmn}, -a_{l'm'n'}]) \\
&= -i\partial_{y'} \Bigg[(\sum_l N_l^2 \sin(k_l x)\sin(k_{l'} x'))(\sum_m N_m^2 \sin(k_m y)\sin(k_{m'} y')) \\
&\quad \times (\sum_n N_n^2 \cos(k_n z)\cos(k_{n'} z')) \Bigg] \\
&= -i\partial_{y'}[\delta(x-x')\delta(y-y')\delta(z-z')] \\
&= i\partial_y \delta(\vec{x}-\vec{y}). \qquad (6.14)
\end{aligned}
$$

Next, E_z with B_y yields

$$
\begin{aligned}
[E_z(\vec{x}), B_y(\vec{y})] &= [-\nabla_\perp \phi(t,x,y,z), \partial_{y'}\partial_{z'}\psi(t,x',y',z') - \partial_t \partial_{x'} \phi(t,x',y',z')] \\
&= \partial_{x'}[\nabla_\perp \phi(t,x,y,z), \partial_t \phi(t,x',y',z')], \qquad (6.15)
\end{aligned}
$$

but here we notice a similar form to (6.13), and so we can immediately write down the answer

$$
\begin{aligned}
[E_z(\vec{x}), B_y(\vec{y})] &= \partial_{x'}(i\delta(\vec{x}-\vec{y})) \\
&= -i\partial_x \delta(\vec{x}-\vec{y}). \qquad (6.16)
\end{aligned}
$$

In a similar way, E_x with B_z comes to

$$
\begin{aligned}
[E_x, B_z] &= [\partial_z \partial_x \phi(t,x,y,z) - \partial_t \partial_y \psi(t,x,y,z), -\nabla_\perp \psi(t,x',y',z')] \\
&= \partial_y [\partial_t \psi(t,x,y,z), \nabla_\perp \psi(t,x',y',z')] \\
&= i\partial_y \sum_{l,m,n} \sum_{l',m',n'} \frac{N_{lmn} N_{l'm'n'} |k_{\perp'}| \sqrt{\omega_{lmn}}}{2|k_\perp| \sqrt{\omega_{l'm'n'}}} \\
&\quad \times \cos(k_l x) \cos(k_{l'} x') \cos(k_m y) \cos(k_{m'} y') \sin(k_n z) \sin(k_{n'} z') \\
&\quad \times ([-b_{lmn}, b^\dagger_{l'm'n'}] + [b^\dagger_{lmn}, b_{l'm'n'}]) \\
&= \; - i\partial_y \delta(x-x')\delta(y-y')\delta(z-z') \\
&= \; - i\partial_y \delta(\vec{x} - \vec{y}),
\end{aligned}
$$

and E_y with B_z to

$$
\begin{aligned}
[E_y, B_z] &= [\partial_t \partial_x \psi(t,x,y,z) + \partial_y \partial_z \phi(t,x,y,z), -\nabla_\perp \psi(t,x',y',z')] \\
&= \; - \partial_x [\partial_t \psi(t,x,y,z), \nabla_\perp \psi(t,x',y',z')] \\
&= i\partial_x \delta(\vec{x} - \vec{y}),
\end{aligned}
$$

so we must finally check E_x with B_y and E_y with B_x. We find

$$\begin{aligned}
[E_x, B_y] &= [\partial_z\partial_x\phi(t,x,y,z) - \partial_t\partial_y\psi(t,x,y,z), \partial_{y'}\partial_{z'}\psi(t,x',y',z') - \partial_t\partial_{x'}\phi(t,x',y',z')] \\
&= -\partial_z[\partial_x\phi, \partial_t\partial_{x'}\phi'] - \partial_{z'}[\partial_t\partial_y\psi, \partial_{y'}\psi'] \\
&= -i\partial_z \sum_{l,m,n}\sum_{l',m',n'} \frac{N_{lmn}N_{l'm'n'}\sqrt{\omega_{l'm'n'}}}{2|k_\perp||k_{\perp'}|\sqrt{\omega_{lmn}}} \\
&\quad \times k_l k_{l'} \cos(k_l x)\cos(k_{l'}x')\sin(k_m y)\sin(k_{m'}y')\cos(k_n z)\cos(k_{n'}z') \\
&\quad \times \Big([a_{lmn}, a^\dagger_{l'm'n'}] + [a^\dagger_{lmn}, -a_{l'm'n'}]\Big) \\
&\quad - i\partial_{z'} \sum_{l,m,n}\sum_{l',m',n'} \frac{N_{lmn}N_{l'm'n'}\sqrt{\omega_{lmn}}}{2|k_\perp||k_{\perp'}|\sqrt{\omega_{l'm'n'}}} \\
&\quad \times k_m k_{m'} \cos(k_l x)\cos(k_{l'}x')\sin(k_m y)\sin(k_{m'}y')\sin(k_n z)\sin(k_{n'}z') \\
&\quad \times \Big([-b_{lmn}, b^\dagger_{l'm'n'}] + [b^\dagger_{lmn}, b_{l'm'n'}]\Big) \\
&= -i\partial_z \sum_{lmn} N^2_{lmn} \cos(k_l x)\cos(k_l x')\sin(k_m y)\sin(k_m y')\cos(k_n z)\cos(k_n z') \\
&= -i\partial_z \delta(\vec{x} - \vec{y}),
\end{aligned} \tag{6.17}$$

and E_y with B_x follows similarly.

To compare with the cylindrical case treated in the next section, we point out that any commutations involving the components of either the electric or magnetic field in the direction of the "scalar" potentials almosted immediately yielded the desired delta function relations. Commutations with neither component in the Hertz potential directions were much more subtle and involved.

2. Vacuum Expectation Values

To calculate the vacuum expectation value of the energy density, we must calculate the vacuum expectation values of the square of each component of the electromagnetic field. As an example, let $\vec{x} = (x, y, z)$ and $\vec{y} = (x', y', z')$ and consider the component of the electric field in the z-direction. Then,

$$\begin{aligned}
\langle E_z(\vec{x})E_z(\vec{y})\rangle = \langle &\sum_{lmn}\sum_{l'm'n'} \frac{N_{lmn}N_{l'm'n'}k_\perp^2 k_{\perp'}^2}{2|k_\perp||k_{\perp'}|\sqrt{\omega_{lmn}\omega_{l'm'n'}}} \\
&\times \sin(k_l x)\sin(k_{l'}x')\sin(k_m y)\sin(k_{m'}y')\cos(k_n z)\cos(k_{n'}z') \\
&\times \big(a_{lmn}a_{l'm'n'}e^{-i(\omega_{lmn}+\omega_{l'm'n'})t} + a_{lmn}a^\dagger_{l'm'n'}e^{i(\omega_{l'm'n'}-\omega_{lmn})} \\
&+ a^\dagger_{lmn}a_{l'm'n'}e^{i(\omega_{lmn}-\omega_{l'm'n'})} + a^\dagger_{lmn}a^\dagger_{l'm'n'}e^{i(\omega_{lmn}+\omega_{l'm'n'})}\big)\rangle \\
= &\sum_{lmn}\frac{N_{lmn}^2}{2\omega_{lmn}}k_\perp^2 \sin(k_l x)\sin(k_l x')\sin(k_m y)\sin(k_m y')\cos(k_n z)\cos(k_n z').
\end{aligned}$$

Similarly for the magnetic field in the z-direction,

$$\begin{aligned}
\langle B_z(\vec{x})B_z(\vec{y})\rangle =&\langle\nabla_\perp\psi(t,x,y,z)\nabla_{\perp'}\psi(t,x',y',z')\rangle \\
=&\langle\sum_{lmn}\sum_{l'm'n'} \frac{N_{lmn}N_{l'm'n'}k_\perp^2 k_{\perp'}^2}{2|k_\perp||k_{\perp'}|\sqrt{\omega_{lmn}\omega_{l'm'n'}}} \\
&\times \cos(k_l x)\cos(k_{l'}x')\cos(k_m y)\cos(k_{m'}y')\sin(k_n z)\sin(k_{n'}z') \\
&\times \big(b_{lmn}b_{l'm'n'}e^{-i(\omega_{lmn}+\omega_{l'm'n'})t} + b_{lmn}b^\dagger_{l'm'n'}e^{i(\omega_{l'm'n'}-\omega_{lmn})} \\
&+ b^\dagger_{lmn}b_{l'm'n'}e^{i(\omega_{lmn}-\omega_{l'm'n'})} + b^\dagger_{lmn}b^\dagger_{l'm'n'}e^{i(\omega_{lmn}+\omega_{l'm'n'})}\big)\rangle \\
=&\sum_{lmn}\frac{N_{lmn}^2}{2\omega_{lmn}}k_\perp^2 \cos(k_l x)\cos(k_l x')\cos(k_m y)\cos(k_m y')\sin(k_n z)\sin(k_n z')
\end{aligned}$$

Because of the degeneracy of the eigenfunctions in each dimension, the other four expectation values in the other two directions can be calculated just as easily by considering scalar Hertz potential constructions in those dimensions.

B. Perfectly Conducting Cylinder

Consider a hollow cylinder of length L with circular cross section of radius R, perfectly conducting boundary, and no electromagnetic sources. This system can be solved completely with two scalar Hertz potentials, one satisfying a Dirichlet condition on the circular boundary and a Neumann condition on the axial boundary, and the other satisfying opposite conditions (i.e., Neumann at the radial boundary and Dirichlet on the caps). [5] [11] For reference, the classical solution given by normal modes is

$$\phi(t,\rho,\varphi,z) = \sum_{\substack{j=0,l=1\\ n=-\infty}}^{\infty} \frac{1}{{\lambda_{jn}^{D}}^{2}} \cos(k_l z) J_n(\lambda_{jn}^{D}\rho) \times \left(E_{jln}^{+} e^{i(n\varphi-\omega_{jln}t)} + E_{jln}^{-} e^{-i(n\varphi-\omega_{jln}t)}\right) \tag{6.18}$$

$$\psi(t,\rho,\varphi,z) = \sum_{\substack{j=0,l=1\\ n=-\infty}}^{\infty} \frac{1}{{\lambda_{jn}^{N}}^{2}} \sin(k_l z) J_n(\lambda_{jn}^{N}\rho) \times \left(B_{jln}^{+} e^{i(n\varphi-\omega_{jln}t)} + B_{jln}^{-} e^{-i(n\varphi-\omega_{jln}t)}\right) \tag{6.19}$$

where $k_l = \frac{\pi}{L} l$, λ satisfies the conditions $J_n(\lambda_{jn}^{D} R) = 0$ for ϕ and $J_n'(\lambda_{jn}^{N} R) = 0$ for ψ, and we remark for completeness that $\lambda_{jn}^2 = \omega_{jln}^2 - k_l^2$.

As in the Cartesian case, to produce quantum operators for the electromagnetic fields satisfying the known commutation relations we must quantize the scalar fields in a slightly non-canonical way. Using the same eigenvalue definitions as above, the quantized fields are

$$\phi(t,\rho,\varphi,z) = \sum_{\substack{j=0,l=1\\ n=-\infty}}^{\infty} \frac{N_{jln}}{\lambda^D_{jn}\sqrt{2\omega_{jln}}}\cos(k_l z)J_n(\lambda^D_{jn}\rho) \times \left(a_{jln}e^{-i(n\varphi-\omega_{jln}t)} + a^\dagger_{jln}e^{i(n\varphi-\omega_{jln}t)}\right) \tag{6.20}$$

$$\psi(t,\rho,\varphi,z) = \sum_{\substack{j=0,l=1\\ n=-\infty}}^{\infty} \frac{N_{jln}}{\lambda^N_{jn}\sqrt{2\omega_{jln}}}\sin(k_l z)J_n(\lambda^N_{jn}\rho) \times \left(b_{jln}e^{-i(n\varphi-\omega_{jln}t)} + b^\dagger_{jln}e^{i(n\varphi-\omega_{jln}t)}\right) \tag{6.21}$$

where a, b are lowering operators and $a^\dagger, b^\dagger$ are raising operators for the ϕ, ψ scalar fields, respectively.

1. Stress-Energy Tensor

The energy density may be calculated from the stress-energy tensor. This is, in turn, calculated by finding vacuum expectation values of various products of the components of the electromagnetic field. For the calculations below, we consider the product of field elements for different spacetime events, namely $(x^0, \vec{x}) = (t, \rho, \varphi, z)$ and $(y^0, \vec{y}) = (t', \rho', \varphi', z')$, but we take $t = t'$, and we let $\langle\cdot\rangle$ here denote the vacuum expectation value. For the electric field in the axial direction, we obtain

$$\langle E_z(\vec{x})E_z(\vec{y})\rangle = \langle(-\Delta_\perp\phi(\vec{x}))(-\Delta_\perp\phi(\vec{y}))\rangle \tag{6.22}$$

$$= \sum_{j,l,n}^{\infty} \frac{N^2_{jln}}{2}\frac{{\lambda^D_{jn}}^2}{\omega_{jln}}\cos(k_l z)\cos(k_l z')J_n(\lambda^D_{jn}\rho)J_n(\lambda^D_{jn}\rho'). \tag{6.23}$$

At this point, the ideal next step would be the construction of scalar Hertz potentials in the other dimensions, as was possible in the Cartesian case. It is, in fact, this desire that initiated the search for the general formulation presented herein. However, since such a construction was not obtained, we note that the remaining vacuum expectation values can either follow a lengthy computation using the above construction, or be computed as in [5].

2. Commutation Relations

To make sure that the Hertz construction doesn't miss any quantum states, we now check the commutation relations for the components of the electric and magnetic fields. If any states are missing, the calculated sums will not converge to the proper variations of the Dirac delta function.

$$
\begin{aligned}
[E_\rho(x^\mu), B_\rho(y^\mu)] =& [\partial_z\partial_\rho\phi(x^\mu) - \frac{1}{\rho}\partial_t\partial_\varphi\psi(x^\mu), \frac{1}{\rho'}\partial_{t'}\partial_{\varphi'}\phi(y^\mu) + \partial_{z'}\partial_{\rho'}\psi(y^\mu)] \\
=& \frac{1}{\rho'}[\partial_z\partial_\rho\phi(x^\mu), \partial_{t'}\partial_{\varphi'}\phi(y^\mu)] - \frac{1}{\rho}[\partial_t\partial_\varphi\psi(x^\mu), \partial_{z'}\partial_{\rho'}\psi(y^\mu)] \\
=& \frac{1}{\rho'}\sum_{jln}^{\infty}\sum_{j'l'n'}^{\infty}\frac{N_{jln}N_{j'l'n'}n'\omega_{j'l'n'}}{2\lambda^D_{jn}\lambda^D_{j'n'}}k_l\sin(k_l z)\cos(k_{l'}z')(\partial_\rho J_n(\lambda^D_{jn}\rho))J_{n'}(\lambda^D_{j'n'}\rho') \\
& \times [a_{jln}e^{i(n\varphi-\omega_{jln}t)} + a^\dagger_{jln}e^{-i(n\varphi-\omega_{jln}t)}, a_{j'l'n'}e^{i(n'\varphi'-\omega_{j'l'n'}t')} + a^\dagger_{j'l'n'}e^{-i(n'\varphi'-\omega_{j'l'n'}t')}] \\
& - \frac{1}{\rho}\sum_{jln}^{\infty}\sum_{j'l'n'}^{\infty}\frac{N_{jln}N_{j'l'n'}n\omega_{jln}}{2\lambda^N_{jn}\lambda^N_{j'n'}}k_{l'}\sin(k_l z)\cos(k_{l'}z')J_n(\lambda^N_{jn}\rho)(\partial_{\rho'}J_{n'}(\lambda^N_{j'n'}\rho')) \\
& \times [b_{jln}e^{i(n\varphi-\omega_{jln}t)} + b^\dagger_{jln}e^{-i(n\varphi-\omega_{jln}t)}, b_{j'l'n'}e^{i(n'\varphi'-\omega_{j'l'n'}t')} + b^\dagger_{j'l'n'}e^{-i(n'\varphi'-\omega_{j'l'n'}t')}] \\
=& \frac{1}{\rho'}\sum_{jln}^{\infty}\sum_{j'l'n'}^{\infty}\frac{N_{jln}N_{j'l'n'}n'\omega_{j'l'n'}}{2\lambda^D_{jn}\lambda^D_{j'n'}}k_l\sin(k_l z)\cos(k_{l'}z')(\partial_\rho J_n(\lambda^D_{jn}\rho))J_{n'}(\lambda^D_{j'n'}\rho') \\
& \times (e^{i([n\varphi-n'\varphi']-[\omega_{jln}t-\omega_{j'l'n'}t'])}[a_{jln}, a^\dagger_{j'l'n'}] + e^{-i([n\varphi-n'\varphi']-[\omega_{jln}t-\omega_{j'l'n'}t'])}[a^\dagger_{jln}, a_{j'l'n'}]) \\
& - \frac{1}{\rho}\sum_{jln}^{\infty}\sum_{j'l'n'}^{\infty}\frac{N_{jln}N_{j'l'n'}n\omega_{jln}}{2\lambda^N_{jn}\lambda^N_{j'n'}}k_{l'}\sin(k_l z)\cos(k_{l'}z')J_n(\lambda^N_{jn}\rho)(\partial_{\rho'}J_{n'}(\lambda^N_{j'n'}\rho')) \\
& \times (e^{i([n\varphi-n'\varphi']-[\omega_{jln}t-\omega_{j'l'n'}t'])}[b_{jln}, b^\dagger_{j'l'n'}] + e^{-i([n\varphi-n'\varphi']-[\omega_{jln}t-\omega_{j'l'n'}t'])}[b^\dagger_{jln}, b_{j'l'n'}]) \\
=& \frac{1}{\rho'}\sum_{jln}^{\infty}\frac{N^2_{jln}n\omega_{jln}}{2\lambda^{D\,2}_{jn}}k_l\sin(k_l z)\cos(k_l z')(\partial_\rho J_n(\lambda^D_{jn}\rho))J_n(\lambda^D_{jn}\rho') \\
& \times (2i\sin(n(\varphi-\varphi'))) \\
& - \frac{1}{\rho}\sum_{jln}^{\infty}\frac{N^2_{jln}n\omega_{jln}}{2\lambda^{N\,2}_{jn}}k_l\sin(k_l z)\cos(k_l z')J_n(\lambda^N_{jn}\rho)(\partial_{\rho'}J_n(\lambda^N_{jn}\rho')) \\
& \times (2i\sin(n(\varphi-\varphi'))) \\
=& \; 0
\end{aligned}
$$

All same-field (e.g. $[E_\rho, E_\varphi]$) and same-dimension (e.g. $[E_\varphi, B_\varphi]$) commutations follow with similar sine function cancellation. The final test is off-diagonal, cross-field commutation.

$$
\begin{aligned}
[E_\rho(x^\mu), B_\varphi(y^\mu)] &= [\partial_z\partial_\rho\phi(x^\mu) - \frac{1}{\rho}\partial_t\partial_\varphi\psi(x^\mu), \frac{1}{\rho'}\partial_{\varphi'}\partial_{z'}\psi(y^\mu) - \partial_t\partial_\rho\phi(y^\mu)] \\
&= [\partial_z\partial_\rho\phi(x^\mu), -\partial_t\partial_{\rho'}\phi(y^\mu)] + [-\frac{1}{\rho\rho'}\partial_t\partial_\varphi\psi(x^\mu), \partial_{\varphi'}\partial_{z'}\psi(y^\mu)] \\
&= \Bigg[\partial_z \sum_{j,l,n}^{\infty} \frac{N_{jln}}{\lambda^D_{jn}\sqrt{2\omega_{jln}}}\cos(k_l z)\partial_\rho J_n(\lambda^D_{jn}\rho)(a_{jl}e^{i(n\varphi-\omega_{jln}t)} + a^\dagger_{jl}e^{-i(n\varphi-\omega_{jln}t)}), \\
&\quad i\sum_{j',l',n'}^{\infty} \frac{N_{j'l'n'}\sqrt{\omega_{j'l'n'}}}{\lambda^D_{j',n'}\sqrt{2}}\cos(k_{l'}z)\partial_{\rho'}J_{n'}(\lambda^D_{j'n'}\rho') \\
&\quad \times(-a_{j'l'}e^{i(n'\varphi'-\omega_{j'l'n'}t)} + a^\dagger_{j'l'}e^{-i(n'\varphi'-\omega_{j'l'n'}t)})\Bigg] \\
&\quad + \frac{1}{\rho\rho'}\Bigg[i\sum_{j,l,n}^{\infty} \frac{N_{jln}\sqrt{\omega_{jln}}}{\lambda^N_{jn}\sqrt{2}}\sin(k_l z)J_n(\lambda^N_{jn}\rho)\partial_\varphi(-b_{jl}e^{i(n\varphi-\omega_{jln}t)} + b^\dagger_{jl}e^{-i(n\varphi-\omega_{jln}t)}), \\
&\quad \partial_{z'}\sum_{j',l',n'}^{\infty} \frac{N_{j'l'n'}}{\lambda^N_{j',n'}\sqrt{2\omega_{j'l'n'}}}\sin(k'_l z')J_{n'}(\lambda^N_{j'n'}\rho') \\
&\quad \times\partial_{\varphi'}(b_{j'l'}e^{i(n'\varphi'-\omega j'l't)} + b^\dagger_{j'l'}e^{-i(n'\varphi'-\omega_{j'l'n'}t)})\Bigg] \\
&= i\partial_z\sum_{j,l,n}^{\infty}\frac{N^2_{jln}}{\lambda^{D\,2}_{jn}}\cos(k_l z)\cos(k_l z')\partial_\rho J_n(\lambda^D_{jn}\rho)\partial_{\rho'}J_n(\lambda^D_{jn}\rho')(e^{in(\varphi-\varphi')} + e^{-in(\varphi-\varphi')}) \\
&\quad + \frac{i}{\rho\rho'}\partial_{z'}\sum_{j,l,n}^{\infty}\frac{N^2_{jln}}{\lambda^{N\,2}_{jn}}\sin(k_l z)\sin(k_l z')J_n(\lambda^N_{jn}\rho)J_n(\lambda^N_{jn}\rho')\partial_\varphi\partial_{\varphi'}(e^{in(\varphi-\varphi')} + e^{-in(\varphi-\varphi')}) \\
&= 2i\partial_z\sum_{j,l,n}^{\infty}\cos(k_l z)\cos(k_l z') \\
&\quad \times(\frac{N^2_{jln}}{\lambda^{D\,2}_{jn}}\partial_\rho J_n(\lambda^D_{jn}\rho)\partial_{\rho'}J_n(\lambda^D_{jn}\rho')e^{in(\varphi-\varphi')} \\
&\quad + \frac{N^2_{jln}}{\rho\rho'\lambda^{N\,2}_{jn}}J_n(\lambda^N_{jn}\rho)J_n(\lambda^N_{jn}\rho')\partial_\varphi\partial_{\varphi'}e^{in(\varphi-\varphi')})
\end{aligned}
$$

We pause here to note the following. For notational simplicity, let

$$S = i\partial_z \sum_{j,l,n}^{\infty} \cos(k_l z)\cos(k_l z') \times \left(\frac{N_{jln}^2}{{\lambda_{jn}^D}^2} \partial_\rho J_n(\lambda_{jn}^D \rho) \partial_{\rho'} J_n(\lambda_{jn}^D \rho') e^{in(\varphi-\varphi')} + \frac{N_{jln}^2}{\rho\rho' {\lambda_{jn}^N}^2} J_n(\lambda_{jn}^N \rho) J_n(\lambda_{jn}^N \rho') \partial_\varphi \partial_{\varphi'} e^{in(\varphi-\varphi')} \right).$$

We also point out that the normalization factor N_{jln} depends only on the Bessel functions, and is thus actually independent of l. Then

$$\begin{aligned} S =& (i\partial_z \sum_{l}^{\infty} \cos(k_l z)\cos(k_l z')) \\ & \times (\sum_{j,n}^{\infty} \frac{N_{jln}^2}{{\lambda_{jn}^D}^2} \partial_\rho J_n(\lambda_{jn}^D \rho) \partial_{\rho'} J_n(\lambda_{jn}^D \rho') e^{in(\varphi-\varphi')} + \frac{N_{jln}^2}{\rho\rho' {\lambda_{jn}^N}^2} J_n(\lambda_{jn}^N \rho) J_n(\lambda_{jn}^N \rho') \partial_\varphi \partial_{\varphi'} e^{in(\varphi-\varphi')}) \\ =& (i\partial_z \delta(z-z')) \\ & \times \partial_\rho \partial_{\rho'} \sum_{j,n}^{\infty} \frac{N_{jln}^2}{{\lambda_{jn}^D}^2} J_n(\lambda_{jn}^D \rho) J_n(\lambda_{jn}^D \rho') e^{in(\varphi-\varphi')} + \frac{1}{\rho\rho'} \partial_\varphi \partial_{\varphi'} \sum_{j,n}^{\infty} \frac{N_{jln}^2}{{\lambda_{jn}^N}^2} J_n(\lambda_{jn}^N \rho) J_n(\lambda_{jn}^N \rho') e^{in(\varphi-\varphi')} \end{aligned}$$

The final step in this calculation involves recognizing the summations as Green's functions for the wave equation in the plane in polar coordinates with Dirichlet and Neumann boundary conditions. In the case of Neumann boundary conditions, we note that no standard Green's function exists, as the Laplacian contains a vanishing eigenvalue, so what we mean here by Green's function is instead the inverse operator for the Laplacian restricted to the space spanned by eigenfunctions of nonvanishing eigenvalue. These functions come out to be

$$G_0(\rho,\varphi;\rho',\varphi') = \frac{1}{4\pi}\log\left(\frac{\rho^2}{R^2}+\frac{\rho'^2}{R^2}-2\frac{\rho\rho'}{R^2}\cos(\varphi-\varphi')\right) \tag{6.24}$$

$$G_D(\rho,\varphi;\rho',\varphi') = G_0(\rho,\varphi;\rho',\varphi') - \frac{1}{4\pi}\log\left(1+\frac{\rho^2\rho'^2}{R^4}-2\frac{\rho\rho'}{R^2}\cos(\varphi-\varphi')\right) \tag{6.25}$$

$$\begin{aligned} G_N(\rho,\varphi;\rho',\varphi') = G_0(\rho,\varphi;\rho',\varphi') &+ \frac{1}{4\pi}\log\left(1+\frac{\rho^2\rho'^2}{R^4}-2\frac{\rho\rho'}{R^2}\cos(\varphi-\varphi')\right) \\ &+ \frac{\rho^2+\rho'^2}{4\pi R^2} - \frac{3}{8\pi}, \end{aligned} \tag{6.26}$$

as we will show from a more general construction in an annulus case (which turns the cylinder into a coaxial cable of finite length) below. Returning to the S calculation with these in hand,

$$\begin{aligned} S =&(i\partial_z\delta(z-z')) \times \left(\partial_\rho\partial_{\rho'}G_D(\rho,\varphi,\rho',\varphi') + \frac{1}{\rho\rho'}\partial_\varphi\partial_{\varphi'}G_N(\rho,\varphi,\rho',\varphi')\right) \\ =&i\partial_z\delta(\vec{x}-\vec{y}) \end{aligned} \tag{6.27}$$

by a computation using the Mathematica computer algebra system.

These commutations turn out to be the most computationally involved. For relations between axial components of the electric or magnetic field with cross-sectional components, the computation follows much more easily and in fact generalizes to cylinders of any cross section. Such calculation yields the desired relation

$$[E_i(t,\vec{x}), B_j(t',\vec{y})] = i\epsilon_{ij}{}^k\partial_{x^k}\delta(\vec{x}-\vec{y}).$$

This means all EM modes are accounted for, and any further calculations involving the scalar Hertz potentials, such as the vacuum expectation values of their second derivatives, will produce quantities directly relevant to the electromagnetic field.

3. Cylinder with Non-Trivial Topology – Perfectly Conducting Coaxial Cable

By adding a second cylinder of equal length and smaller radius, with perfectly conducting boundary, inside the first, we find the cavity between them forms a computationally simple non-trivial topology. The classical scalar field solutions change from Bessel functions of the first kind to a linear combination of Bessel functions of the first and second kind, but other than a change of normalization constants, no other significant change occurs.

However, since it is a non-trivial topology, we expect possible solutions to the EM field that cannot be represented with Hertz potentials, scalar or otherwise. Thus, in the calculation of the commutation relations, we expect not to obtain (6.4). Relations involving the axial components of the electric and magnetic fields remain unchanged, and so it is the commutation relation involving the Green's functions from above that we expect to change. So for this, we must find the Green's functions for Dirichlet and Neumann boundary conditions on these surfaces.

For Dirichlet boundary conditions, we look to solve the differential equation

$$-\Delta G(\rho, \varphi; \rho'; \varphi') = \frac{\delta(\rho - \rho')}{\rho}\delta(\varphi - \varphi'). \tag{6.28}$$

We can expand in terms of an eigenfunction basis in the angular coordinates using the relation $\delta(\varphi - \varphi') = \sum_n u_n(\varphi)u_n^*(\varphi')$ and obtain the equation

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-\frac{1}{\rho}\partial_\rho \rho \partial_\rho g_n(\rho; \rho') + \frac{n^2}{\rho^2})e^{in\varphi}e^{-in\varphi'} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\rho}\delta(\rho - \rho')e^{in\varphi}e^{-in\varphi'}.$$

As the exponential functions are linearly independent, the n-th term in the right-hand summation must match the n-th term in the left-hand summation. This yields

$$-\partial_\rho \rho \partial_\rho g_n(\rho;\rho') + \frac{n^2}{\rho} = \delta(\rho - \rho'). \tag{6.29}$$

This is solved differently for $n = 0$ and $n \neq 0$, so first consider $n \neq 0$. For $\rho > \rho'$, the general solution is of the form

$$g_n(\rho;\rho') = A_n \rho^n + B_n \rho^{-n},$$

and $\rho < \rho'$, of the form

$$g_n(\rho;\rho') = C_n \rho^n + D_n \rho^{-n}.$$

We require g_n to be continuous when $\rho = \rho'$, so

$$A_n(\rho')^n + B_n(\rho')^{-n} = C_n(\rho')^n + D_n(\rho')^{-n}, \tag{6.30}$$

and integrating (6.29) over an infinitesimal interval around ρ' requires

$$A_n(\rho')^n - B_n(\rho')^{-n} - C_n(\rho')^n + D_n(\rho')^{-n} = -\frac{1}{n}. \tag{6.31}$$

Boundary conditions yield two more equations, namely

$$B_n = -A_n R_2^{2n} \tag{6.32}$$

$$D_n = -C_n R_1^{2n} \tag{6.33}$$

where R_2 is the radius of the outer cylinder and R_1 is the radius of the inner cylinder.

Putting together these four equations in four unknown coefficients provides the solution

$$g_n(\rho;\rho') = \frac{R_1^{2n}(\frac{\rho_>}{\rho_<})^n - (\rho_>\rho_<)^n - (R_1R_2)^{2n}(\rho_>\rho_<)^{-n} + R_2^{2n}(\frac{\rho_<}{\rho_>})^n}{2n(R_2^{2n} - R_1^{2n})}, \tag{6.34}$$

$$= \frac{1}{2n}\left[\left(\frac{\rho_<}{\rho_>}\right)^n - \left(\frac{\rho_>\rho_<}{R_2^2}\right)^2 - \left(\frac{R_1^2}{\rho_>\rho_<}\right)^n\right]$$
$$+ \frac{1}{2n(1-\left(\frac{R_1^{2n}}{R_2^{2n}}\right))}\left[\left(\frac{R_1^2\rho_>}{R_2^2\rho_<}\right)^n - \left(\frac{R_1^2\rho_>\rho_<}{R_2^4}\right)^n - \left(\frac{R_1^4}{R_2^2\rho_>\rho_<}\right)^n + \left(\frac{R_1^2\rho_<}{R_2^2\rho_>}\right)^n\right]$$
$$= \frac{1}{2n}\left[\left(\frac{\rho_<}{\rho_>}\right)^n - \left(\frac{\rho_>\rho_<}{R_2^2}\right)^2 - \left(\frac{R_1^2}{\rho_>\rho_<}\right)^n\right]$$
$$+ \frac{1}{2n}\sum_{s=0}^{\infty}\left[\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)^n - \left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)^n - \left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)^n + \left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)^n\right]$$

where, for $\rho > \rho'$, $\rho_> = \rho$ and $\rho_< = \rho'$, while for $\rho < \rho'$, $\rho_> = \rho'$ and $\rho_< = \rho$.

For $n = 0$, the equation we find is

$$-\partial_\rho \rho \partial_\rho g_0(\rho;\rho') = \delta(\rho - \rho')$$

and we use the same basic prescription as above. When $\rho > \rho'$, the general solution takes the form

$$g_0(\rho;\rho') = A_0 + B_0\log(\rho),$$

and for $\rho < \rho'$,

$$g_0(\rho;\rho') = C_0 + D_0\log(\rho).$$

And again, boundary conditions, continuity, and the differential condition yield the four unknown coefficients, giving a final solution as

$$g_0(\rho;\rho') = \frac{\log(\frac{\rho_>}{R_2})\log(\frac{\rho_<}{R_1})}{\log(\frac{R_1}{R_2})}. \tag{6.35}$$

Taking the summation of all of these solutions yields

$$
\begin{aligned}
G_D(\rho,\varphi;\rho',\varphi') = {} & \frac{\log(\frac{\rho_>}{R_2})\log(\frac{\rho_<}{R_1})}{2\pi\log(\frac{R_1}{R_2})} \\
& + \frac{1}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{in(\varphi-\varphi')}\frac{1}{2n}\left[\left(\frac{\rho_<}{\rho_>}\right)^n - \left(\frac{\rho_>\rho_<}{R_2^2}\right)^2 - \left(\frac{R_1^2}{\rho_>\rho_<}\right)^n\right] \\
& + \frac{e^{in(\varphi-\varphi')}}{2n}\sum_{s=0}^{\infty}\left[\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)^n - \left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)^n\right. \\
& \left. - \left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)^n + \left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)^n\right].
\end{aligned}
$$

By a simple repeat of the above solution technique, one can find that the first term appearing in the summation over n is in fact the free Green's function

$$
G_0(\rho,\varphi;\rho',\varphi') = \frac{1}{4\pi}\log\left[\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\varphi-\varphi')\right].
$$

The next two terms can then be summed in a similar way, and can be interpretted as corrections due to, in the first case, a disk of radius R_2 with Dirichlet boundary, and in the second case, the plane less a disk of radius R_1 with Dirichlet boundary. These are then

$$
\Gamma_{R_2} = \frac{1}{4\pi}\log\left[\frac{R_2^2}{\rho_>^2} + \frac{\rho_<^2}{R_2^2} - 2\frac{\rho_<}{\rho_>}\cos(\varphi-\varphi')\right] \tag{6.36}
$$

$$
\Gamma_{R_1} = \frac{1}{4\pi}\log\left[\frac{\rho_>^2}{R_1^2} + \frac{R_1^2}{\rho_<^2} - 2\frac{\rho_>}{\rho_<}\cos(\varphi-\varphi')\right], \tag{6.37}
$$

and with these terms computed, we can pull the summation over s outside of the summation over n to obtain

$$G_D(\rho,\varphi;\rho',\varphi') = \frac{\log(\frac{\rho_>}{R_2})\log(\frac{\rho_<}{R_1})}{2\pi\log(\frac{R_1}{R_2})} + G_0 - \Gamma_{R_2} - \Gamma_{R_1} + \sum_{s=0}^{\infty}\frac{1}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{in(\varphi-\varphi')}}{2n}\left[\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)^n - \left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)^n - \left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)^n + \left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)^n\right].$$

The terms under the summation in s follow the same pattern as the free Green's function; they are each geometric series of either a ratio or product of $\rho_<$ and $\rho_>$ and have the required prefactor of $(2n)^{-1}$. So these can then separately be summed to produce an infinite sum of logarithms,

$$\begin{aligned} G_D(\rho,\varphi;\rho',\varphi') = {} & \frac{\log(\frac{\rho_>}{R_2})\log(\frac{\rho_<}{R_1})}{2\pi\log(\frac{R_1}{R_2})} + G_0 - \Gamma_{R_2} - \Gamma_{R_1} \\ & + \sum_{s=0}^{\infty}\frac{1}{4\pi}\left[\log\left(1+\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)^2 - 2\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)\cos(\varphi-\varphi')\right)\right. \\ & - \log\left(1+\left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)^2 - 2\left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)\cos(\varphi-\varphi')\right) \\ & - \log\left(1+\left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)^2 - 2\left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)\cos(\varphi-\varphi')\right) \\ & \left. + \log\left(1+\left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)^2 - 2\left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)\cos(\varphi-\varphi')\right)\right]. \end{aligned} \tag{6.38}$$

The Green's function for the Neumann boundary conditions follows very similarly, except here we wish to solve the equation

$$-\Delta G(\rho,\varphi;\rho',\varphi') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi') - \frac{1}{\pi(R_2^2-R_1^2)}, \tag{6.39}$$

where the last term on the right-hand side represents the removal of the subspace of eigenfunctions with vanishing eigenvalue from the domain of G.

When taking the angular eigenfunction expansion, the constant term only appears for $n = 0$, and so for $n \neq 0$ the solution proceeds exactly as the Dirichlet case, excepting that the boundary conditions change the sign on the right-hand side of equations (6.32) and (6.33). This yields the solution for $n \neq 0$ as

$$\begin{aligned} g_n(\rho;\rho') &= \frac{R_1^{2n}(\frac{\rho_>}{\rho_<})^n + (\rho_>\rho_<)^n + (R_1R_2)^{2n}(\rho_>\rho_<)^{-n} + R_2^{2n}(\frac{\rho_<}{\rho_>})^n}{2n(R_2^{2n} - R_1^{2n})}, \qquad (6.40) \\ &= \frac{1}{2n}\left[\left(\frac{\rho_<}{\rho_>}\right)^n + \left(\frac{\rho_>\rho_<}{R_2^2}\right)^2 + \left(\frac{R_1^2}{\rho_>\rho_<}\right)^n\right] \\ &\quad + \frac{1}{2n}\sum_{s=0}^{\infty}\left[\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)^n + \left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)^n + \left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)^n + \left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)^n\right]. \end{aligned}$$

For the $n = 0$ case, we obtain the equation

$$-\partial_\rho\rho\partial_\rho g_0(\rho;\rho') = \delta(\rho - \rho') - \frac{2}{R_2^2 - R_1^2}\rho,$$

which, for $\rho \neq \rho'$ is an inhomogeneous O.D.E. with particular solution $\frac{\rho^2}{2(R_2^2 - R_1^2)}$, so for $\rho > \rho'$ we find

$$g_0(\rho;\rho') = A_0 + B_0\log(\rho) + \frac{\rho^2}{2(R_2^2 - R_1^2)} \qquad (6.41)$$

and for $\rho < \rho'$

$$g_0(\rho;\rho') = C_0 + D_0\log(\rho) + \frac{\rho^2}{2(R_2^2 - R_1^2)}. \qquad (6.42)$$

Here, the continuity condition and integration condition are degenerate, and so we must determine one of the coefficients, say A_0, by some other means. Since we require G to be orthogonal to the subspace of solutions with vanishing eigenvalue, this means the integral of G_N over the annulus must vanish. With this the final solution becomes

$$
\begin{aligned}
g_0(\rho;\rho') = {} & \frac{\rho_>^2+\rho_<^2}{2(R_2^2-R_1^2)} + \frac{\log(\rho_<)(R_1^4-R_1^2R_2^2)+\log(\rho_>)(R_1^2R_2^2-R_2^4)}{(R_1^2-R_2^2)^2} \\
& - \frac{3(R_2^2+R_1^2)}{4(R_2^2-R_1^2)} + \frac{R_2^4\log(R_2)-R_1^4\log(R_1)}{(R_2^2-R_1^2)^2}.
\end{aligned} \tag{6.43}
$$

Thus, for Neumann boundary conditions in the annulus, the Green's function is

$$
\begin{aligned}
G_N(\rho,\varphi;\rho',\varphi') = {} & \frac{\rho_>^2+\rho_<^2}{4\pi(R_2^2-R_1^2)} + \frac{\log(\rho_<)(R_1^4-R_1^2R_2^2)+\log(\rho_>)(R_1^2R_2^2-R_2^4)}{2\pi(R_1^2-R_2^2)^2} \\
& - \frac{3(R_2^2+R_1^2)}{8\pi(R_2^2-R_1^2)} + \frac{R_2^4\log(R_2)-R_1^4\log(R_1)}{2\pi(R_2^2-R_1^2)^2} \\
& + G_0 + \Gamma_{R_2} + \Gamma_{R_1} \\
& + \sum_{s=0}^{\infty}\frac{1}{4\pi}\left[\log\left(1+\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)^2 - 2\left(\frac{R_1^{2s+2}\rho_>}{R_2^{2s+2}\rho_<}\right)\cos(\varphi-\varphi')\right)\right. \\
& + \log\left(1+\left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)^2 - 2\left(\frac{R_1^{2s+2}\rho_>\rho_<}{R_2^{2s+4}}\right)\cos(\varphi-\varphi')\right) \\
& + \log\left(1+\left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)^2 - 2\left(\frac{R_1^{2s+4}}{R_2^{2s+2}\rho_>\rho_<}\right)\cos(\varphi-\varphi')\right) \\
& \left. + \log\left(1+\left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)^2 - 2\left(\frac{R_1^{2s+2}\rho_<}{R_2^{2s+2}\rho_>}\right)\cos(\varphi-\varphi')\right)\right].
\end{aligned} \tag{6.44}
$$

A similar Mathematica calculation for the commutation relation as in the trivial topology case yields a result that does not reduce to the delta function; however it is not readily apparent that this represents the expected missing modes.

CHAPTER VII

CONCLUSION AND FUTURE WORK

In Cartesian, axial cylindrical, and radial spherical coordinates, the scalar Hertz potentials represent a concrete realization of a concept long theorized in the physics community: that of a complete formulation of electromagnetic fields directly in terms of their two degrees of freedom. A differential geometric formulation offers the only hope of generalizing these results to arbitrary coordinate systems and beyond to curved spacetime, however the process is not straightforward.

The presentation given in this thesis has shown how the most obvious and direct formulation fails. At this point, we expect unexplored gauge transformations to provide the necessary freedoms that will produce Hertz potentials in other, more complicated coordinate systems, if such a description is possible at all.

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