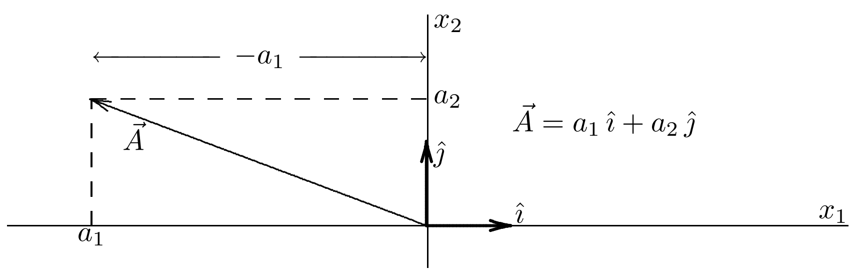
Chapter 4

Bases

4.1 The Basis Concept: Independence and Span

Let's look again at the crucial drawing from Sec. 1.1:



Its point is that every vector $\vec{A}$ in a plane can be written as a linear combination of two "basic" vectors, $\hat{\imath}$ and $\hat{\jmath}$:

$$\vec{A} = a_1\,\hat{\imath} + a_2\,\hat{\jmath},$$

where a_1 and a_2 are real numbers. (In the example, we have $a_1 = -8$, $a_2 = 3$.) In this section we clarify the properties of the pair of vectors $\{\hat{\imath}, \hat{\jmath}\}$ that makes it possible to use them in this way. This geometrical conception of a basis is good to keep in mind.* However, to give a broader view of why bases are so important, we prefer to introduce the concept in a more applied-algebraic way.

* There is one way in which this drawing may be misleading, however. The vectors in a basis are not required to be orthogonal (mutually perpendicular), nor to be of unit length. In fact, in an abstract vector space as defined in Sec. 3.1, "orthogonal" and "length" have no meaning. We have been using these concepts in the familiar way in the context of $\mathbf{R}^2$ and $\mathbf{R}^3$, but their formal definition will not show up until Chapter 6. At that point it will transpire that bases composed of orthogonal unit vectors are indeed especially nice, but they are not the only bases that are useful and necessary.

One of the principal skills and habits acquired early on by a student of mathematics is *simplifying expressions*. In elementary algebra you acquired the reflex, when confronted with an expression like

$$t + 3t^2 - (t-1)(t+2) + 3(t+1)^2 - 4t + 6,$$

of expanding products and combining terms so that there is (at most) one term for each power of t:

$$5t^2 + 2t + 11.$$

Similarly, in a calculus or differential-equations course, you probably came to regard the expressions

$$2e^t + 3e^{-t} \quad \text{and} \quad 5\cosh t - \sinh t$$

as satisfactory answers, but realized that

$$e^t + 2\sinh t + 4e^{-t}$$

is somehow half-baked. In fact, all three of these functions are equal; the last one can be converted to either of the other two by means of the formulas relating hyperbolic functions to exponential functions:

$$\cosh t \equiv \frac{e^t + e^{-t}}{2}, \qquad e^t = \cosh t + \sinh t,$$
$$\sinh t \equiv \frac{e^t - e^{-t}}{2}, \qquad e^{-t} = \cosh t - \sinh t.$$

Ordinarily we use only one of these function sets or the other, not both at once. (Hyperbolic functions will be further reviewed later in this section.)

The purpose of simplifying an expression is far more than to save ink when it is recopied. The expression is more meaningful and useful to us when it has been worked into some kind of standard form. In particular, if you are presented with two different expressions, you frequently need to know whether they are actually equal (i.e., represent the same function (or other mathematical object)) or are genuinely different. (Every student has faced this problem with "answers in the back of the book".) The most effective method to resolve that question is to have an agreed-upon *normal form* for expressions of the type concerned, and to convert each expression to that normal form. The normal form of an expression is, by definition, unique, and mathematically equivalent to the original expression. Therefore, the two starting expressions are mathematically equal if and only if their normal forms are *identical*.

Perhaps the most frequently occurring situation of this nature is that where the mathematical objects involved are vectors of some kind, and the expressions are, or can be simplified into, linear combinations of such vectors. The normal-form problem, then, is to find *a list of vectors such that every vector in the space can be written as a linear combination of the vectors in the list in exactly one way.* We could take this requirement as the definition of a *basis*. However, we shall follow tradition by giving a definition of basis that is easier to verify in concrete cases, and then proving the normal-form property as a theorem.

We wish to construct a definition of "basis", to be imposed on a list of vectors, that guarantees that every vector can be expressed in terms of those in the list *in at least one way* and *in at most one way.* These are separate problems, and so we need to define a separate concept to handle each. (It will turn out in the end, however, that there is an unexpected relation between them — see the summary theorem in Sec. 4.3.)

Span

The problem of "at least one way" can be handled for now just by a definition. (We will need to study later how to determine in practice whether the definition is satisfied by a given list of vectors.)

Definition 1a: If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are vectors in $\mathcal{V}$, then the *[linear] span* of $\mathcal{S} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is the set of all linear combinations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots = \sum_{j=1}^{n} c_j\vec{v}_j\,.$$

Example: Recall the parametric formula for a plane through the origin, $\vec{x} = s\vec{u} + t\vec{v}$. This reveals the plane as the span of the two vectors $\vec{u}$ and $\vec{v}$.

Remark: The span of any set of vectors is itself a vector space. It is an example of a *subspace* of $\mathcal{V}$ — see Sec. 5.1.

The definition of a span continues to make sense for an *infinite* set of vectors, $\mathcal{S}$. The *span* of $\mathcal{S}$ (abbreviated $\operatorname{span}\mathcal{S}$) is the set of all linear combinations of vectors from $\mathcal{S}$. For example, the space of all harmonic polynomials (polynomials that satisfy Laplace's equation, $\nabla^2 p = 0$) is the span of the set of all homogeneous harmonic polynomials of all degrees (see Sec. 5.1). Note that although we're now talking about infinite sets of vectors, we are still considering only *finite* sums in defining "linear combination", "span", and (later) "subspace". In many applications one really does want to consider infinite sums, but to do so one needs to study convergence of infinite series of vectors, and that is out of bounds for this course.

The word "span" can be a verb as well as a noun:

Definition 1b: A set $\mathcal{T}$ in $\mathcal{V}$ (i.e., $\mathcal{T} \subseteq \mathcal{V}$) is *spanned by* $\vec{v}_1, \ldots, \vec{v}_n$ (which are vectors in $\mathcal{T}$) if every $\vec{x} \in \mathcal{T}$ is a linear combination of those vectors:

$$\vec{x} = \sum_{j=1}^{n} c_j \vec{v}_j \,.$$

(One also turns this sentence around and says that the n vectors *span* the set, or that the list of vectors *spans* the set.)

The most important situation for the employment of this definition is that where $\mathcal{T}$ is $\mathcal{V}$ itself. We restate the definition:

Corollary: The set of vectors $\mathcal{S} \equiv \{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathcal{V}$ spans $\mathcal{V}$ if and only if every $\vec{y}$ in $\mathcal{V}$ can be expressed in *at least one way* as a linear combination of the vectors $\vec{v}_j$.

Examples. Let us find some sets that span $\mathcal{P}_2$ (the space of quadratic polynomials introduced in Sec. 3.1).

1. The obvious choice is $\{t^2, t, 1\}$. By definition, $\mathcal{P}_2$ is the linear combinations of these three functions.

2. A less obvious choice is $\{t, t-5, 2t+1, (t+1)^2\}$. With a little bit of algebra you can verify that every quadratic polynomial $p(t) \equiv at^2 + bt + c$ can be written as

$$p(t) = a(t+1)^2 - (10b + 2c)t + b(t-5) + (5b + c - a)(2t+1).$$

   Incidentally, it could also be written as

$$p(t) = a(t+1)^2 - (b + 2c)t + (c - a)(2t+1) \; [+\, 0(t-5)],$$

   so the coefficients in a linear combination need not be uniquely determined. This is a spanning set but not a basis.

3. The set $\{t^2-2, t^2+2, 2\}$ *does not* span $\mathcal{P}_2$, because there is no way to get terms linear in t in linear combinations of these three polynomials.

REMARK: Some books use "generate" to mean "span" in the sense of Definition 1b.

This subsection has been a bare introduction to the span concept. We'll return to it in more depth in the next chapter, as a special case of the concept of "subspace".

LINEAR INDEPENDENCE

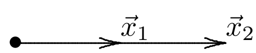
It is necessary to approach the issue of "at most one way" by a more indirect route.

Definition 2a: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is *linearly* **de***pendent* if there is some nontrivial linear combination of them that vanishes:

$$r_1\vec{v}_1 + \cdots + r_k\vec{v}_k = \vec{0},$$

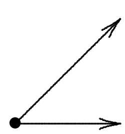
where not all of the scalars r_j are 0.

EXAMPLES. These two sets are each dependent, with the coefficients $r_1 = -2$, $r_2 = 1$:



$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

In contrast, here are two sets that are linearly *in*dependent:



$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The definition can be restated in the negative this way:

Definition 2b: A set of vectors is *linearly* **in***dependent* if: Whenever

$$r_1\vec{v}_1 + \cdots + r_k\vec{v}_k = \vec{0},$$

it follows that $r_1 = 0$, $r_2 = 0$, ..., $r_k = 0$.

Linear independence is one of the most useful and important concepts in linear algebra. From now on we'll usually omit the word "linearly" when there is no danger of confusion with some other kind of independence. Let's explore the definition:

1. A set of *two* vectors, $\{\vec{x}, \vec{y}\}$, is dependent if and only if one of them is a multiple of the other: We have $r\vec{x} + s\vec{y} = 0$ and either $r \neq 0$ or $s \neq 0$ (or both). Therefore either $\vec{x} = -\frac{s}{r}\,\vec{y}$ or $\vec{y} = -\frac{r}{s}\,\vec{x}$ (or both). (This proves "only if". The converse ("if") is even easier, since we don't have to divide by anything.) REMARK: If one of the scalars *is* 0, then the *other* vector must be $\vec{0}$. Example: If $0\vec{x} + 5\vec{y} = 0$, then $\vec{y} = 0$. The latter is indeed a multiple of $\vec{x}$: $\vec{y} = 0\vec{x}$.

2. A set consisting of a single *nonzero* vector, $\{\vec{v}\}$, is independent: $r\vec{v} \neq 0$ unless $r = 0$.

3. The set $\{\vec{0}\}$ is dependent: $5 \cdot \vec{0} = \vec{0}$ but $5 \neq 0$.

4. Any set containing $\vec{0}$ is dependent. [Why?]

5. How do we generalize (1) to more than two vectors? One's first guess might be that in a dependent set, some vector is a multiple of one of the others, but that is wrong:

Theorem 1: A set containing more than one vector is dependent if and only if one vector in the set is a linear combination of the others.

PROOF: Let $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = 0$ and at least one $r_j \neq 0$, say r_2. Then we can solve for $\vec{v}_2$:

$$\vec{v}_2 = -\frac{1}{r_2}(r_1\vec{v}_1 + r_3\vec{v}_3 + \cdots).$$

(Clearly, we could solve for the vector multiplied by *any* of the nonzero coefficients.) Conversely, if $\vec{y} = \sum_{j=1}^{n} r_j\vec{v}_j$, then $\sum_{j=1}^{n} r_j\vec{v}_j + (-1)\vec{y} = 0$, and so $\{\vec{v}_1, \ldots, \vec{v}_n, \vec{y}\}$ is dependent.

Note that in this theorem the qualification "more than one vector" is needed because of the case (3).

A familiar example of an independent set in a function space is provided by the power functions:

Proposition: In the vector space $C[a, b]$ of continuous functions on $[a, b]$, or the space $\mathcal{P}$ of polynomials, for any n the set $\{1, t, \ldots, t^n\}$ is linearly independent.

PROOF: Suppose the contrary; then $c_0\,1 + c_1 t + c_2 t^2 + \ldots + c_n t^n = 0$, for some numbers c with $c_0^2 + c_1^2 + \ldots + c_n^2 \neq 0$ (a quick way of saying that the coefficients are not all zero). Thus we have a polynomial $p_{(n)}(t)$ of degree n,

$$p_{(n)}(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n,$$

that has every $t \in [a, b]$ as a solution. But according to the *fundamental theorem of algebra* $p_{(n)}(t)$ has no more than n real solutions (*roots*). (See Sec. 8.1 for a complete statement of this theorem.) Thus we arrive at a contradiction.

ALTERNATIVE VERSION OF THE LAST STEP OF THE PROOF: For simplicity of notation consider the case $n = 2$; other cases obviously work out the same way. We have a polynomial $p(t) = c_0 + c_1 t + c_2 t^2$ that is equal to 0 for all $t \in [a, b]$. But then

$$p'(t) = c_1 + 2c_2 t \quad \text{and} \quad p''(t) = 2c_2$$

must also be identically 0 in the interval. Thus $c_2 = 0$; but then cascading back up the equations we see that $c_1 = 0$ and $c_0 = 0$.

REMARK: In dealing with linear combinations of the power functions t^j (alias polynomials), one usually starts numbering the coefficients at 0 instead of at 1. Clearly this makes no difference, except in counting the number of terms.

The next theorem shows that independence is exactly the property we're looking for.

Theorem 2: The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are (i.e., the set $\mathcal{S} \equiv \{\vec{v}_1, \ldots, \vec{v}_k\}$ is) independent if and only if every $\vec{y}$ in the vector space can be expressed in *at most one way* as a linear combination of the vectors $\vec{v}_j$.

PROOF: Suppose $\vec{y} = \sum_{j=1}^{k} c_j \vec{v}_j$ and also $\vec{y} = \sum_{j=1}^{k} d_j \vec{v}_j$. Then

$$0 = \vec{y} - \vec{y} = \sum_{j=1}^{k} (c_j - d_j)\vec{v}_j\,.$$

Thus independence implies that

$$c_j - d_j = 0 \quad \text{for all } j.$$

That is, $c_j = d_j$ — the two linear combinations are the same. Conversely, if the $\vec{v}$s are dependent, then one of them, say $\vec{v}_1$, is a linear combination of the others:

$$1\vec{v}_1 = \vec{v}_1 = \sum_{j=2}^{k} c_j \vec{v}_j \, .$$

Here are *two* ways of representing $\vec{v}_1$ as a linear combination of vectors in the set $\mathcal{S}$!

REMARK: Although we stated the theorem for a finite set $\mathcal{S}$, there is nothing in the proof that is invalid for an infinite set. (Recall that linear combinations are finite sums, so the two sums at the beginning of the proof are finite. We regard two such sums as the same if they differ only by the omission of terms with coefficient 0.) See also Exercise 4.1.7.

In this theorem, the qualification "at most" is needed because if $\mathcal{S}$ doesn't span the vector space, some vectors will have *no* expression as a linear combination of the $\vec{v}$s. To get rid of the "at most" we put *independence* and *spanning* together in the definition of a *basis*.

Bases

Definition 3: A set (or a list)† of vectors, $\mathcal{S} = \{\vec{v}_1, \ldots, \vec{v}_n\}$, is a *basis*‡ for a vector space $\mathcal{V}$ if (1) they are linearly independent, and (2) their span is $\mathcal{V}$.

Theorem 3: $\mathcal{S}$ (a subset of $\mathcal{V}$, where $\mathcal{V} \neq \{0\}$) is a basis for $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ can be expressed in *precisely one way* as a linear combination of the vectors in $\mathcal{S}$.

PROOF: Independence is equivalent to "at most one way" (by Theorem 2). Spanning is equivalent to "at least one way" (by the corollary to Definition 1).

Examples I. Here are some bases for $\mathcal{P}_2$ and some sets that fail to be bases for $\mathcal{P}_2$. In each case we sketch a set in $\mathbf{R}^2$ that is analogous. (The analogies are not exact, because the spaces have different dimensions. If anything in this list is mysterious now, come back and read it again after reading Sec. 4.3.)

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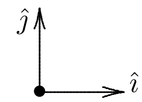
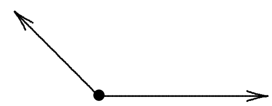
† See the Remark at the end of this section.

‡ The plural is "bases", with a long *E*.

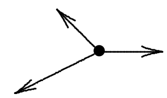
1) $\{1, t, t^2\}$ is a basis — in fact, a "natural" or obvious one.



2) $\{t, t+1, (t-3)^2\}$ is a basis — a randomly chosen one. (We shall not stop now to verify that this is a basis. Some methods for deciding such questions will come very soon.)



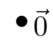
3) $\{1, t, t+1, t^2\}$ is not a basis — it is not independent.



4) $\{t, t^2\}$ is not a basis — it doesn't span.

Each of the next three examples is not independent *and* does not span:

5) $\{t, 2t\}$ is not a basis. (The closest analogue in $\mathbf{R}^2$ is the set consisting solely of the zero vector.)

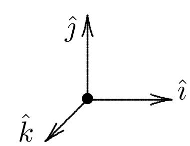


6) $\{t, t-1, t+1\}$ is not a basis.

7) $\{t, t-1, t+1, 1\}$ is not a basis.

Finally,

8) $\{1, t, t^2, t^3\}$ is not a basis for $\mathcal{P}_2$, because its span is too big — t^3 is not in the space $\mathcal{P}_2$.



Examples II. Here are some bases for the space of solutions of the differential equation

$$\frac{d^2y}{dt^2} - 9y = 0.$$

1) $y_+ = e^{3t}, \quad y_- = e^{-3t}.$
2) $y_1 = \cosh(3t), \quad y_2 = \frac{1}{3}\sinh(3t).$

The basis $\{y_1, y_2\}$ has an advantage over the basis $\{y_+, y_-\}$. These basis functions are chosen because they satisfy

$$y_1(0) = 1, \quad y_1'(0) = 0, \qquad y_2(0) = 0, \quad y_2'(0) = 1.$$

Suppose that we need to solve the initial-value problem

$$\frac{d^2y}{dt^2} - 9y = 0, \qquad y(0) = x, \quad y'(0) = v.$$

Then we can immediately write the answer down as

$$y(t) = x\,y_1(t) + v\,y_2(t)$$

without any further algebra. This is why hyperbolic functions were invented!

Review of hyperbolic functions

We can define cosh and sinh as the solutions of $y'' = y$ satisfying

$$\cosh(0) = 1, \quad \frac{d}{dt}\cosh(0) = 0, \qquad \sinh(0) = 0, \quad \frac{d}{dt}\sinh(0) = 1.$$

It follows that they are related to the exponential function by

$$\cosh t \equiv \frac{e^t + e^{-t}}{2}, \qquad e^t = \cosh t + \sinh t,$$
$$\sinh t \equiv \frac{e^t - e^{-t}}{2}, \qquad e^{-t} = \cosh t - \sinh t$$

(the equations on the left usually being taken as the definitions of cosh and sinh). Their derivatives are

$$\frac{d}{dt}\cosh t = \sinh t, \qquad \frac{d}{dt}\sinh t = \cosh t.$$

Thus these functions play exactly the same role relative to $y'' = y$ that cos and sin play relative to the equation $y'' = -y$, and their relationship to the exponential is essentially the same as that of the trigonometric functions but without the imaginary numbers (see Example 4, Sec. 1.1).

In fact, every trigonometric identity has a corresponding hyperbolic identity that looks the same except for a few sign changes. Examples are

$$\cosh^2 t - \sinh^2 t = 1,$$

$$\cosh(-t) = \cosh t, \qquad \sinh(-t) = -\sinh t,$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$$

It is not necessary to memorize all such formulas. The important thing is to recall that they exist and to have some memory of their structure, so that you know what to expect when rederiving them. Then, for example, the formula for $\sinh(x+y)$ can easily be recovered by writing out

$$\sinh(x+y) = \frac{e^{x+y} - e^{-x-y}}{2},$$

substituting $e^x = \cosh x + \sinh x$ etc., and simplifying, using the well known algebraic properties of the exponential function. (In fact, the basic exponential identities are so much simpler than the basic trig identities that often the best way to prove or recall a trigonometric identity is to reexpress it in terms of exponentials via $e^{i\theta} = \cos\theta + i\sin\theta$, etc.)

On the other hand, the qualitative behavior of the hyperbolic functions is very different from the trig functions. The latter oscillate and never have values greater than 1 in absolute value. But $\cosh t$ is never *less* than 1, and

$$\cosh t \to +\infty \quad \text{as } t \to \pm\infty, \qquad \sinh t \to \pm\infty \quad \text{as } t \to \pm\infty,$$

exponentially fast. The function

$$\tanh t \equiv \frac{\sinh t}{\cosh t}$$

has the same qualitative behavior as $\tan^{-1}$ (not $\tan$): a smooth rise between horizontal asymptotes at infinity,

$$\tanh t \to \pm 1 \quad \text{as } t \to \pm\infty.$$

In European books $\cosh t$ and $\sinh t$ are often written $\operatorname{ch} t$ and $\operatorname{sh} t$. (Also, sinh may be informally pronounced “shine” as well as “sinch”.)

Testing a set for linear independence

The following methods are helpful in deciding whether a given finite set of vectors is independent, so we present them now, although they would more logically come after the discussions of dimension, coordinates, subspaces, rank, and determinants that are soon to come.

(A) A set of n vectors in $\mathbf{R}^n$ (*same* n!) is a basis for $\mathbf{R}^n$ if and only the matrix formed from them is nonsingular. (Since the matrix is square, its singularity may be tested by calculating the determinant.) Similarly, if we know that a vector space $\mathcal{V}$ has a basis of n vectors, then another set of n vectors in $\mathcal{V}$ is a basis if and only if the matrix formed from their coordinates with respect to the known basis is nonsingular.

EXAMPLE: $y_1 = e^t - 3e^{-t}$ and $y_2 = e^t + 2e^{-t}$ form a basis for the space of solutions of $\dfrac{d^2y}{dt^2} = y$, because the determinant $\begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} = 5$ is not zero.

(B) Suppose a set $\mathcal{S}$ of vectors are presented to us as linear combinations of a known basis for $\mathcal{V}$. We can use row reduction to test whether $\mathcal{S}$ is linearly independent, and, if it isn't, to find an independent set with the same span. (In particular, if $\mathcal{S}$ spans $\mathcal{V}$, then this new set will be a basis for $\mathcal{V}$.)

EXAMPLE: Let $\mathcal{V} = \mathcal{P}_2$, and let the set $\mathcal{S}$ be $\{\vec{v}_j\}_{j=1}^3$, where

$$\vec{v}_1 = -6t^2 + 9t - 3, \qquad \vec{v}_2 = t^2 + 5t - 6, \qquad \vec{v}_3 = t^2 - 8t + 7.$$

(Thus the "known basis" is $\{t^2, t, 1\}$.) We demonstrate the method:

Algorithm:

1. Form a matrix whose rows are the coordinates (with respect to the given basis) of the vectors in $\mathcal{S}$.

$$\text{In the example:} \quad \begin{pmatrix} -6 & 9 & -3 \\ 1 & 5 & -6 \\ 1 & -8 & 7 \end{pmatrix}$$

2. Reduce the matrix by row operations. (It is not necessary here to clear out nonzero entries above the leading 1s.)

$$\text{In the example:} \quad \begin{pmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

(If we decided to reduce all the way, we would get $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.)

3. If the reduced matrix has no identically zero rows, the original vectors $\mathcal{S}$ were independent.

(Does not apply to the example.)

4. The nonzero rows of the reduced matrix are coordinates of independent vectors whose span is the same as $\operatorname{span}\mathcal{S}$.

$$\text{In the example:} \quad \{\vec{w}_1 = t^2 - \tfrac{3}{2}t + \tfrac{1}{2}, \quad \vec{w}_2 = t - 1\}$$

is a basis for the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. This choice of basis comes from the first row-echelon matrix given under step (2). If, instead, we use the fully reduced matrix given there, we get the equally good basis

$$\{\vec{u}_1 = t^2 - 1, \quad \vec{u}_2 = t - 1\}.$$

Note that these sets are bases for $\operatorname{span}\mathcal{S}$, but *not* bases for $\mathcal{P}_2$!

Example 1. Test for linear independence the vectors

$$\vec{x}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 8 \\ 7 \\ 6 \\ 5 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \vec{x}_4 = \begin{pmatrix} 0 \\ 2 \\ 8 \\ 10 \end{pmatrix}.$$

Solution: The matrix formed from the vectors (as rows) is

$$X = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \\ \vec{x}_4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 8 & 7 & 6 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{pmatrix}.$$

Reduce X:

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ 8 & 7 & 6 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{pmatrix}$$

$$\xrightarrow[\;]{\substack{8(1)+(2)\to(2) \\ (1)+(3)\to(3)}} \begin{pmatrix} -1 & -1 & 1 & 1 \\ 0 & -1 & 14 & 13 \\ 0 & 1 & 4 & 5 \\ 0 & 2 & 8 & 10 \end{pmatrix}$$

$$\xrightarrow[\;]{\substack{(2)+(3)\to(3) \\ (4)-2(3)\to(4)}} \begin{pmatrix} -1 & -1 & 1 & 1 \\ 0 & -1 & 14 & 13 \\ 0 & 0 & 18 & 18 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This shows that only 3 of the original 4 vectors were independent. A basis for the span of these vectors can be read off as

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 14 \\ 13 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Example 2. Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right\}$

SOLUTION: YES — obviously these two vectors are not multiples of each other. More formally,

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{9}{5} \\ 0 & 1 & -\frac{2}{5} \end{pmatrix},$$

and we see that the number of nonzero rows has not decreased as we put the matrix into row echelon form.

(b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 5 \end{pmatrix} \right\}$

SOLUTION: Let's put the vectors into a matrix as rows and reduce:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 1 & 7 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 5 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{11}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix}.$$

So the answer is NO. A basis for the span is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{2}{5} \end{pmatrix} \right\}, \quad \text{or} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{11}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{2}{5} \end{pmatrix} \right\},$$

or any two of the original three vectors (since *in this case* none of the vectors is a multiple of any of the others).

Example 3. For which values α and β will the vectors $\vec{x} = (1, 2, \beta)$, $\vec{y} = (1, 1, 0)$, and $\vec{z} = (\alpha, 1, 1)$ in $\mathbf{R}^3$ be linearly dependent?

SOLUTION: Suppose that $c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = 0$, with $c_1^2 + c_2^2 + c_3^2 \neq 0$. In coordinate form this is equivalent to the homogeneous system of equations

$$c_1\,1 + c_2\,1 + c_3\,\alpha = 0,$$
$$c_1\,2 + c_2\,1 + c_3\,1 = 0,$$
$$c_1\,\beta + c_2\,0 + c_3\,1 = 0.$$

A condition for it to have only the zero solution is that the determinant $\det(A)$ of its coefficient matrix A be different from 0. We have (by cofactors of the middle column)

$$\det(A) = \begin{vmatrix} 1 & 1 & \alpha \\ 2 & 1 & 1 \\ \beta & 0 & 1 \end{vmatrix} = -(2-\beta) + (1-\alpha\beta) = \beta - \alpha\beta - 1.$$

Therefore, we have —

Dependence condition: $\beta - \alpha\beta - 1 = 0$.

Independence condition: $\beta - \alpha\beta - 1 \neq 0$.

We could also find this result by row reduction, without using determinants. The row operations $(2) \to (2) - 2(1)$, $(3) \to (3) - \beta(1)$, $(3) \to (3) - \beta(2)$ reduce the matrix A to

$$\begin{pmatrix} 1 & 1 & \alpha \\ 0 & -1 & 1-2\alpha \\ 0 & 0 & 1+\alpha\beta-\beta \end{pmatrix}.$$

The system will have nontrivial solutions (that is, the vectors will be dependent) exactly when the element in the lower right corner is zero.

Remark: A basis is sometimes thought of as a set and sometimes as a list of vectors. There are two differences between a set and a list: (1) A list has a definite order; a list of the same vectors in a different order therefore counts as a different basis, even though the set of vectors is the same. In the second half of this chapter, when we study the representation of vectors by their coefficients with respect to a basis, the ordering of the vectors in the basis will be significant. (2) A list may contain two elements that are equal. For instance, the row-reduction algorithm above can be applied to find the span of the rows of a matrix even if some of the rows are the same. In that case the number of elements in the set of rows is smaller than the number of rows! The set of rows in that case might be linearly independent, but the list of rows obviously is not.

Exercises

4.1.1 Prove that any set containing the zero vector is dependent.

4.1.2 Tell whether each of these sets is linearly dependent or linearly independent.

(a) $\{(1,0,1),\ (2,3,5),\ (1,1,2)\}$

(b) $\{\,(1,2),\ (1,-1),\ (0,3)\,\}$

4.1.3 Tell whether each of these sets is linearly dependent or linearly independent.

(a) $\{\cosh(x),\ 2\sinh(x),\ 5e^x - 2e^{-x}\}$

(b) $\{t, (t-5)^2, t^2\}$.

4.1.4 Show that $\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for $\mathbf{R}^3$.

4.1.5 Which sets are bases for the span of the functions

$$\{\sin x, \sin x \cos x, \sin(2x)\}\,?$$

Explain.

(A) all 3 functions given
(B) $\{\sin x, \sin(2x)\}$
(C) $\{\sin x, \sin(2x), \sin(3x)\}$
(D) $\{\sin x, \cos x\}$

4.1.6 Let $L\colon \mathcal{D} \to \mathcal{W}$ be a linear function. Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in $\mathcal{D}$. One of the following "theorems" is true and one is false. Prove the true statement and disprove the false one.

(a) If $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, then $L(\vec{v}_1)$ and $L(\vec{v}_2)$ must be linearly independent.

(b) If $L(\vec{v}_1)$ and $L(\vec{v}_2)$ are linearly independent, then $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent.

4.1.7 Does the proof of Theorem 2 provided in this section apply to the set $\mathcal{S} = \{\vec{0}\}$? Is the theorem true for that set?

4.1.8 Prove that the functions $\{e^t, e^{2t}, e^{3t}\}$ are independent (as elements of $\mathcal{C}^2(-\infty,\infty)$, or of $\mathcal{C}^2(a,b)$ over any interval $(a,b) \subseteq \mathbf{R}$). *Possible alternative strategies:*

(A) Set $z = e^t$ and use the corresponding fact for polynomials.

(B) Prove it directly: Evaluate e^{kt} for three different values of t and the three relevant values of k, getting a 3×3 matrix. Show that if the three *functions* are linerly dependent, then the determinant of this matrix must be zero. But then use properties of the exponential function to show that the determinant is not zero.

In the next four exercises, test the given set of vectors for linear independence.

4.1.9 $\vec{v}_1 = (5,7,3,5)$, $\vec{v}_2 = (2,3,1,4)$, $\vec{v}_3 = (3,4,2,1)$

4.1.10 $\vec{v}_1 = (4,1,-3,1)$, $\vec{v}_2 = (2,1,4,1)$, $\vec{v}_3 = (-3,1,-2,1)$

4.1.11 $\vec{v}_1 = (11,9,7,5,3)$, $\vec{v}_2 = (7,2,1,0,0)$, $\vec{v}_3 = (5,3,0,1,0)$, $\vec{v}_4 = (6,0,2,0,1)$

4.1.12 $\vec{v}_1 = (1,2,4,3)$, $\vec{v}_2 = (2,1,3,4)$, $\vec{v}_3 = (1,2,3,4)$, $\vec{v}_4 = (4,3,2,1)$

In the remaining exercises, determine whether the given set is linearly independent; if it is not, find a set of vectors with the same span that *is* independent.

4.1.13 $\vec{v}_1 = (1,1,1)$, $\vec{v}_2 = (2,3,5)$, $\vec{v}_3 = (3,5,9)$

4.1.14 $\vec{v}_1 = (1,1,2)$, $\vec{v}_2 = (1,2,3)$, $\vec{v}_3 = (0,2,4)$

4.1.15 $f_1 = 1$, $f_2 = t+1$, $f_3 = (t-2)^2$, $f_4 = t^3 - 5t$

4.1.16 $f_1 = e^{2t}$, $f_2 = \cosh(2t)$, $f_3 = 3\sinh(2t)$, $f_4 = e^{-3t}$, $f_5 = \cosh(3t)$

4.2 Local Bases Associated with a Coordinate System

The familiar polar coordinate transformation in $\mathbf{R}^2$,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

is an instance of a common situation: In order to adapt our mathematics to the geometry of a particular physical problem, we find it convenient to label the points in space by strings of numbers that are *not* the components of vectors. The polar coordinates (r,θ) do *not* constitute a vector in the usual sense — for example, it makes no physical sense to add two such number pairs, getting $(r_1+r_2, \theta_1+\theta_2)$; this has nothing to do with the true vector sum of the two vectors with those pairs as polar coordinates.

Nevertheless, (r, θ) does range over some region in $\mathbf{R}^2$, so it makes sense (and is sometimes useful) to represent it by a $\mathbf{R}^2$-valued variable, say $\vec{u}$. Then we can summarize the coordinate transformation formulas above in the symbolic form $\vec{x} = T(\vec{u})$, where $T\colon \mathbf{R}^2 \hookrightarrow \mathbf{R}^2$ is a certain nonlinear function.

More generally, in $\mathbf{R}^n$ (or a part of it) a *curvilinear coordinate system* is defined by a nonlinear function of the type

$$\vec{x} \equiv \begin{pmatrix} x \\ y \\ \vdots \end{pmatrix} = T \begin{pmatrix} u \\ v \\ \vdots \end{pmatrix} \equiv T(\vec{u}).$$

From one point of view T is a mapping of one region of $\mathbf{R}^n$ onto another; from another (more pertinent) point of view, however, it simply describes two assignments of strings of numbers to the same fixed point in an n-dimensional space. The coordinates $\vec{u}$ are called *curvilinear* because the axes in the space of the coordinates $(u, v, \ldots) \equiv (u_1, u_2, \ldots)$ and the grid of lines parallel to those axes correspond to curved lines in the space of the (*rectilinear*, or *Cartesian*) coordinates $(x, y, \ldots) \equiv (x_1, x_2, \ldots)$, which we think of as the "physical" space.

It might seem more natural to specify a coordinate transformation by giving the "new" curvilinear coordinates as functions of the "old" rectilinear coordinates, but in most cases in practice it turns out to be more convenient to work in the reverse direction, as we do here.

Let's look closely at the Jacobian matrix

$$T' = \left\{ \frac{\partial x_j}{\partial u_k} \right\} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

(Here we have specialized successively to two dimensions and to the polar example.) Remember the "row before column" rule. Here the dependent ("numerator") variables label the rows, while the independent variables change as one moves from column to column. To get a numerical matrix one must evaluate the partial derivatives at some point $\vec{u}_0$; this will correspond to some definite point $\vec{x}_0 \equiv T(\vec{u}_0)$ in the Cartesian coordinates.

The kth column is a vector, $\vec{c}_k$, *tangent to the kth coordinate curve.* By this we mean the curve through $\vec{x}_0$ along which u_k varies while all the other u_l $(l \neq k)$ are held fixed.

In the polar example, we have

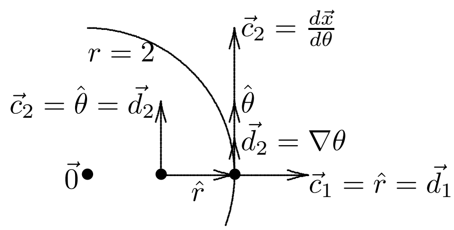
$$\vec{c}_1 = \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \equiv \hat{r},$$

which points along the radial line from the origin to $\vec{x}_0$. It happens to be a unit vector. In contrast,

$$\vec{c}_2 = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$$

is not a unit vector: $\|\vec{c}_2\| = r$. The corresponding unit vector is

$$\hat{\theta} \equiv \frac{\vec{c}_2}{\|\vec{c}_2\|}\,.$$



There is another way to associate a vector with each coordinate in a curvilinear coordinate system: Take its gradient.* Let

$$\vec{d}_k \equiv \nabla u_k = \begin{pmatrix} \frac{\partial u_k}{\partial x} \\ \frac{\partial u_k}{\partial y} \end{pmatrix}.$$

From the general properties of the gradient, we know that this vector is *normal to the surface* $u_k =$ constant. (The "surface" terminology derives from coordinate systems in three-dimensional space. In two dimensions these coordinate "surfaces" are simply curves.) A bit of calculation shows that in polar coordinates

$$\vec{d}_1 = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \hat{r}, \qquad \vec{d}_2 = \begin{pmatrix} -\frac{y}{r^2} \\ \frac{x}{r^2} \end{pmatrix} = \begin{pmatrix} -\frac{\sin\theta}{r} \\ \frac{\cos\theta}{r} \end{pmatrix} = \frac{\hat{\theta}}{r}\,.$$

---

* A gradient vector is more properly written as a row vector, but in this section we think of the gradients as columns so that we can look at them in the same space as the tangent vectors. The possibility of doing this meaningfully depends on regarding the geometry of $\vec{x}$-space (in particular, its dot product — see Chapter 6) as having a direct physical significance.

The drawing shows this entire assemblage of vectors at two points, one of which has $r = 1$ so that all three θ-related vectors are the same.

Most of the coordinate systems commonly used are *orthogonal* — meaning that the *coordinate curves* (u_k varying, all other us constant) are perpendicular to the *coordinate surfaces* (u_k constant, all other us varying). Thus $\vec{d}_k$ and $\vec{c}_k$ are parallel in such a case. In fact, we see in the polar example that their lengths are reciprocal. (This makes sense: If a unit change in θ produces a large change in $\vec{x}$, then a unit change in a component of $\vec{x}$ ought to produce a small change in θ; and this scaling factor clearly is controlled by r.) The inverse function theorem treated in Sec. 5.5 shows that this property holds for all orthogonal coordinate systems.

For a nonorthogonal coordinate system, however, the two associated sets of basis vectors at each point won't be parallel. For example, let

$$x = u + v, \qquad y = v.$$

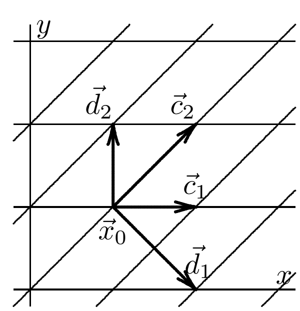
The inverse transformation is

$$u = x - y, \qquad v = y.$$

We easily calculate, for instance,

$$\vec{c}_2 = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \vec{d}_2 = \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We note (see drawing) that $\vec{d}_2$ is perpendicular to $\vec{c}_1$ and that $\vec{d}_1$ is perpendicular to $\vec{c}_2$, but that none of the vectors are parallel and that $\vec{d}_1$ and $\vec{d}_2$ are not perpendicular to each other.



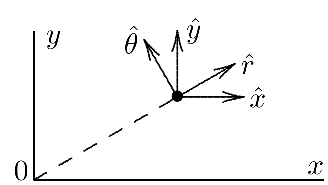
The importance of these local bases is that often it is convenient to express a vectorial physical quantity, such as a velocity or an electric field, in terms of basis vectors that have a special geometrical meaning at a particular point in the physical space of the problem. For example, in studying the motion of a satellite around the earth, the most important direction in space is the direction away from the earth (the radial direction), not one of the Cartesian directions. The unit vector in this direction is $\hat{r}$ (or its analogue in 3-dimensional spherical coordinates). The algebra or calculus of a calculation is likely to be easier, and the physical interpretation more transparent, if one works in polar coordinates instead of Cartesian ones.

Vector-valued functions in curvilinear coordinates (an example)

Let $\vec{g}(\vec{y})$ be a vector field in two dimensions. Let $\vec{y} = T(\vec{u})$ be the expression for Cartesian coordinates in terms of polar coordinates — that is,

$$\vec{y} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \vec{u} = \begin{pmatrix} r \\ \theta \end{pmatrix}.$$

To study $\vec{g}$ from a truly polar point of view, it is not sufficient to express its Cartesian components as functions of r and θ — that is, to study the composition $g \circ T$. One will want to decompose $\vec{g}(\vec{y})$ at each point $\vec{y}$ in terms of its components along the polar unit vectors, $\hat{r}$ and $\hat{\theta}$, introduced above. These polar components are obtained from the Cartesian components by a linear transformation (a rotation), which depends on $\vec{y}$ (in particular, on θ, as is clear from the sketch).



Thus the function of interest is

$$F \equiv U_{\vec{y}(\vec{u})} \circ g \circ T,$$

where U acts linearly on the coordinates of $\vec{g}(\vec{y})$ but also depends nonlinearly on $\vec{y}$ (and hence $\vec{u}$). That is, the elements of the matrix of U are nonlinear functions of the coordinates. That matrix is related to, but not identical to, $T'(\vec{y})$, the Jacobian of the coordinate transformation. As a matter of fact,

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

How this matrix can be deduced from the formulas earlier in this section will become clear later (after Secs. 4.5, 5.5, 6.2). Here we merely want to point out that to compute the derivatives of the polar components with respect to the polar coordinates, one needs to apply the chain rule to all the $\vec{u}$-dependence of F; this means replacing g and T by their differentials and also calculating an additional term by differentiating U with respect to $\vec{u}$.

Exercises

4.2.1 The formulas

$$x = \cosh u \cos v,$$
$$y = \sinh u \sin v$$

define *elliptic coordinates* (u, v) in the x–y plane, with the ranges $u \geq 0$, $0 \leq v < 2\pi$. The curves $u =$ constant > 0 are ellipses; $u = 0$ is a line segment joining the foci at $(-1, 0)$ and $(1, 0)$. The curves $v =$ constant are hyperbolas with those same foci, except for the cases $v = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, which are parts of the x and y axes.

(a) Find the tangent vectors to the coordinate lines $u =$ constant and $v =$ constant (at each point (u, v)).

(b) Verify that these tangent vectors are orthogonal to each other.

(c) Sketch two coordinate lines of each type, and sketch the two tangent vectors at two of the resulting intersection points.

4.2.2 *Parabolic coordinates* (u, v) in $\mathbf{R}^2$ are defined by

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv.$$

(a) Find (as functions of u and v) the Cartesian components of the tangent vectors to the coordinate curves (the curves $v =$ constant and the curves $u =$ constant).

(b) Illustrate the situation with a sketch. (See part (c) of the previous problem.)

4.2.3 Find the tangent vectors to the coordinate curves for spherical coordinates in $\mathbf{R}^3$, defined by†

$$
\begin{aligned}
x &= r \sin\theta \cos\phi, \\
y &= r \sin\theta \sin\phi, \\
z &= r \cos\theta.
\end{aligned}
$$

The remaining exercises deal with the coordinates (u, v) defined by

$$x = u^2, \quad y = u + v, \quad u > 0$$

in the region of $\mathbf{R}^2$ where $x > 0$.

4.2.4 Find the tangent vectors to the coordinate curves. Note that the vectors become linearly dependent when $u = 0$.

4.2.5 The coordinate "surfaces" are actually curves in this case, since the space is two-dimensional. Find the normal vectors to these curves. (Solve the equations for u and v and take gradients.) Note that the vectors "blow up" where $x = 0$.

4.2.6 Sketch a few of the coordinate curves in the (x, y) plane. Sketch the two sets of vectors at two points, one near the line $x = 0$ and one farther away. Comment on what happens as x approaches 0 and as x approaches infinity.

4.3 Dimension

We have frequently had to use the term "dimension" without defining it, relying instead on your intuition built on the elementary geometry of lines, planes, and three-dimensional space. Roughly speaking, the dimension of a space is the number of parameters or coordinates needed to specify a point in the space. We are now able to give a precise definition.

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† This definition is the one used in most physics textbooks (but not in most calculus textbooks, where θ and ϕ are interchanged): θ is the polar angle (colatitude) coming down from the north pole, and ϕ is the azimuthal angle (longitude). It is the definition of spherical coordinates that is used throughout this book.

All the proofs in this section will be delayed to the end, to avoid interfering with the main ideas.

Theorem and Definition: All bases for a given vector space consist of the same number of vectors. This number (which may be "infinity"*) is called the *dimension* of the space. The dimension of $\mathcal{V}$ is abbreviated $\dim \mathcal{V}$.

REMARK: $\{0\}$ is a zero-dimensional vector space. (The null set, $\emptyset$, qualifies as a basis for it.)

Examples:

1. $\dim \mathbf{R}^n = n$. A natural basis is $\{\vec{e}_j\}_{j=1}^n$. A general element of the space has the expansion
$$\vec{x} \equiv (x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j \vec{e}_j$$
in terms of this basis.
2. $\dim \mathcal{P}_n = n + 1$. A natural basis is $\{1, t, \ldots, t^n\}$. (For historical reasons, the basis elements are usually written in the opposite order.)
3. The dimension of the space of solutions of an nth-order homogeneous linear ordinary differential equation is n. (One needs n initial conditions to fix a solution.) A convenient basis is $\{y_0, \ldots, y_{n-1}\}$, where
$$\begin{aligned} &y_0(0) = 1, \quad y_0'(0) = 0, \quad \ldots, \quad y_0^{(n-1)}(0) = 0; \\ &y_1(0) = 0, \quad y_1'(0) = 1, \quad \ldots, \quad y_1^{(n-1)}(0) = 0; \\ &\qquad \cdots \\ &y_{n-1}(0) = 0, \quad y_{n-1}'(0) = 0, \quad \ldots, \quad y_{n-1}^{(n-1)}(0) = 1. \end{aligned}$$
EXAMPLE OF THE EXAMPLE: For $\dfrac{d^2y}{dt^2} - \omega^2 y = 0$ this basis is
$$y_0(t) = \cosh \omega t, \qquad y_1(t) = \frac{1}{\omega} \sinh \omega t.$$
The reason why this basis is convenient is that the initial data of a solution y are the coefficients in the linear combination
$$y(t) = \sum_{j=0}^{n-1} c_j y_j(t).$$

---

* Readers who know about *transfinite numbers* are warned not to interpret any statements in this book in those terms. We do not distinguish between different sizes of infinite sets.

That is, $c_0 = y(0)$, $c_1 = y'(0)$, etc. (This is why the functions cosh and sinh were invented!) Of course, in some application one may want to prescribe initial data at some t other than 0; then a different basis will be used.

4. In ordinary geometry, points have dimension 0, lines have dimension 1, planes have dimension 2, and physical space as a whole has dimension 3. (See Sec. 5.1.)

5. $\mathcal{P}, \mathcal{C}(-\infty,\infty)$, and $\mathcal{C}^1(-\infty,\infty)$ are *infinite-dimensional* since *no* finite set spans one of them.

A decent treatment of bases in infinite-dimensional spaces requires a distinction between finite linear combinations and infinite sums. Also, there may be several different definitions of convergence for infinite sums, resulting in different definitions of what constitutes a basis. The power functions, $\{t^n\}_{n=0}^{\infty}$, form a basis for $\mathcal{P}$ in the sense of finite linear combinations. Interesting bases for most function spaces, however, require infinite sums. For example, as already hinted in an earlier discussion of the heat equation, the simplest partial differential equations have solutions that are *Fourier series* of various types. A typical category of Fourier series comprises the sums of the form

$$\sum_{n=1}^{\infty} c_n \sin(nx). \qquad (*)$$

It turns out that every differentiable function on the interval $0 \le x \le \pi$ has a Fourier series $(*)$ that converges *pointwise* — that is, for each fixed x the series of numbers $(*)$ converges to $f(x)$. In a certain sense, therefore, the sine functions $\{\sin nx\}$ form a basis for the space $\mathcal{C}^1[0,\pi]$ of differentiable functions on that interval. On the other hand, the space $\mathcal{C}[0,\pi]$, consisting of functions that are (perhaps) merely continuous, contains functions whose Fourier series do not converge pointwise; they nevertheless converge in a weaker sense ("in the mean"), which we don't have time to explain here. "Weaker" means that *more* series qualify as convergent, hence *more* functions are in the span of the sines; this allows the sines to be regarded as a basis for the space $\mathcal{C}$.

Summary theorem on dimension, independence, and spanning

Theorem: Let $\mathcal{S}$ be a set of k vectors in an n-dimensional vector space $\mathcal{V}$.

(1) If $k < n$, then $\mathcal{S}$ can't span $\mathcal{V}$; $\mathcal{S}$ may or may not be independent.

(2) If $k > n$, then $\mathcal{S}$ can't be independent; $\mathcal{S}$ may or may not span $\mathcal{V}$.

(3) If $k = n$ and $n < \infty$, then either

(A) $\mathcal{S}$ is a basis for $\mathcal{V}$ (i.e., $\mathcal{S}$ is independent *and* spans),

or

(B) $\mathcal{S}$ *neither* spans *nor* is independent.

To show that a set *with the right number of elements* is a basis for a finite-dimensional space, therefore, it is enough to check *one* of the two properties, independence or spanning. Usually independence is easier to show, since it reduces to an algebraic condition on the basis vectors themselves rather than a relationship to all the other vectors in the space.

Lemmas and proofs

In this subsection we shall prove the basic dimension theorems stated earlier in the section. Along the way we state and prove some other theorems that are important in their own right.

We begin with a consequence of the row-reduction algorithm of Sec. 2.1. Recall that a linear equation is called *homogeneous* if the "constant term" is zero, as in

$$3x - y + z = 0.$$

Lemma 1: If a system of homogeneous linear algebraic equations contains fewer equations than unknowns, then it has (many) solutions in which the unknowns are not all equal to zero. (Such solutions are called "nontrivial".)

Proof: The reduction of the augmented matrix of any system leads to a matrix with these properties ("row echelon form"): (1) The first nonzero element in any row is a 1. (There may also be rows at the bottom that consist entirely of zeros.) (2) The number of leading zeros in the row is strictly increasing as one moves down the matrix; in particular, no two rows have their leading 1s in the same column. If the system is homogeneous, then all the numbers in the last column are 0 throughout the reduction, so a leading 1 is never in the last column. Therefore, each row of the reduced matrix represents an equation that can be solved for a different one of the unknowns. If there are more unknowns than equations, at least one of the unknowns is not determined by this procedure and enters the general solution as an arbitrary parameter — certainly not always 0.

Lemma 2: If $\mathcal{V}$ has a basis consisting of n vectors ($n < \infty$), then any subset of $\mathcal{V}$ with more than n elements is dependent.

PROOF: Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be the basis and $\{\vec{x}_1, \ldots, \vec{x}_p\}$ $(p > n)$ be the larger set. The $\vec{x}$s have expansions

$$\vec{x}_j = \sum_{k=1}^{n} c_{kj} \vec{v}_k \,.$$

We must investigate the equation

$$\sum_{j=1}^{p} r_j \vec{x}_j = 0.$$

It implies

$$\begin{aligned} 0 &= \sum_{j=1}^{p} \sum_{k=1}^{n} r_j c_{kj} \vec{v}_k \\ &= \sum_{k=1}^{n} \left(\sum_{j=1}^{p} c_{kj} r_j \right) \vec{v}_k \,. \end{aligned}$$

Since the $\vec{v}$s are independent, it follows that

$$\sum_{j=1}^{p} c_{kj} r_j = 0 \quad \text{for } k = 1, \ldots, n.$$

This is a linear system of the type described in Lemma 1, so it has nontrivial solutions $\{r_j\}$. This proves that the $\vec{x}$s are dependent, by definition.

We are now ready to prove the theorem that made the definition of "dimension" possible.

PROOF THAT ALL BASES HAVE THE SAME SIZE: If $\mathcal{V}$ has two bases, one with n vectors and one with p vectors, where $n < p < \infty$, then we get an immediate contradiction with Lemma 2. If the larger basis is infinite, apply Lemma 2 to a large finite subset of that basis and get the same contradiction.

At the last step of the proof we tacitly used one of these obvious consequences of the definitions in Sec. 4.1:

Proposition 1:

(a) If a set $\mathcal{S}$ is independent, then any subset of $\mathcal{S}$ is also independent.

(b) If a set $\mathcal{S}$ spans $\mathcal{V}$, then any superset of $\mathcal{S}$ (i.e., a set of which $\mathcal{S}$ is a subset) also spans $\mathcal{V}$.

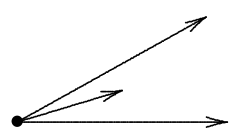
The picture we have built up is not quite complete: We have not yet proved that every vector space actually has a basis. Nor have we proved that existence of a finite spanning set implies existence of a finite basis. But these facts follow from the first half of the following theorem.

Proposition 2:

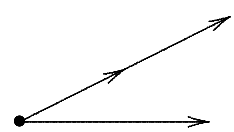
(a) Any set of vectors can be made independent (if it isn't already) by throwing away vectors — without changing the span of the set. In particular, any spanning set can be made into a basis by discarding vectors from it.

(b) Any independent set can be made into a basis (if it isn't one already) by adding vectors.

Examples of (a): Consider three vectors in a plane:



or



In the first case, any one of the three can be discarded, leaving an independent set that spans the plane. In the second case, either of the two parallel vectors can be discarded; in this case, however, the third vector is essential — omitting it would leave a set that does not span the plane (and is dependent, as well).

Partial proof of (a): We confine attention to the case that the set is *finite*:

$$\mathcal{S} = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}.$$

(To avoid fussing over a trivial special case, we also assume that $\mathcal{S}$ contains more than just the zero vector.) If $\mathcal{S}$ is not already independent, then (by Theorem 1 of Sec. 4.1) one of its elements is a linear combination of the others, say

$$\vec{x}_3 = r_1\vec{x}_1 + r_2\vec{x}_2 + r_4\vec{x}_4 + \cdots.$$

This formula can be used to replace $\vec{x}_3$ in any linear combination of the vectors in $\mathcal{S}$ so as to get a linear combination of the remaining vectors.

Therefore, the span of $\mathcal{S}$ does not decrease when the redundant vector is discarded. Repeating this process as many times as necessary, we must eventually reach an independent set.

Partial proof of (b): We shall treat only the case of a finite-dimensional vector space. If the set $\mathcal{S} = \{\vec{x}_1, \dots \vec{x}_k\}$ is not a basis (doesn't span the vector space $\mathcal{V}$), then there is a vector $\vec{v} \in \mathcal{V}$ that is not a linear combination of the vectors in $\mathcal{S}$. Therefore, if we add $\vec{v}$ to $\mathcal{S}$ in the role of $\vec{x}_{k+1}$, the extended set remains independent (see Theorem 1 of Sec. 4.1 and its proof). If the new set is not yet a basis, add another vector, and so on. When the number of vectors reaches the dimension of $\mathcal{V}$, the set must be a basis.

With the aid of Proposition 2 (even in its restricted form, referring to finite sets in finite-dimensional spaces), it should be easy to fill in the details of the proof of the summary theorem.

Exercises

4.3.1 Write out a proof of Proposition 1.

4.3.2 Prove the "summary theorem".

4.3.3 Find the dimension of the span of each of these sets. (Review Sec. 4.1 for the techniques.)

(a) $\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ \alpha \end{pmatrix}$ (for a real parameter α)

(c) $(1, 1, 1), \quad (1, 2, 3), \quad (4, 5, 6)$

(d) $(4, 5, 9), \quad (2, 3, 4), \quad (0, 1, -1)$

4.3.4 Find the dimension of the span of each of these sets.

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

(b) $(1, 2, 3), \quad (4, 5, 9)$

(c) $(1, 1, 1), \quad (2, 3, 1), \quad (4, 5, 3), \quad (-1, 0, -2)$

(d) $(5, 6, -2), \quad (\alpha, -3, 1)$ (for a real parameter α)

4.3.5 What is the dimension of the subset of $\mathbf{R}^3$ consisting of solutions of the equation

$$x + y + z = 0\,?$$

4.3.6 Consider a system of 2 homogeneous equations in 3 unknowns.

(a) Suppose that the row echelon form of the augmented matrix is of the type

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \end{pmatrix}.$$

(The asterisks represent numbers whose values are irrelevant.) What is the dimension of the subset of $\mathbf{R}^3$ consisting of solutions of the system?

(b) Give an example of a row echelon form that would yield a different dimension.

4.3.7 Consider a system of 3 homogeneous equations in 3 unknowns.

(a) Suppose that the row echelon form of the augmented matrix is of the type

$$\begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What is the dimension of the subset of $\mathbf{R}^3$ consisting of solutions of the system?

(b) Give an example of a row echelon form that would yield a larger dimension.

4.3.8 In a plane, sketch examples of the six situations allowed by the summary theorem on dimension:

(a) two vectors that are linearly independent and span the plane

(b) two vectors that are not independent and don't span the plane

(c) three vectors that are not independent and do span the plane

(d) three vectors that are not independent and don't span the plane

(e) one vector that constitutes an independent set and does not span the plane

(f) one vector that constitutes a dependent set and does not span the plane

In the remaining exercises, find the dimension of the span of the given list of vectors, and construct a basis for that span. (Electronic assistance is permitted.)

4.3.9 $(-1, 1, -3, 0), \quad (2, 1, 3, 1), \quad (1, 2, 0, 1)$

4.3.10 $(1, -3, 2, 9, -5), \quad (0, -2, 1, 5, -3), \quad (1, 1, 0, -1, 1), \quad (2, 0, 1, 3, -1)$

4.3.11 $(1, 5, -3, 1), \quad (4, 5, 3, -1), \quad (-1, 1, -3, 1), \quad (2, 1, 3, -1)$

4.3.12 $(1, 2, -1, 1), \quad (1, 1, 3, 1), \quad (1, 2, 1, 0), \quad (1, 1, 1, 1)$

4.3.13 $(1, 1, 1, 0), \quad (1, -1, 0, 1), \quad (2, 0, 1, 1), \quad (1, 0, -1, 0)$

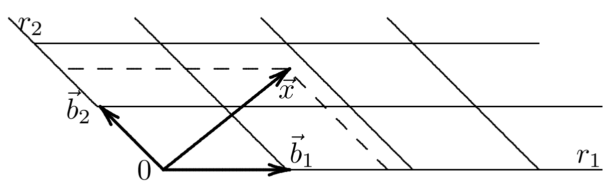
4.3.14 $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -2 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

4.4 Coordinates with Respect to a Basis

If $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for $\mathcal{V}$, then we know that every $\vec{x}$ in $\mathcal{V}$ has a *unique* expansion

$$\vec{x} = \sum_{j=1}^{n} r_j \vec{b}_j \,.$$

The numbers r_j are the *coordinates* of $\vec{x}$ with respect to that basis. This generalizes the usual notion of the (rectilinear) coordinates of a vector in space with respect to a coordinate system (a grid defined by basis vectors).



Coordinates are also called *(expansion) coefficients* or *components*, although it is more correct to reserve the word "component" for a corresponding *vectorial* piece of $\vec{x}$, one of the terms $r_j \vec{b}_j$.

From now on we shall think of the vectors in a basis as having a definite order. (The basis is a *list*, not just a set.) Then the coordinates r_j have a definite order, and constitute a vector in $\mathbf{R}^n$.

Notational remark: When we are doing all our calculations with respect to one fixed basis, it is natural and helpful to call the coordinates x_j instead of r_j, so that the notation always shows *which vector* $\vec{x}$ we're talking about. However, in the near future we'll be considering the coordinates of the *same* vector with respect to more than one basis. If we were to use the letter x for all of its coordinates, we would have to resort to notations like x'_j or $x_{j'}$ to distinguish the coordinates belonging to different bases. For now, we prefer to use different letters, say r_j and s_j, for coordinates relative to different bases.

If the basis is the *natural basis* in $\mathbf{R}^n$, then the coordinates of a vector (which is a sequence of numbers in this case) are the numbers in the sequence itself:

$$(x, y) = x\vec{e}_1 + y\vec{e}_2 \equiv x\hat{\imath} + y\hat{\jmath}.$$

For a different basis in $\mathbf{R}^n$, the coordinates will be different numbers.

Example. Consider the basis $\{\vec{b}_1 = (1, 2), \quad \vec{b}_2 = (1, 1)\}$. Each (x, y) has the form $c\vec{b}_1 + d\vec{b}_2$. What are c and d? [Example of the example: $(5, -1) = -6(1, 2) + 11(1, 1)$.]

Solution, Method 1: Write out

$$\begin{aligned}(x, y) &= c(1, 2) + d(1, 1)\\ &= (c + d, 2c + d).\end{aligned}$$

This gives two equations to be solved for the two unknowns, c and d:

$$\begin{aligned}x &= c + d,\\ y &= 2c + d.\end{aligned}$$

This can be solved once and for all — for all (x, y)— by inverting $\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$. The inverse matrix is $\begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix} \equiv C$. That is,

$$\begin{aligned}c &= -x + y,\\ d &= 2x - y.\end{aligned}$$

Solution, Method 2: We know that $(x, y) = x\vec{e}_1 + y\vec{e}_2$. If we had expressions for the $\vec{e}$s in terms of the $\vec{b}$s, we could plug them into this, combine terms, and be done. What we actually have are the expressions going in the reverse direction:

$$\begin{aligned}\vec{b}_1 &= \vec{e}_1 + 2\vec{e}_2\,,\\ \vec{b}_2 &= \vec{e}_1 + \vec{e}_2\,.\end{aligned}$$

We can solve these for $\vec{e}_1$ and $\vec{e}_2$. The unknowns here are vectors, not numbers, but the algebra of solving the linear system is the same as always. Note, incidentally, that the linear system involved is the transpose of the one in Method 1 — that is, its matrix is $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Therefore, the inverse matrix, $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$, is the transpose of the inverse we found in the other method. That is,

$$\begin{aligned} \vec{e}_1 &= -\vec{b}_1 + 2\vec{b}_2, \\ \vec{e}_2 &= \quad \vec{b}_1 - \quad \vec{b}_2 \,. \end{aligned}$$

Plugging into $\vec{x} = x\vec{e}_1 + y\vec{e}_2$ as planned, we get

$$\vec{x} = (y - x)\vec{b}_1 + (2x - y)\vec{b}_2 \,.$$

This equation is equivalent to the formulas for c and d that we found by the other method.

Continuation of example: In order to pass from this example to a general theorem, it is necessary to restate the results in a form that is cluttered with more subscripts. Let $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ be the expression of a generic vector $\vec{x}$ as a linear combination of the natural basis vectors, and let $\vec{x} = r_1\vec{b}_1 + r_2\vec{b}_2$ be the expansion in terms of the new $\vec{b}$ basis (that is, $r_1 \equiv c$, $r_2 \equiv d$). Our problem was to find formulas for the rs in terms of the xs. Our result was that

$$r_j = C_{j1}x_1 + C_{j2}x_2 \quad \text{for } j = 1 \text{ and } 2,$$

for a certain matrix C,

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Along the way we found that

$$\vec{e}_k = C_{1k}\vec{b}_1 + C_{2k}\vec{b}_2 \quad \text{for } k = 1 \text{ and } 2. \tag{$*$}$$

(There were also two other pairs of linear formulas that are the inverses of these two.) The critical thing to notice is that the subscripts in ($*$) are "twisted" relative to the format for linear formulas that we have regarded as standard since the beginning of Chapter 2. If we are to regard ($*$) as a linear system in the usual sense, we must identify the matrix appearing in it as C^{t} rather than C.

By calculations just like those in the example, but stated in full generality, one can prove the master theorem on change of basis:

Theorem 1: Given two bases, $\{\vec{v}_j\}$ and $\{\vec{w}_j\}$, if

$$\vec{x} = \sum_{k=1}^{n} r_k \vec{v}_k \qquad \text{and} \qquad \vec{x} = \sum_{j=1}^{n} s_j \vec{w}_j$$

(the rs and ss being defined by these equations), and if

$$\vec{v}_k = \sum_{j=1}^{n} C_{jk} \vec{w}_j \,, \tag{†}$$

then

$$s_j = \sum_{k=1}^{n} C_{jk} r_k \,. \tag{‡}$$

(Note that the subscripts in (†) are "twisted" in the sense that the summed index on the matrix elements is not the one adjacent to the vector's coordinates, while the subscripts in (‡) are untwisted!) In other words, if we call the $\vec{v}$s the "old basis" and the $\vec{w}$s the "new basis", then the relation among the four matrices in the problem is:

C	expresses *new coordinates* in terms of *old coordinates*;
C^{-1}	expresses *old coordinates* in terms of *new coordinates*;
C^{t}	expresses *old basis* in terms of *new basis*;
$(C^{-1})^{\mathrm{t}} = (C^{\mathrm{t}})^{-1}$	expresses *new basis* in terms of *old basis.*

Remark 1: The statement "C expresses new coordinates in terms of old coordinates" is shorthand for the more accurate statement, "The linear formula whose coefficient matrix is C (namely, (‡)) expresses the new coordinates as functions of the old coordinates." Another way of saying the same thing is "C maps the old coordinates to the new coordinates." Notice that "new" and "old" have changed places here, since the domain and codomain have exchanged roles relative to the mathematical verb of the sentence. Of course, exactly analogous remarks apply to the other three statements in the theorem.

Remark 2: The matrix $(C^{-1})^{\mathrm{t}} = (C^{\mathrm{t}})^{-1}$ is sometimes called the *contragredient* of C. Note that the operation of taking the contragredient commutes with matrix multiplication, since the reversal of factor order associated with inversion cancels that due to transposition.

REMARK 3: Another way of looking at the relationships in Theorem 1 is to regard each basis as a *row vector* whose elements are vectors. (Thus this is a monstrosity like the top row of the determinant defining a cross product.) When a row vector is multiplied on the *right* by a square matrix, the result is a new row vector of the same size. The formula (†) is of that form:

$$(\vec{v}_1, \ldots, \vec{v}_n) = (\vec{w}_1, \ldots, \vec{w}_n)C. \qquad (\dagger')$$

This new form of the basis transformation formula does not overtly involve any twist or transpose. The corresponding coordinate transformation in the opposite direction is, of course,

$$\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = C \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \qquad \text{or } \vec{s} = C\vec{r}. \qquad (\ddagger')$$

Theorem 1 should be studied until you are sure you understand why it is true. However, *memorizing* the statements at the end is not necessarily recommended!. It is safer to approach each concrete problem with care and common sense. Usually you are given one of the four relations and you need to find one of the other three. You figure out (by a calculation like those in the example) what matrix does what you want to do, and then you test that it does indeed do so. (If you feel more comfortable with partial derivatives than with indexed symbols, you may find the following remark useful.)

Remark (Linear algebra as multivariable calculus): If we regard the ss as functions of the rs, then (from (‡) and its inverse)

$$C_{jk} = \frac{\partial s_j}{\partial r_k} \qquad \text{and} \qquad (C^{-1})_{kj} = \frac{\partial r_k}{\partial s_j}\,.$$

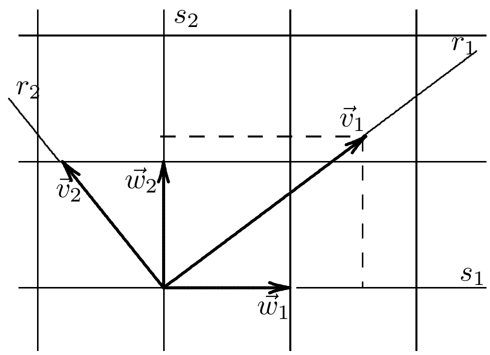
(This was the essence of Exercise 3.4.10.) Thus (†) can be rewritten

$$\vec{v}_k = \sum_{j=1}^{n} \frac{\partial s_j}{\partial r_k}\, \vec{w}_j\,. \qquad (\dagger'')$$

Now recall that the multivariable chain rule tells us that partial derivatives transform this way under a change of coordinate system:

$$\frac{\partial}{\partial r_k} = \sum_{j=1}^{n} \frac{\partial s_j}{\partial r_k}\, \frac{\partial}{\partial s_j}\,.$$

Thus ($\dagger''$) says that basis vectors "behave like" partial-derivative operators when coordinates are changed. (Indeed, many mathematicians nowadays insist that the basis vectors *are* partial-derivative operators, but that is a piece of metaphysics that you are not obligated to accept at the moment.) The point is that if you have thoroughly absorbed the multivariable chain rule from third-semester calculus, but you feel bewildered by the matrix C and its three cousins, then writing formulas in terms of partial derivatives may help you get them right!



To get a geometrical interpretation of ($\dagger''$), suppose that $\{\vec{w}_j\}$ is the natural basis for $\mathbf{R}^n$, so that each s_j is simply a standard Cartesian coordinate in $\mathbf{R}^n$. Then the r_k s are a set of "oblique" coordinates. The numbers $\frac{\partial s_j}{\partial r_k}$ are the Cartesian coordinates of the basis vector along the r_k axis, $\vec{v}_k$. Therefore, they tell how much each quantity s_j changes when r_k increases by one unit, all the other $r_{k'}$ s being held fixed. Of course, that is exactly what a partial derivative ought to do!

Example. As discussed in Sec. 4.1, two standard bases for the solution space of $y'' = 9y$ are

$$\mathcal{A} = \{\vec{a}_1 = e^{3t}, \vec{a}_2 = e^{-3t}\} \qquad \text{and} \qquad \mathcal{B} = \{\vec{b}_1 = \cosh(3t), \vec{b}_2 = \tfrac{1}{3}\sinh(3t)\}.$$

Find the matrix that expresses the coordinates of an arbitrary vector (solution) with respect to the $\mathcal{A}$ basis in terms of its coordinates with respect to the $\mathcal{B}$ basis.

Solution: From the definition of the hyperbolic functions, we have

$$\vec{b}_1 = \tfrac{1}{2}\vec{a}_1 + \tfrac{1}{2}\vec{a}_2\,,$$
$$\vec{b}_2 = \tfrac{1}{6}\vec{a}_1 - \tfrac{1}{6}\vec{a}_2\,.$$

From here there are many equivalent ways to proceed:

Method 1: Therefore, by Theorem 1,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} \end{pmatrix}^{t} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix}$$

maps the $\mathcal{B}$-coordinates into the $\mathcal{A}$-coordinates, as demanded.

Method 2: Therefore (see Remark 3),

$$(\vec{b}_1, \vec{b}_2) = (\vec{a}_1, \vec{a}_2) \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix},$$

and the matrix appearing in this equation is the desired one.

Method 3: Therefore, if $y = x_1\vec{a}_1 + x_2\vec{a}_2$ and also $y = y_1\vec{b}_1 + y_2\vec{b}_2$, then

$$y = y_1\left(\tfrac{1}{2}\vec{a}_1 + \tfrac{1}{2}\vec{a}_2\right) + y_2\left(\tfrac{1}{6}\vec{a}_1 - \tfrac{1}{6}\vec{a}_2\right) = \left(\tfrac{1}{2}y_1 + \tfrac{1}{6}y_2\right)\vec{a}_1 + \left(\tfrac{1}{2}y_1 - \tfrac{1}{6}y_2\right)\vec{a}_2\,.$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

(This method amounts to reproving Theorem 1 in this particular case.)

Theorem 2: Given a basis, the resulting correspondence between vectors and their coordinates is an isomorphism (that is, it's linear, one-to-one, and onto; see Sec. 5.2 for definitions). Thus every n-dimensional vector space is isomorphic to $\mathbf{R}^n$.

Example. Let $(-4, 3, 2, -1, 1)$ and $(-1, -2, 2, 1, -3)$ be the coordinates of two vectors, $\vec{x}$ and $\vec{y}$, with respect to a basis $\{\vec{v}_1, \ldots, \vec{v}_5\}$ of a 5-dimensional vector space, $\mathcal{V}$. Find the coordinates of the vector $3\vec{x} - 5\vec{y}$ in the same basis.

Solution: The talk about an abstract basis is just a smokescreen. You can do the arithmetic on the coordinate vectors as usual. Since $3\vec{x} = (-12, 9, 6, -3, 3)$ and $-5\vec{y} = (5, 10, -10, -5, 15)$, the vector $3\vec{x} - 5\vec{y} = 3\vec{x} + (-5)\vec{y}$ has the coordinates $(-7, 19, -4, -8, 18)$.

The matrix of a linear function with respect to a basis

Theorem 2 has tremendous practical significance. It allows one to carry out many calculations involving vectors in an abstract space $\mathcal{V}$ in terms of ordinary numbers. (We've been doing this tacitly all along in our examples involving the polynomial spaces, $\mathcal{P}_{n-1}$.) In particular, *every linear function* *from one finite-dimensional vector space into another can be represented by a matrix*:

Theorem 3 (kth-Column Rule): Let $\mathcal{D}$ be an n-dimensional space, and $\mathcal{W}$ a p-dimensional space. Choose a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathcal{D}$ and a basis $\{\vec{w}_1, \ldots \vec{w}_p\}$ for $\mathcal{W}$. Let L be a linear function on $\mathcal{D}$ into $\mathcal{W}$. Then there is a $p \times n$ matrix A_L such that for each $\vec{x} \in \mathcal{D}$ the coordinates of $L(\vec{x})$ with respect to the $\vec{w}$-basis are obtained from the coordinates of $\vec{x}$ with respect to the $\vec{v}$-basis by multiplication by A_L:

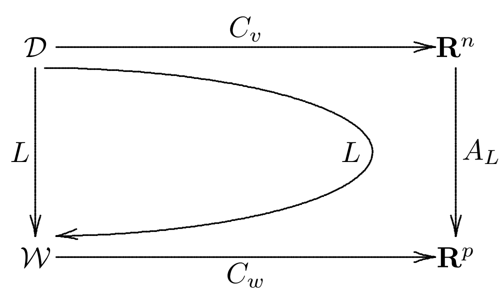
$$\vec{x} \equiv \sum_{k=1}^{n} x_k \vec{v}_k\,, \qquad \vec{L}(\vec{x}) \equiv \sum_{j=1}^{p} L(\vec{x})_j \vec{w}_j\,,$$

$$L(\vec{x})_j = \sum_{k=1}^{n} A_{jk} x_k$$

where $A_L = \{A_{jk}\} \quad (1 \le j \le p,\ 1 \le k \le n)$. Furthermore, the elements of the kth column of A_L are the coordinates of $\vec{L}(\vec{v}_k)$ with respect to the $\vec{w}$-basis:

$$A_{jk} = L(\vec{v}_k)_j\,.$$

Here's the picture.



The matrix A_L represents a linear function from $\mathbf{R}^n$ to $\mathbf{R}^p$; let's denote that function also by A_L. Then L itself equals $C_w^{-1} \circ A_L \circ C_v$, where $C_v\colon \mathcal{D} \to \mathbf{R}^n$ and $C_w\colon \mathcal{W} \to \mathbf{R}^p$ are the coordinate isomorphisms. Those isomorphisms "translate" everything about the spaces $\mathcal{D}$ and $\mathcal{W}$ into numerical terms; in particular, they translate L into a matrix.

Proof of theorem: The mapping $A_L = C_w \circ L \circ C_v^{-1}$ of the $\{x_k\}$ onto the $\{L(\vec{x})_j\}$ is linear, because it is a composition of linear maps. Apply the elementary kth-column rule (Theorem 2 in Sec. 3.2) to get its matrix.

The basic theorems about matrix representations of linear functions from $\mathbf{R}^n$ to $\mathbf{R}^p$ now carry over to arbitrary finite-dimensional vector spaces. In particular,

$$A_{L\circ K} = A_L A_K\,;$$

$$A_{L^{-1}} = (A_L)^{-1} \qquad \text{when they exist;}$$

$$A_{rL+K} = rA_L + A_K\,.$$

Please note that in our recent discussions we have been considering two *different* situations in each of which we've been dealing with two bases. First, we considered two different bases for the same vector space, and studied how the coordinates of a fixed vector changed when we abandoned one basis in favor of the other. Second, we considered *two* vector spaces, each equipped with its own basis, and a linear function from one space into the other; we studied how the coordinates of the image (output) vector in the codomain depend on the coordinates of the input vector in the domain. (Actually, these two spaces could be the same space; the important point is that two different vectors, the input and output, are involved.) In the next section we will put these two ingredients together; therefore, we'll need to deal with as many as four bases at once.

Examples of finding the matrix of a transformation

Example 1. Let $\mathcal{D} = \mathcal{P}_3$ and $\mathcal{W} = \mathcal{P}_3$. Let $D = d/dt$, the operation of differentiation. Use the basis $\{t^3, t^2, t, 1\}$ in both $\mathcal{D}$ and $\mathcal{W}$. Then the first column of the matrix of D is the list of coefficients of $D(t^3) = 3t^2$, namely, $(0,3,0,0)^{\mathrm{t}}$. Treating the other columns in the same way, we see that

$$\begin{matrix} t^3 \to \\ t^2 \to \\ t^1 \to \\ t^0 \to \end{matrix} \begin{pmatrix} \top & \top & \top & \top \\ D(t^3) & D(t^2) & D(t) & D(1) \\ | & | & | & | \\ \bot & \bot & \bot & \bot \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(The basis functions have been written at the left here merely as a one-time reminder of the meaning of each row: The output of the calculation described by the first row is the coefficient of t^3 in the answer, etc.) Thus, for example, if $p(t) = 3t^3 - 5t + 1$, then

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ 0 \\ -5 \end{pmatrix},$$

so $p'(t) = [Dp](t) = 9t^2 - 5$ (which agrees with a direct calculation of p').

Example 2. Let $\mathcal{D} = \mathcal{P}_3$ and $\mathcal{W} = \mathcal{P}_2$. Let $D = d/dt$ again. Use the basis $\{t^3, t^2, t, 1\}$ for $\mathcal{D}$ and $\{t^2, t, 1\}$ for $\mathcal{W}$. Then the matrix for D is easily seen to be

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

— just the matrix from the previous example without its top row. Note that the matrix corresponding to a given linear function can depend upon which vector space is regarded as its codomain!

An alternative way of solving problems like these is simply to calculate $L(\vec{x})$ for an arbitrary $\vec{x} \in \mathcal{D}$ and read off the coefficients. (In the present case, $L = D$ and $\mathcal{D} = \mathcal{P}_3$.) Adopting a notation suggested by our general discussion of bases, we calculate

$$\begin{aligned} D(r_1 t^3 + r_2 t^2 + r_3 t + r_4) &= 3r_1 t^2 + 2r_2 + r_3 \\ &\equiv s_1 t^2 + s_2 t + s_3 . \end{aligned}$$

Thus $s_1 =$ coefficient of $t^2 = 3r_1$, etc. From the three equations for the ss we can read off the matrix.

Example 3. Let $\mathcal{D}$ be the vector space spanned by the functions

$$U_1 = 1, \quad U_2 = \cos t, \quad U_3 = \cos 2t, \quad U_4 = \cos 3t.$$

Let L be the differential operator d^2/dt^2, which is a linear function on $\mathcal{D}$ into $\mathcal{D}$. What is the matrix of L, with respect to the basis $\{U_1, U_2, U_3, U_4\}$ at both ends of the transformation? Answer:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}.$$

Example 4. Let $\mathcal{D}$ and L be as above. Let us find the matrix of L with respect to the basis

$$V_1 = 1, \quad V_2 = \cos t, \quad V_3 = \cos^2 t, \quad V_4 = \cos^3 t$$

(used at both ends of the transformation L). We note first of all that

$$U_3 \equiv \cos 2t = 2V_3 - V_1\,, \\ U_4 \equiv \cos 3t = 4V_4 - 3V_2\,, \\ U_1 = V_1\,, \qquad U_2 = V_2\,,$$

and hence

$$V_3 = \tfrac{1}{2}U_3 + \tfrac{1}{2}U_1 = \tfrac{1}{2}\cos 2t + \tfrac{1}{2}, \\ V_4 = \tfrac{1}{4}U_4 + \tfrac{3}{4}U_2 = \tfrac{1}{4}\cos 3t + \tfrac{3}{4}\cos t.$$

This verifies that the new set is indeed a basis for the same space, $\mathcal{D}$. It also enables us to calculate easily

$$\frac{d^2V_1}{dt^2} = 0, \qquad \frac{d^2V_2}{dt^2} = -V_2\,, \\ \frac{d^2V_3}{dt^2} = -2\cos 2t = -4V_3 + 2V_1\,, \\ \frac{d^2V_4}{dt^2} = -\frac{9}{4}\cos 3t - \frac{3}{4}\cos t = -9V_4 + 6V_2\,.$$

Therefore, the matrix is

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}.$$

Of course, we could also use the U basis for $\mathcal{D}$ in its role as domain while using the V basis for $\mathcal{D}$ in its role as codomain (or vice versa), if we had a reason for doing so. This would give still a different matrix for L.

Example 5 (an inverse problem)**.** Let

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix}, \qquad \vec{b}_1 = e^t, \quad \vec{b}_2 = e^{2t}, \quad \vec{b}_3 = e^{3t}.$$

Construct the linear operator L whose matrix in the basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is equal to A.

SOLUTION: Your first reaction should be to demand a clarification of the problem: $\mathcal{B}$ is a basis for what? According to Exercise 4.1.8, the set $\mathcal{B}$ is linearly independent. It is therefore a basis for the vector space that it spans — that is, the space $\mathcal{S}$ of all functions of the form

$$\begin{aligned} f(t) &= r_1\vec{b}_1 + r_2\vec{b}_2 + r_3\vec{b}_3 \\ &= r_1 e^t + r_2 e^{2t} + r_3 e^{3t}. \end{aligned}$$

Let $\vec{f} = (r_1, r_2, r_3)^{\mathrm{t}} \in \mathbf{R}^3$ be the coordinate vector corresponding to a function f. Then

$$A\vec{f} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 - 2r_2 + r_3 \\ r_1 + r_2 - 2r_3 \\ -2r_1 + r_2 + r_3 \end{pmatrix}.$$

So

$$\begin{aligned} L(f)(t) &= (r_1 - 2r_2 + r_3)\vec{b}_1 + (r_1 + r_2 - 2r_3)\vec{b}_2 + (-2r_1 + r_2 + r_3)\vec{b}_3 \\ &= (r_1 - 2r_2 + r_3)e^t + (r_1 + r_2 - 2r_3)e^{2t} + (-2r_1 + r_2 + r_3)e^{3t}. \end{aligned}$$

This formula completely describes a linear operator $L\colon \mathcal{S} \to \mathcal{S}$.

Example 6. A linear mapping $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ is defined by

$$[L(p)](t) = (t^2 - 4)p''(t) + tp'(t) - 4p(t).$$

Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{P}_2$.

SOLUTION: Calculate

$$\begin{aligned} L(t^2) &= (t^2 - 4)(2) + t(2t) - 4t^2 = -8, \\ L(t) &= 0 + t - 4t = -3t, \\ L(1) &= -4. \end{aligned}$$

Therefore, the matrix is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ -8 & 0 & -4 \end{pmatrix}.$$

Leaving or returning to the natural basis

Very frequently (for example, in the matrix diagonalization problems of Chapter 8) one needs to find a change-of-basis matrix when one of the bases is the natural basis in $\mathbf{R}^n$ and the other basis vectors are given to us by their natural representations in $\mathbf{R}^n$. In these problems there is a shortcut around the general theory embodied in Theorem 1:

Theorem 4: Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be n independent column vectors in $\mathbf{R}^n$. Form a matrix M by stacking these vectors together:*

$$M \equiv \begin{pmatrix} \top & \top & & \top \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \bot & \bot & & \bot \end{pmatrix}.$$

Then M is the matrix that maps coordinates with respect to the new $(\vec{v})$ basis into coordinates with respect to the natural basis.

Proof: The assertion is easily seen to be true for the basis vectors themselves, which have expansions like $\vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n$. Hence by the kth-column rule it is true of all vectors.

Exercises

4.4.1 Write out a proof of Theorem 1. (Feel free to use the notation of Remark 3 if you prefer.)

4.4.2 Let us use the standard basis $\{t^n, t^{n-1}, \ldots, t, 1\}$ for each polynomial space $\mathcal{P}_n$. Find the matrix representing each of these linear functions with respect to the standard bases.

(a) $L(p) = p'' + p'$ as an operator from $\mathcal{P}_2$ into $\mathcal{P}_1$.

(b) $L(p) = p'' - 5p'$ as an operator from $\mathcal{P}_2$ into $\mathcal{P}_2$.

4.4.3 Define the linear operator $L\colon \mathcal{P}_4 \to \mathcal{P}_4$ by

$$Lp(t) = 3p(t) - p'(t).$$

(As usual, p' is the derivative of the polynomial p.) Find the matrix that represents L with respect to the standard basis $\{t^4, t^3, t^2, t, 1\}$.

4.4.4 Express $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

* For example, if $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $M = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

4.4.5 Let $D \equiv \frac{d}{dt}$ be the operator of differentiation, acting on the space spanned by the functions $\{\vec{b}_1 \equiv e^t,\ \vec{b}_2 \equiv e^{-t}\}$.

(a) Find the matrix representing D with respect to the basis $\{\vec{b}_1, \vec{b}_2\}$.

(b) Find the matrix representing D with respect to the basis

$$\{\vec{v}_1 \equiv \cosh t,\ \vec{v}_2 \equiv \sinh t\}.$$

4.4.6 An operator $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ is defined by

$$[L(p)](t) = t^2 p''(t) + p'(t) + 2p(t).$$

Find the matrix of L with respect to the standard basis, $\{t^2, t, 1\}$, of the polynomial space $\mathcal{P}_2$.

4.4.7 Consider the vectors $\vec{v}_1 = \binom{1}{2}$, $\vec{v}_2 = \binom{-2}{1}$ in $\mathbf{R}^2$.

(a) Show that $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbf{R}^2$.

(b) Express $\vec{v}_1$ and $\vec{v}_2$ as linear combinations of the natural basis vectors, $\hat{e}_1 = \binom{1}{0}$ and $\hat{e}_2 = \binom{0}{1}$.

(c) Express the natural basis vectors in terms of $\vec{v}_1$ and $\vec{v}_2$.

(d) Find the coordinates of an arbitrary vector $\vec{x} = \binom{x}{y}$ with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

4.4.8 Let $\mathcal{V}_1$ be the vector space span$\{\cos t,\ \sin t\}$, and $\mathcal{V}_2$ be the space

$$\operatorname{span}\{1,\ \cos t,\ \sin t,\ \cos(2t),\ \sin(2t)\}.$$

(a) Show that the linear operator $L(p)(t) \equiv p'(t) + (\sin t)p(t)$ maps $\mathcal{V}_1$ into $\mathcal{V}_2$.

(b) Find the matrix representing this linear function (with respect to the bases listed in the definitions of the two spaces).

4.4.9 Consider these two bases for $\mathcal{P}_2$:

$$\mathcal{E} \equiv \{1, t, t^2\}, \qquad \mathcal{B} \equiv \{1 - t, 1 + t, 2t^2\}.$$

(a) Find the change-of-basis matrix that takes coordinates relative to $\mathcal{B}$ into coordinates relative to $\mathcal{E}$.

(b) Find the change-of-basis matrix that takes coordinates relative to $\mathcal{E}$ into coordinates relative to $\mathcal{B}$.

4.4.10 Show that $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis (for $\mathbf{R}^3$) and find the coordinates of the vector $\vec{x}$ in that basis.

(a) $\vec{b}_1 = (-1, 2, 1)$, $\vec{b}_2 = (3, -2, 1)$, $\vec{b}_3 = (2, 3, 4)$; $\vec{x} = (4, 3, 6)$.

(b) $\vec{b}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$, $\vec{b}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$; $\vec{x} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$.

4.4.11 Show that $\{\vec{b}_1, \dots\}$ is a basis (for $\mathbf{R}^4$) and find the coordinates of the vector $\vec{x} = (-2, -3, 0, 3)$ in that basis.

$$\vec{b}_1 = (1, 3, 2, 1), \quad \vec{b}_2 = (3, 2, 1, 1), \quad \vec{b}_3 = (1, 1, 2, 3), \quad \vec{b}_4 = (1, 1, 1, 2).$$

4.4.12 Show that $\{\vec{b}_1, \dots\}$ is a basis (for $\mathbf{R}^4$) and find the coordinates of the vector $\vec{x} = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 4 \end{pmatrix}$ in that basis.

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{b}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

4.4.13 Find the matrix (with respect to standard bases) of the linear operator $L\colon \mathcal{P}_1 \to \mathcal{P}_2$ defined by

$$L(p)(t) = \int_0^t p(\tau)\, d\tau + p'(t).$$

4.4.14 Find the formulas ($x_1 = \dots r_1 \dots r_2 \dots r_3 \dots$, $x_2 = \dots$, $x_3 = \dots$) for converting the coordinates of an arbitrary vector $\vec{x} \in \mathbf{R}^3$ with respect to the basis in Exercise 4.4.10(a) into its coordinates with respect to the natural basis.

4.4.15 Find the formulas ($r_1 = \dots x_1 \dots x_2 \dots x_3 \dots$, $r_2 = \dots$, $r_3 = \dots$) for converting the coordinates of an arbitrary vector $\vec{x} \in \mathbf{R}^3$ with respect to the natural basis into its coordinates with respect to the basis in Exercise 4.4.10(b).

4.4.16 Prove Theorem 2.

In the remaining exercises, the set $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis (for the appropriate $\mathbf{R}^n$), and so is the set $\{\vec{b}'_1, \vec{b}'_2, \vec{b}'_3\}$. Find the formulas ($x_1 = \ldots x'_1 \ldots x'_2 \ldots,\ x_2 = \ldots,\ \ldots$) for converting the coordinates of an arbitrary vector with respect to the primed basis into coordinates with respect to the unprimed basis. (Electronic matrix inversion is allowed.)

4.4.17 $\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \vec{b}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ \vec{b}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix};$

$$\vec{b}'_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ \vec{b}'_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix},\ \vec{b}'_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

4.4.18 $\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \vec{b}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},\ \vec{b}_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix};$

$$\vec{b}'_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},\ \vec{b}'_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix},\ \vec{b}'_3 = \begin{pmatrix} -6 \\ 1 \\ 1 \end{pmatrix}.$$

4.4.19 $\vec{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad \vec{b}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \vec{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\quad \vec{b}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};$

$$\vec{b}'_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\quad \vec{b}'_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad \vec{b}'_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \vec{b}'_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

4.4.20 $\vec{b}_1 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix},\ \vec{b}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix},\ \vec{b}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix},\ \vec{b}_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix};$

$$\vec{b}'_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix},\ \vec{b}'_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix},\ \vec{b}'_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix},\ \vec{b}'_4 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

4.5 Change of Basis

As Examples 3 and 4 of the preceding section show, the matrix representation of a linear function depends upon the basis used for the domain and the basis used for the codomain. Let us study the effect of changing each of these. We must combine the two big theorems of the previous section (one about the matrix representation of a linear function and one about the effect of a basis change on the coordinates of vectors). As before, we'll discuss an example in detail and then state the corresponding result for the most general case.

Let $\mathcal{D} = \mathcal{P}_2$ be the domain and $\mathcal{W} = \mathcal{P}_1$ the codomain. Let $L\colon \mathcal{D} \to \mathcal{W}$ be the differentiation operator, d/dt. Thus

$$\begin{aligned} L(p) &= L(r_1 t^2 + r_2 t + r_3) \\ &= 2r_1 t + r_2 \\ &\equiv b_1 t + b_2 \,, \end{aligned} \qquad (*)$$

where the last line will be our notation (in this section) for a general element of $\mathcal{P}_1$ in the standard form. Incidentally, please be reminded that when the polynomial $r_1 t^2 + r_2 t + r_3$ appears as the argument of L, it is a *pattern* denoting the *function* p in a vector space of polynomial functions; it does *not* denote a *number* which is the value of the polynomial for some particular t. (It would be nonsensical, for example, to use the chain rule to calculate something labeled "dL/dt".)

Let us first consider the natural bases

$$\{\vec{v}_j\}_{j=1}^{3} \equiv \{t^2, t, 1\} \text{ for } \mathcal{D}, \qquad \{\vec{w}_j\}_{j=1}^{2} \equiv \{t, 1\} \text{ for } \mathcal{W}.$$

The matrix of L with respect to these bases is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \equiv A$$

(by the kth-column rule or by inspection of $(*)$). That is,

$$b_j = \sum_{k=1}^{3} A_{jk} r_k \,, \quad \text{or} \quad \vec{b} = A\vec{r},$$

where $\vec{b} \in \mathbf{R}^2$ and $\vec{r} \in \mathbf{R}^3$.

Now let's introduce a new basis at the domain end:

$$\{\vec{u}_j\}_{j=1}^3 = \{(t-1)^2, (t-1), 1\}.$$

(This sort of basis change for polynomials is likely to occur in practice, since it just corresponds to a change of origin for the variable t.) Any element of $\mathcal{D}$ can be written as

$$s_1(t-1)^2 + s_2(t-1) + s_3\,.$$

We want the matrix (or the set of equations) that expresses b_1 and b_2 in terms of s_1, s_2, and s_3. If we had the rs expressed in terms of the ss, we could plug into $(*)$ to get our answer. What do we know that is related to this? Well, we can quickly get the expression of the $\vec{u}$s in terms of the $\vec{v}$s:

$$\begin{aligned} \vec{u}_1 &= t^2 - 2t + 1 \\ &= \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3\,, \\ \vec{u}_2 &= \vec{v}_2 - \vec{v}_3\,, \\ \vec{u}_3 &= \vec{v}_3\,. \end{aligned}$$

That is,

$$\vec{u}_j = \sum_{k=1}^3 M_{kj}\vec{v}_k\,, \qquad M \equiv \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix};$$

that is, the *new* basis is given in terms of the *old* basis by the *transpose* of M (since we deliberately put a "twist" in the order of the indices of the formula defining M). Looking back at the theorem covering this situation, we see that M itself gives the *old* coordinates in terms of the *new* ones:

$$r_j = \sum_{k=1}^3 M_{jk}s_k\,, \quad \text{or} \quad \vec{r} = M\vec{s}.$$

(We'll rederive this from scratch in a second, so as to practice what we preach.) Therefore,

$$\vec{b} = A\vec{r} = AM\vec{s},$$

so the matrix of L with respect to bases $\{\vec{u}_j\} \subset \mathcal{D}$ and $\{\vec{w}_j\} \subset \mathcal{W}$ is

$$AM = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}.$$

That is,

$$L\big(s_1(t-1)^2 + s_2(t-1) + s_3\big) = 2s_1 t + (-2s_1 + s_2).$$

Let's work through the connection between the basis change and the coordinate change: We have two ways of expanding a general vector,

$$p(t) = \sum_{j=1}^{3} r_j \vec{v}_j = \sum_{k=1}^{3} s_k \vec{u}_k \,.$$

We rename the indices in the formula for the $\vec{u}$s so that we can substitute it into the second form:

$$\vec{u}_k = \sum_{j=1}^{3} M_{jk} \vec{v}_j \,.$$

Thus

$$\begin{aligned} p(t) &= \sum_{k=1}^{3} s_k \sum_{j=1}^{3} M_{jk} \vec{v}_j \\ &= \sum_{j=1}^{3} \left[\sum_{k=1}^{3} M_{jk} s_k \right] \vec{v}_j \,. \end{aligned}$$

Comparing with the first expression for p, we see that r_j equals the quantity in square brackets. That is the formula we wanted to derive.

Now that we know how to change basis in the domain, it should be child's play to change basis in the codomain. Let

$$\begin{aligned} \vec{t}_1 &= t + 1 = \vec{w}_1 + \vec{w}_2 \,, \\ \vec{t}_2 &= 2t - 1 = 2\vec{w}_1 - \vec{w}_2 \,. \end{aligned}$$

We want formulas for the cs, defined by

$$L(p) = \sum_{j=1}^{2} c_j \vec{t}_j \,.$$

By the same principle as before,

$$\text{if } \vec{w}_j = \sum_{k=1}^{2} H_{kj} \vec{t}_k \,, \quad \text{then } c_j = \sum_{k=1}^{2} H_{jk} b_k \,.$$

We therefore solve for the $\vec{w}$s in terms of the $\vec{t}$s:

$$\vec{w}_1 = \tfrac{1}{3}(\vec{t}_1 + \vec{t}_2), \qquad \vec{w}_2 = \tfrac{1}{3}(2\vec{t}_1 - \vec{t}_2); \qquad H = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Then $\vec{c} = H\vec{b}$ and $\vec{b} = A\vec{r}$, so $\vec{c} = HA\vec{r}$;

$$HA = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \end{pmatrix}.$$

This is the matrix of L with respect to the bases $\{\vec{v}_j\}$ and $\{\vec{t}_j\}$.

Finally, we might want the matrix of L with respect to the bases $\{\vec{u}_j\}$ and $\{\vec{t}_j\}$. To get this we combine the two previous procedures: Multiply A by M on the right to change basis in the domain, and multiply it by H on the left to change basis in the codomain.

$$HAM = \frac{1}{3}\begin{pmatrix} -2 & 2 & 0 \\ 4 & -1 & 0 \end{pmatrix}.$$

That is,

$$L\bigl(s_1(t-1)^2 + s_2(t-1) + s_3\bigr) = \tfrac{1}{3}(-2s_1 + 2s_2)(t+1) + \tfrac{1}{3}(4s_1 - s_2)(2t-1),$$

which you are invited to check by direct calculation of the derivative.

General theorem on change of basis

Theorem: Let $\{\vec{v}_j\}_{j=1}^n$ be a basis for $\mathcal{D}$ and $\{\vec{w}_j\}_{j=1}^p$ a basis for $\mathcal{W}$. Let L be a linear function from $\mathcal{D}$ into $\mathcal{W}$ whose matrix with respect to these bases is A. Let $\{\vec{u}_j\}_{j=1}^n$ be a new basis for $\mathcal{D}$ related to the old one by a matrix G:

$$\vec{v}_j = \sum_{j=1}^n G_{kj}\,\vec{u}_k\,.$$

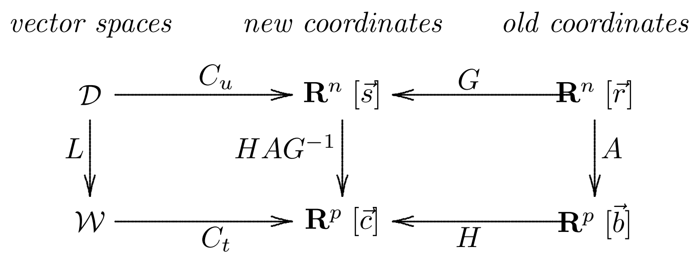
Let $\{\vec{t}_j\}_{j=1}^p$ be a new basis for $\mathcal{W}$ related to the old one by a matrix H:

$$\vec{w}_j = \sum_{j=1}^p H_{kj}\,\vec{t}_k\,.$$

[In other words, multiplication of a column vector by the matrix G or H changes *old* coordinates to *new* coordinates (for $\mathcal{D}$ or $\mathcal{W}$ respectively).] Then the matrix of L with respect to the new bases is

$$HAG^{-1}.$$

This theorem is the distillation of what we learned in the foregoing lengthy example. Note that M of the example is called G^{-1} in the theorem. Here is a diagram of the situation:



Again, we remark that it is probably less error-prone to work out the answer from scratch in each concrete case than to memorize the theorem in detail. (This remark does not apply when a large number of similar problems must be solved at one sitting — see below.) The formula HAG^{-1} is worse than useless unless you have also memorized the *precise* conventions used to define H and G! The matrix which first arises in your problem as a representation of a basis change in the codomain may be the inverse, transpose, or contragredient of H, rather than H itself. Why, then, did we bother to write down the theorem? First, a thorough understanding of the basic fact described in the theorem is essential: A problem of this type always has a solution, and the solution is given by a simple matrix multiplication. Second, independent of any conventions, the formula HAG^{-1} expresses certain more specific facts which are quite helpful in getting the correct answer: Basis changes in the domain and codomain are implemented by matrix multiplication on the right and left, respectively, and one of the matrices will carry an inverse if your conventions relate "old" and "new" coordinates consistently in the two cases. Finally, if you have several problems of the same type to do at once, then if you set up a consistent notation and carefully express the theorem in that notation, the formula corresponding to HAG^{-1} will give you a powerful and efficient way to calculate all the answers! The optimal methodology for solving practical mathematical problems is always some intermediate mixture of the abstract and the concrete, which varies from situation to situation (as well as from person to person).

The theorem has a very important special case:

Corollary: Suppose that $\mathcal{W}$ and $\mathcal{D}$ are the same space, and that, at any given time, we want to use the same basis for this space in both of its roles (domain and codomain). Then, whenever this basis is changed, in such a way that old coordinates are transformed into new coordinates by a matrix H, the matrix representing a linear function L changes from A to HAH^{-1}.

Definition: The operation of replacing A by HAH^{-1} is called a *similarity transformation* of the matrix A.

Example: Recall our two recent examples of matrix representations of a linear function in the vector space spanned by the functions $\{\cos nt\}_{n=0}^{3}$. We calculated both matrices directly, by the kth-column rule. Now let's verify that one of those matrices can be obtained from the other by the appropriate similarity transformation.

Previously we observed that the set

$$U_1 = 1, \quad U_2 = \cos t, \quad U_3 = \cos 2t, \quad U_4 = \cos 3t.$$

and the set

$$V_1 = 1, \quad V_2 = \cos t, \quad V_3 = \cos^2 t, \quad V_4 = \cos^3 t$$

are bases for the same vector space, $\mathcal{D}$. We also observed that the matrix of the operator

$$L \equiv \frac{d^2}{dt^2}$$

with respect to the U basis (used in $\mathcal{D}$ as both domain and codomain) is

$$A \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

Finally, we observed that

$$\begin{aligned} U_1 &= V_1\,, \\ U_2 &= V_2\,, \\ U_3 &= -V_1 + 2V_3\,, \\ U_4 &= -3V_2 + 4V_4\,, \end{aligned}$$

so that, inversely,

$$\begin{aligned} V_1 &= U_1\,, \\ V_2 &= U_2\,, \\ V_3 &= \tfrac{1}{2}U_1 + \tfrac{1}{2}U_3\,, \\ V_4 &= \tfrac{3}{4}U_2 + \tfrac{1}{4}U_4\,. \end{aligned}$$

Now let's find the matrix that represents L when we use the V basis in the codomain, but continue to use the U basis in the domain. (There is no law against using two different bases for the same space at the same time for distinct purposes.) Let's reason this out carefully. If we start with the U-coordinates of an arbitrary vector, $\vec{x}$, then applying A will give us the U-coordinates of $L(\vec{x})$. What we want is the V-coordinates of $L(\vec{x})$. But to express V-coordinates in terms of U-coordinates, we should use the transpose of the matrix that expresses the U basis vectors in terms of the V vectors. (If you prefer the other language: To map U-coordinates to V-coordinates, we should use the transpose of the matrix that maps the Vs into the Us.) From the formulas for the change of the basis vectors, we see that this transposed matrix is

$$H \equiv \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Thus the matrix we desire is

$$HA = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 27 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -36 \end{pmatrix}.$$

Next, let's suppose that we decide to use the V basis in the domain also. Then the first step of the calculation should be to translate the V-coordinates of the input vector $\vec{x}$ into U-coordinates; we can then apply to the result the matrix HA we just found. The coordinate transformation matrix we want is the transpose of the matrix expressing the V vectors in terms of the U vectors; alternatively, it is the inverse of the matrix expressing the V-coordinates in terms of the U-coordinates — that is, the inverse of the matrix H we found before. Either way, we arrive at the matrix

$$H^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Thus the matrix of L with respect to the V basis (in both domain and codomain) is

$$HAH^{-1} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}.$$

This is the same result we found before by the kth-column rule.

Of course, we could use the V basis for the domain but the U basis for the codomain; then the matrix of L would be AH^{-1}.

More examples

Example 1. With respect to a basis $\{\vec{e}_1, \vec{e}_2\}$, a certain linear transformation has the matrix $A = \begin{pmatrix} 5 & 2 \\ 8 & 4 \end{pmatrix}$. Find the matrix of the transformation with respect to the basis $\{\vec{f}_1 = 6\vec{e}_1 + 4\vec{e}_2,\ \vec{f}_2 = 2\vec{e}_1 + \vec{e}_2\}$.

Solution: The transposed matrix $S = \begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix}$ expresses the old $(\vec{e})$ coordinates in terms of the new $(\vec{f})$ cordinates (as can be easily checked by feeding in the new coordinates $\binom{1}{0}$ of $\vec{f}_1$, etc.). Its inverse is

$$S^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & -2 \\ -4 & 6 \end{pmatrix}.$$

Now we can find the matrix B of the transformation in the new basis $\{\vec{f}_1, \vec{f}_2\}$ by the formula $B = S^{-1}AS$:

$$B = -\frac{1}{2}\begin{pmatrix} 1 & -2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -11 & -6 \\ 28 & 16 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -66-24 & -22-6 \\ 168+64 & 56+16 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -90 & -28 \\ 232 & 72 \end{pmatrix} = \begin{pmatrix} 45 & 14 \\ -116 & -36 \end{pmatrix}.$$

Example of the example: A certain linear transformation maps the space of solutions of the differential equation $y'' + 4y = 0$ into itself, and its matrix with respect to the basis $\left\{e^{2t}, e^{-2t}\right\}$ is A. What is the matrix of this transformation with respect to the basis $\left\{6e^{2t} + 4e^{-2t},\ 2e^{2t} + e^{-2t}\right\}$?

Example 2. Let $A\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear operator given by the matrix

$$\overline{A} = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

with respect to the natural basis. Find the matrix of this operator in the basis

$$\vec{f}_1 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}, \quad \vec{f}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{f}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

SOLUTION, METHOD 1: Let us express $\vec{f}_1, \vec{f}_2, \vec{f}_3$ in terms of the natural basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$\begin{aligned}\vec{f}_1 &= 5\hat{e}_1 - 3\hat{e}_2 + 2\hat{e}_3\,,\\ \vec{f}_2 &= 2\hat{e}_1 + 0\hat{e}_2 + \hat{e}_3\,,\\ \vec{f}_3 &= 2\hat{e}_1 + 0\hat{e}_2 - \hat{e}_3\,.\end{aligned}$$

(In passing we note that since

$$\det\begin{pmatrix}5 & -3 & 2\\ 2 & 0 & 1\\ 2 & 0 & -1\end{pmatrix} = -(-3)\begin{vmatrix}2 & 1\\ 2 & -1\end{vmatrix} = -12 \neq 0,$$

$\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ really is a basis for $\mathbf{R}^3$.) In order to find $A\vec{f}_1, A\vec{f}_2, A\vec{f}_3$, we apply $\overline{A}$ to the basis column vectors:

$$\overline{A}\begin{pmatrix}5\\ -3\\ 2\end{pmatrix} = \begin{pmatrix}0 & 1 & 4\\ 0 & 1 & 0\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}5\\ -3\\ 2\end{pmatrix} = \begin{pmatrix}5\\ -3\\ 2\end{pmatrix},$$

$$\overline{A}\begin{pmatrix}2\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}4\\ 0\\ 2\end{pmatrix}, \quad \overline{A}\begin{pmatrix}2\\ 0\\ -1\end{pmatrix} = \begin{pmatrix}-4\\ 0\\ 2\end{pmatrix}.$$

This means that $A\vec{f}_1 = 1\vec{f}_1\,, A\vec{f}_2 = 2\vec{f}_2\,, A\vec{f}_3 = -2\vec{f}_3$. Therefore, the matrix being sought is the diagonal matrix

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -2\end{pmatrix}.$$

(Later (Chap. 8) we will find out that $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are called *eigenvectors* of the operator A.)

SOLUTION, METHOD 2: Following the prescription at the end of the previous section, we stack the given basis vectors together to make a matrix,

$$M \equiv \begin{pmatrix}\top & \top & \top\\ \vec{f}_1 & \vec{f}_2 & \vec{f}_3\\ \bot & \bot & \bot\end{pmatrix} = \begin{pmatrix}5 & 2 & 2\\ -3 & 0 & 0\\ 2 & 1 & -1\end{pmatrix}.$$

A moment's thought confirms that this is the matrix that maps coordinates with respect to the new $(\vec{f})$ basis into coordinates with respect to the natural basis (not vice versa!). Therefore, the new matrix is $M^{-1}\overline{A}M$, for we want to translate an input vector into natural coordinates, calculate the operator A in those terms, and then translate the result in the reverse direction. Evaluating the matrix inverse and product will give the same answer as the other method.

Example 3. The matrix

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

represents a linear operator $L\colon \mathbf{R}^3 \to \mathbf{R}^2$ with respect to the natural bases. What matrix represents L if we switch to the basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

for $\mathbf{R}^2$? (The basis for the domain is still the natural one.)

Solution: Since the basis change is in the codomain (postprocessing the output of the calculation), it will be implemented by multiplying M on the left by some 2×2 matrix. Let us call the new basis vectors

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

In terms of the natural basis vectors, we have

$$\begin{aligned} \vec{b}_1 &= \hat{e}_1 + \hat{e}_2 \,, \\ \vec{b}_2 &= -\hat{e}_1 + \hat{e}_2 \,. \end{aligned}$$

Therefore (by whichever rule of thumb you prefer from Sec. 4.4),

$$C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

maps $\vec{b}$-coordinates into natural coordinates. (Most quickly, this matrix is obtained simply by "stacking the new basis vectors together".) Thus

$$C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

maps natural coordinates to $\vec{b}$-coordinates. To solve our problem we need to postprocess the natural-basis calculation of L with C^{-1}. Therefore, the desired matrix is

$$C^{-1}M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 1 & 5 \\ 1 & -3 & 3 \end{pmatrix}.$$

Transforming row vectors

An important special case of the transformation of a matrix under change of basis is that where the matrix A has only one row. Recall that such a matrix represents a linear function $L: \mathcal{D} \to \mathbf{R}$ (also called a *linear functional*). Let's apply the general theorem on change of basis to this case, with the basis change in the domain implemented by a matrix G as before, and no basis change in the one-dimensional codomain: With respect to the new basis, L is represented by the matrix $B = AG^{-1}$. This is to be contrasted with the transformation law for column vectors, $\vec{s} = G\vec{r}$.

To check the consistency of these two formulas, consider the number $L(\vec{x})$, where $\vec{x}$ is an abstract vector represented by the column vector $\vec{r}$ with respect to the old basis and by the column vector $\vec{s}$ with respect to the new basis. This number must remain unchanged, no matter what basis is used! Sure enough,

$$L(\vec{x}) = A\vec{r} = AG^{-1}G\vec{r} = B\vec{s}.$$

Written out in numbers, the transformation rule for column vectors is

$$s_j = \sum_{k=1}^{n} G_{jk} r_k \,,$$

and the one for row vectors is

$$B_j = \sum_{k=1}^{n} A_k {G^{-1}}_{kj} = \sum_{k=1}^{n} (G^{\mathrm{t}})^{-1}{}_{jk} A_k .$$

Recall (refer back to the statement of the general theorem on change of basis, whose notation we are scrupulously observing in this discussion) that

$$\vec{v}_j = \sum_{j=1}^{n} G_{kj}\, \vec{u}_k \,,$$

where the $\vec{v}$s are the *old* basis and the $\vec{u}$s are the *new* basis. In other words, to go from the old basis to the new we need the formula

$$\vec{u}_j = \sum_{k=1}^{n} (G^{\mathrm{t}})^{-1}{}_{jk}\, \vec{v}_k \,.$$

We notice something rather interesting: When we consistently express the new objects in terms of the old ones, the formula for transformation of a row vector (the matrix of a linear functional) and the formula for the transformation of the basis vectors themselves look very much alike, and both involve the contragredient (inverse transpose) of the matrix in the transformation law for the column vectors. (This fact was already noted from a different point of view in Remark 3 of the previous section.)

In older books row vectors were called "covariant vectors" because they transform in the same way as the basis vectors, and column vectors were called "contravariant vectors" because they transform "in the opposite direction" from the basis. From a modern point of view (although the covariant–contravariant terminology survives) it is better to think of the transformation rule for the column vectors as the fundamental one and the two contragredient rules as derived from it. Also, please keep in mind that, no matter how much alike the formulas look, the terms in the basis-change formula are *vectors* and the terms in the row-vector transformation rule are *numbers*.

Finally, we point out again that the gradient of a function is most properly thought of as a row vector. Given a function $f(r_1, \ldots, r_n)$, its gradient is a list of its partial derivatives with respect to the variables r_j. If we now decide to switch to the new coordinates s_j related to the rs as in the discussion above, then we must transform the derivatives according to the chain rule:

$$\frac{\partial f}{\partial s_j} = \sum_k \frac{\partial r_k}{\partial s_j} \frac{\partial f}{\partial r_k} = \sum_k (G^{\mathrm{t}})^{-1}{}_{jk} \frac{\partial f}{\partial r_k} \,.$$

This is the row-vector transformation law (see the discussion of linear algebra as multivariable calculus in the preceding section). The components of a tangent vector to a curve, $\vec{r}(t)$, in contrast, transform as a column vector. This distinction carries over into curvilinear coordinates: A gradient vector, or any other row vector, is most naturally expressed as a linear combination of the basis vectors called $\vec{d}_k$ in Sec. 4.2, which are the gradients of the curvilinear coordinates with respect to the original Cartesian coordinates. A tangent vector, or any other column vector, is most naturally expressed as a linear combination of the basis vectors called $\vec{c}_k$ in Sec. 4.2, which are the tangent vectors to the coordinate curves. When these things are done, the components of the vectors transform "properly" — that is, covariantly or contravariantly, respectively, as the chain rule prescribes — when the coordinate system is changed.

Exercises

4.5.1 We know that $\{\sinh t, \cosh t\}$ and $\vec{v}_1 = e^t$, $\vec{v}_2 = 2e^t - e^{-t}$ are two bases for the same vector space of functions. If $f(t)$ is an arbitrary element of that space, let us define coordinates by

$$f = r_1\vec{v}_1 + r_2\vec{v}_2 = c_1 \sinh t + c_2 \cosh t.$$

(a) Find the formulas expressing r_1 and r_2 in terms of c_1 and c_2.

(b) Find the matrix representing the operator $L = \frac{d}{dt}$ with respect to the hyperbolic basis.

(c) Find (directly) the matrix representing L with respect to the $\vec{v}$ basis.

(d) Use the results of (a) and (b) to get the answer to (c) in a different way.

4.5.2 In Exercise 4.1.4 we showed that

$$\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is a basis for $\mathbf{R}^3$. The function $F\colon \mathbf{R}^3 \to \mathbf{R}^2$ is represented by the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 0 \end{pmatrix}$ with respect to the natural bases. What matrix represents this function when the new basis is used for the domain in place of the natural basis?

4.5.3 A linear function $L\colon \mathbf{R}^3 \to \mathbf{R}^2$ is represented with respect to the natural bases by the matrix $M = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$. Find the matrix of L with respect to the bases $\{\hat{e}_3\,, \hat{e}_2\,, \hat{e}_1\}$ for $\mathbf{R}^3$ (the natural basis in reverse order) and $\left\{\vec{v}_1 = \binom{1}{1}, \vec{v}_2 = \binom{1}{0}\right\}$ for $\mathbf{R}^2$.

4.5.4 $L\colon \mathcal{V} \to \mathcal{V}$ is a certain linear operator from a two-dimensional vector space into itself. Its matrix with respect to a basis $\{\vec{v}_1, \vec{v}_2\}$ is $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. If we change basis in $\mathcal{V}$ to $\{\vec{w}_1 \equiv \vec{v}_1 + 2\vec{v}_2, \vec{w}_2 \equiv 3\vec{v}_1 + \vec{v}_2\}$, what is the new matrix?

4.5.5 Let $L\colon \mathbf{R}^2 \to \mathbf{R}^3$ have the matrix $\begin{pmatrix} 3 & -1 \\ -1 & 2 \\ -3 & 2 \end{pmatrix}$ with respect to the natural basis. Find the matrix that represents L with respect to the new bases

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

$$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

4.5.6 With respect to the natural bases, the matrix $A \equiv \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 0 \end{pmatrix}$ defines a linear function $f\colon \mathbf{R}^3 \to \mathbf{R}^2$.

(a) What matrix represents f when the basis for $\mathbf{R}^2$ (the codomain) is changed to

$$\left\{ \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}?$$

(b) What matrix represents f when the basis for $\mathbf{R}^3$ (the domain) is changed to

$$\left\{ \vec{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}?$$

(The basis for the codomain is still the natural one.)

(c) Find the matrix when both these basis changes are made.

4.5.7 A linear function $G\colon \mathbf{R}^2 \to \mathbf{R}^2$ has the matrix $B = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$ with respect to the natural basis. Find the matrix of G with respect to the basis $\left\{ \vec{b}_1 = \binom{1}{-1}, \vec{b}_2 = \binom{1}{1} \right\}$. (Use the new basis for both domain and codomain.)

4.5.8 Find the matrix of the linear operator A in the basis $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$, if the matrix in the natural basis is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

and

$$\vec{f}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \vec{f}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{f}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

4.5.9 Find the matrix of the operator A in the natural basis, if the matrix in the basis $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$\vec{b}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ \vec{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \vec{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

4.5.10 Let C be the change-of-basis matrix from one basis to another.

(a) How would C change if you interchanged two vectors of the first basis? SUGGESTION: Think of the change of basis as a linear function $\mathbf{R}^n \to \mathbf{R}^n$, and the interchange as a basis change in the domain of that linear function.

(b) How would C change if you interchanged two vectors of the second basis?

4.5.11 How would a change-of-basis matrix change if you decided to write the vectors of both bases in reverse order? SUGGESTION: Think of the change of basis as a linear function $\mathbf{R}^n \to \mathbf{R}^n$, and the reversal as a basis change in the domain and codomain of that linear function.

In the remaining exercises, computer or calculator use is expected.

4.5.12 Find the matrix of L with respect to the primed basis for $\mathbf{R}^3$ given in Exercise 4.4.18 if its matrix with respect to the unprimed basis in that exercise is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

4.5.13 Find the matrix of L with respect to the unprimed basis for $\mathbf{R}^3$ given in Exercise 4.4.17 if its matrix with respect to the primed basis in that exercise is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

4.5.14 Find the matrix of L with respect to the primed basis for $\mathbf{R}^4$ given in Exercise 4.4.20 if its matrix with respect to the unprimed basis in that exercise is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

4.5.15 Find the matrix of L with respect to the unprimed basis for $\mathbf{R}^4$ given in Exercise 4.4.19 if its matrix with respect to the primed basis in that exercise is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$