Problem: Approximate the integral \( \int_0^1 e^{x^2} \, dx \) to 2 significant figures.

Method: We will approximate the integrand by a Taylor polynomial — so that we can integrate it! — and use Taylor’s theorem with remainder to justify the answer.

Substituting \( x^2 \) for \( z \) in the Taylor expansion of \( e^z \), we get

\[
e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + R_3(x^2).
\]

We can write out as many terms as we need; if this turns out to be not enough, we’ll come back for more later. Therefore,

\[
\int_0^1 e^{x^2} \, dx = \int_0^1 \left[ 1 + x^2 + \cdots \right] \, dx \\
= \left[ x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} \right]_0^1 + \int_0^1 R_3(x^2) \, dx. \\
= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \int_0^1 e^c \frac{x^8}{24} \, dx.
\]

At the last step we have used the standard formula for the Taylor remainder (“Version 2” in the terminology of this Web page), and the fact that \( f^{(4)}(z) = e^z \) if \( f(z) = e^z \). The number \( c \) may depend on \( x \), but in this problem it is always guaranteed to be between 0 and 1; therefore, \( e^c \) is less than \( e \), and the remainder term in the integral is less than

\[
e \int_0^1 \frac{x^8}{24} \, dx = \frac{e}{216}.
\]

So, doing the arithmetic with the fractions, we conclude that

\[
\int_0^1 e^{x^2} \, dx \approx 1.45714 \approx 1.46
\]

with a maximum error of

\[
\frac{e}{216} = 0.01258.
\]
So the accuracy is right on the edge of what we wanted. To be safe, one should go back and include one more term in the approximation:

\[
\int_0^1 e^{x^2} \, dx \approx \cdots + \frac{1^9}{216} \approx 1.46
\]

with a maximum error of

\[
\int_0^1 R_4(x^2) \, dx \leq \frac{e}{11 \times 5!} \approx 0.002.
\]

You might compare this result with the value for the integral calculated numerically by Maple.