Chapter 8

Eigenvectors and Diagonalization

8.1 Eigenvalues and Eigenvectors

When a linear function maps a vector space into itself, essentially it moves the vectors around in the space, and a description of the function amounts to a description of how the vectors are moved around. This contrasts with the situation where the domain and codomain of the function are different spaces: In that case, since the basis in the domain and the basis in the codomain can be chosen independently, and since one basis is as good as another from an abstract point of view, all linear functions of the same rank are essentially alike in terms of abstract structure. (For example, any nonsingular matrix can be row-reduced to the identity matrix. This amounts to choosing the basis vectors in the codomain to be the images under the linear function of the basis vectors in the domain.)

When the codomain is the domain, all bases are *not* alike. It turns out that there are certain directions in the domain along which the action of a given linear function is particularly simple. Instead of being "turned" to point in a new direction, vectors in these directions are merely stretched or shrunk (or perhaps reflected). These are called *eigenvectors*, and various things related to these preferred directions are labelled by the prefix "eigen-". In the best circumstances, one can choose a basis for the domain consisting entirely of eigenvectors, and the study of the linear function in question is thereby tremendously simplified.

This kind of analysis of a linear function is of extreme importance in many applications of linear algebra. For example, changing variables to an eigenbasis simplifies a system of linear ordinary differential equations to a set of separate equations that can each be solved by elementary means; the resulting basic solutions are called "normal modes", and regarding the motion of a physical system as a linear combination of its normal modes is of great conceptual as well as calculational value. Eigenvectors of differential operators in infinite-dimensional vector spaces are important in solving partial differential equations; again, the solutions can be constructed as superpositions of normal modes associated with these eigenfunctions. The best-known eigenfunction expansion is the *Fourier series*, whose terms are eigenfunctions of the second-derivative operator.

Accordingly, let $\mathcal{D}$ be a vector space and let L be a linear function on $\mathcal{D}$ into $\mathcal{D}$. For us, $\mathcal{D}$ will be finite-dimensional. With respect to a basis, L is represented by a square matrix, A.

Definition: If $L(\vec{v}) = \lambda\vec{v}$ (where λ is a scalar) and $\vec{v} \neq 0$, then

$\vec{v}$ is an *eigenvector* of L and λ is an *eigenvalue* of L.

Other terms used for "eigenvalue" are *proper value* and *characteristic value*, and similarly for the vectors. (In past generations there were linguistic purists who objected to combining German and English roots in the same word, but they lost the argument; today "eigen-" is simply a technical prefix in the English language.)

Note: The vector $\vec{0}$ is *not* an eigenvector, even though $L(\vec{0}) = 0\,\vec{0}$ (or $262\,\vec{0}$, for that matter). But the scalar 0 *can* be an eigenvalue. Indeed, $L(\vec{v}) = 0\vec{v}$ precisely when $\vec{v}$ is in the kernel of L; the nonzero elements of the kernel are thus eigenvectors with eigenvalue 0.

Example 1: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ defines an L on $\mathbf{R}^3$ (or on any 3-dimensional space equipped with a basis) for which the first basis element, $\vec{e}_1$, is an eigenvector with eigenvalue 2; $\vec{e}_2$ is an eigenvector with eigenvalue 1; and $\vec{e}_3$ is one with eigenvalue 0. Similarly, it is trivial to read off the eigenvalues and eigenvectors of any diagonal matrix.

Example 2: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ defines an $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ for which $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with $\lambda = 1$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with $\lambda = -1$. (Check this.)

Example 3: Linear operators on infinite-dimensional spaces also can have eigenvectors and eigenvalues: $\sin\omega t$ and $\cos\omega t$ are eigenvectors of $L = \frac{d^2}{dt^2}$ with $\lambda = -\omega^2$. This special relationship of the trigonometric functions (and, even more, the exponential functions) to the differentiation operator is the reason why those functions are so prominent in the solution of differential equations with constant coefficients. (The space $\mathcal{D}$ in this example is best taken to be the space $C^\infty(-\infty,\infty)$ of all "smooth" functions — those which can be differentiated arbitrarily many times.)

Why eigenvectors are nice

As long as we can deal only with eigenvectors of L, computations with L are trivial. (More precisely, they are reduced to one-dimensional problems.) For instance, if

$$L(\vec{v}) = \lambda\vec{v},$$

then

$$L^2(\vec{v}) \equiv L(L(\vec{v})) = \lambda^2\vec{v}.$$

(In terms of diagonal matrices,

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

for example.) Similarly,

$$L^{-1}(\vec{v}) = \frac{1}{\lambda}\,\vec{v}$$

(if L^{-1} exists — and hence $\lambda \neq 0$). Indeed, any *rational function* of a real variable (i.e., a quotient of polynomials) gives rise to a rational function of a matrix variable and can be evaluated on eigenvectors simply by applying that function to the eigenvalue. ($\frac{P}{Q}$ is interpreted as PQ^{-1}, which is the same as $Q^{-1}P$ in this case.)

We can go even further and define nonrational functions of L by equations such as

$$\sqrt{L}(\vec{v}) \equiv \sqrt{\lambda}\,\vec{v}, \qquad e^L(\vec{v}) \equiv e^\lambda\vec{v}.$$

These define the operators $\sqrt{L}$ and e^L on eigenvectors. But suppose that we have a *basis* for $\mathcal{D}$ consisting *entirely* of eigenvectors:

$$\mathcal{D} = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}, \qquad L(\vec{v}_j) = \lambda_j\vec{v}_j\,.$$

(The possibility that some of the λ's are equal (say $\lambda_1 = \lambda_3$) is allowed.) Then every vector $\vec{x}$ in $\mathcal{D}$ has a unique expansion

$$\vec{x} = \sum_{j=1}^{n} x_j\vec{v}_j\,,$$

so

$$L(\vec{x}) = \sum_{j=1}^{n} (\lambda_j x_j)\vec{v}_j\,, \qquad L^2(\vec{x}) = \sum_j (\lambda_j{}^2 x_j)\vec{v}_j\,,$$

etc. Thus all the rational functions of L can be computed. Furthermore, we can *define* the nonrational functions for all $\vec{x} \in \mathcal{D}$ by equations such as

$$e^L(\vec{x}) = \sum_{j=1}^{n} \left(e^{\lambda_j} x_j\right) \vec{v}_j\,.$$

The most important application of this construction is this: $e^{tL}(\vec{x})$ is the solution of the differential equation

$$\frac{d\vec{y}}{dt} = L(\vec{y})$$

with the initial condition $\vec{y}(0) = \vec{x}$. If $\mathcal{D} = \mathbf{R}^n$, such a vectorial differential equation is an abbreviation for a *system* of ODEs. For example, if L is the operator represented by the diagonal matrix in Example 1 above, then the system is

$$y_1' = 2y_1\,, \qquad y_2' = y_2\,, \qquad y_3' = 0y_3 = 0,$$

and the solution with $\vec{y}(0) = \vec{x}$ is

$$y_1 = e^{2t}x_1\,, \qquad y_2 = e^t x_2\,, \qquad y_3 = e^{0t}x_3 = x_3\,,$$

in accordance with the general claim.

An infinite-dimensional generalization of this enables one to solve certain partial differential equations. As previously mentioned, every "nice" function on the interval $0 < x < \pi$ can be written as an infinite linear combination

$$f(x) = \sum_{j=1}^{\infty} b_j \sin(jx),$$

called a *Fourier series*. We have just noted that the "basis" functions $\{\sin jx\}$ are eigenvectors of the operator $L = d^2/dx^2$. So we can form

$$u(t,x) \equiv \left(e^{tL} f\right)(x) \equiv \sum_{j=1}^{\infty} e^{-j^2 t} b_j \sin(jx),$$

and it is the solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(the *heat conduction equation*) with the boundary conditions

$$u(0,x) = f(x) \qquad \text{and} \qquad u(t,0) = 0 = u(t,\pi).$$

(We have not stated what the space $\mathcal{D}$ is in this example, for good reason; the technicalities would lead us far afield.)

Unfortunately, the basis we start with in a problem usually is not an eigenbasis of the operator of interest. However, if we can find an eigenbasis, then we can use the techniques of change-of-basis from Chapter 4 to translate everything into coordinates with respect to that basis, where the calculations become easy.

Theorem: If a basis of eigenvectors $\{\vec{v}_j\}$ for L exists, then the matrix D of L with respect to that basis is diagonal, and the jth diagonal element, D_{jj}, equals λ_j. For any function $f\colon \mathbf{R} \hookrightarrow \mathbf{R}$ such that all the eigenvalues of L belong to the domain of f, the operator $f(L)$ is defined, the matrix of $f(L)$ with respect to the eigenbasis is diagonal, and its jth diagonal element is $f(\lambda_j)$.

Theorem If we start with another ("old") basis and the corresponding matrix A for L, then $D = U^{-1}AU$, where U is the matrix whose jth column contains the coordinates of new basis vector $\vec{v}_j$ with respect to the old basis. Thus we can compute L with respect to the old basis using the formula $A = UDU^{-1}$.

Proof: This is a corollary of the theory in Secs. 4.4–5. Let's contemplate it just long enough to verify that we have the matrices going in the right direction: U as described in the theorem clearly maps the coordinates of an arbitrary vector $\vec{x}$ with respect to the new basis into its coordinates with respect to the old basis. (Think of what it does to the coordinates of the new basis vectors themselves, for example.) Therefore AU gives us the coordinates of $L(\vec{x})$ with respect to the old basis, and applying U^{-1} to that result gives us the coordinates with respect to the new basis. Thus $U^{-1}AU$ does precisely what D is supposed to do.

How to find the eigenvalues and eigenvectors (when they exist)

We will work in a particular "old" basis. In other words, we identify $\mathcal{D}$ with $\mathbf{R}^n$ and identify L with its matrix A (with respect to the natural basis).

We want to solve the equation $A\vec{v} = \lambda\vec{v}$ for both $\vec{v}$ and λ simultaneously. The equation can be rewritten

$$(A - \lambda)\vec{v} = \vec{0}.$$

(Here the term λ stands for the scalar λ times the identity matrix.) This homogeneous linear system has nontrivial solutions if and only if the matrix $A - \lambda$ is singular — that is,

$$\det(A - \lambda) = 0.$$

This is called the *characteristic* (or *secular*) equation for the matrix A (or the linear function L), and its left-hand side is the *characteristic polynomial* of A. It is a polynomial in λ of degree n if A is $n \times n$. Therefore:

Theorem: λ is an eigenvalue of A if and only if it is a root of the characteristic equation of A.

EXAMPLE: $$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}.$$

$$\begin{vmatrix} -1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 6 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-1)^3(\lambda+1)(\lambda-1)(\lambda+2).$$

Therefore,* the eigenvalues of A are -1, 1, and -2. (In general, *the eigenvalues of a triangular matrix are its diagonal elements.*) The order in which we list and number the eigenvalues is rather arbitrary; let's say $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = -2$.

Now, for each eigenvalue λ_j, one should solve $(A-\lambda_j)\vec{v} = \vec{0}$ to obtain the corresponding eigenvectors. There will always be at least a one-dimensional subspace of them, since $A - \lambda_j$ is singular (hence has a nontrivial nullity).

EXAMPLE: Consider the case of the eigenvalue $\lambda_2 = 1$ in the previous example.

$$0 = \begin{pmatrix} -1-1 & 2 & 3 \\ 0 & 1-1 & 6 \\ 0 & 0 & -2-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

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* It should be noted that in general the characteristic polynomial will not automatically factor, as it does for a triangular matrix. In this book we restrict attention to 3×3 examples whose characteristic cubic equation is easy to solve, but nature is not always so cooperative.

Therefore, $v_3 = 0$, v_2 is arbitrary, and $v_1 = v_2$. That is, the λ_2-eigenvectors are the multiples of the vector

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad (\lambda_2 = 1).$$

Similarly, one finds that the λ_1-eigenvectors are multiples of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (\lambda_1 = -1),$$

and the λ_3-eigenvectors are multiples of

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad (\lambda_3 = -2).$$

Finally, we can form the change-of-basis matrix

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

and verify that $D = U^{-1}AU$ is diagonal:

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

This is the result that we expected. In fact, since we know in advance what D is supposed to be, the easier way to check our algebra is to verify the equation $A = UDU^{-1}$, rather than $D = U^{-1}AU$. (Multiplying by a diagonal matrix is easier than multiplying by a matrix with many nonzero entries off-diagonal.)

In this example the eigenvalues were distinct. If some number appears more than once as a root of the characteristic equation, the linear system for its eigenvectors may have more than one linearly independent solution. (In the more elementary terminology of Chapter 2, the general solution may involve more than one parameter.)

Can it be done?

The remaining question is whether we can always be sure that the foregoing procedure will "work": Given L (or A, its matrix), is a basis of eigenvectors of L guaranteed to exist? In other words, can every A be *diagonalized* by a similarity transformation, $A \to U^{-1}AU$?

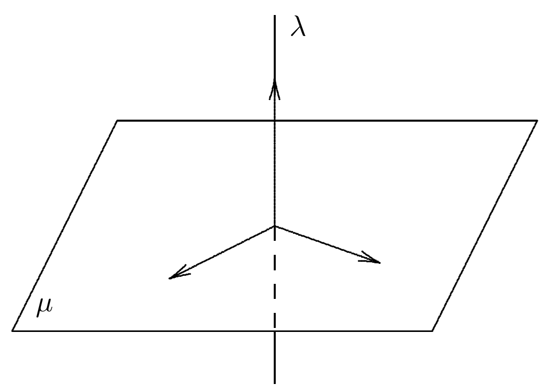
To answer this we need some preliminary theorems:

Theorem: The eigenvectors corresponding to one eigenvalue λ, together with the zero vector, form a subspace.

Proof: $L(\vec{v}_1) = \lambda\vec{v}_1$ and $L(\vec{v}_2) = \lambda\vec{v}_2$ imply $L(r\vec{v}_1 + \vec{v}_2) = \lambda(r\vec{v}_1 + \vec{v}_2)$.

Theorem: Eigenvectors corresponding to distinct λs are linearly independent. (Thus linear combinations of them will *not* be eigenvectors.)

For example, $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$ has a 1-dimensional subspace of λ-eigenvectors and a 2-dimensional subspace of μ-eigenvectors. (In the drawing an arbitrary basis for each subspace is shown.)



We can choose a basis for each eigensubspace. The question is whether all the eigenvectors together span the whole space $\mathcal{D}$. Equivalently, do the dimensions of all the eigensubspaces add up to n, the dimension of $\mathcal{D}$? (Clearly the total can't be more than n, but it might be less.)

Answers to the question:

1. If all n roots of the characteristic equation $\det(A-\lambda) = 0$ are *real and distinct*, **YES**. Each λ_j has an eigenvector, and they are independent; since there are n of them, they form a basis.

2. If the characteristic equation has *nonreal roots*, **NO** — unless one is working in a complex vector space. Note that a *real* matrix may indeed have complex roots to its characteristic polynomial — for example,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda_1 = i, \quad \lambda_2 = -i.$$

If we enlarge our space from $\mathbf{R}^n$ to $\mathbf{C}^n$, then such a matrix has complex eigenvalues and complex eigenvectors. In this example,

$$\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}; \qquad U = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

3. What about *multiple real roots*? Recall the **Fundamental theorem of algebra**:

   Every polynomial in a variable λ can be factored as

$$\text{const.}\, (\lambda - \lambda_1)^{p_1} (\lambda - \lambda_2)^{p_2} \cdots (\lambda - \lambda_r)^{p_r},$$

   where $\sum_{j=1}^{r} p_j$ equals the degree of the polynomial. (Thus r is less than or equal to the degree.) Here some of the λ_j may be complex, even if the original polynomial is real.

   To see the implications of this, let's look at some 2×2 cases:

   (a) $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$. The characteristic equation is $(\mu - \lambda)^2 = 0$. In this case the answer is clearly **YES**; the μ-eigenspace is all of $\mathbf{R}^2$.

   (b) $\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$. The characteristic equation is still $(\mu - \lambda)^2 = 0$. So μ is the only possible eigenvalue. Let's find its eigenvectors:

$$\begin{pmatrix} \mu - \mu & 1 \\ 0 & \mu - \mu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   The augmented matrix of the homogeneous system is

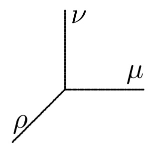
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

   so the general solution of the system is $\binom{x}{y}$ with $y = 0$ and x arbitrary. Thus the *only* eigenvectors are the multiples of $\binom{1}{0}$. Therefore, **NO** eigenbasis exists!

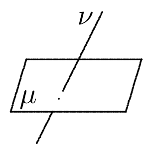
In general, if $(\lambda - \mu)^p$ appears in the characteristic polynomial, you *may* have p independent μ-eigenvectors, but you *may* have fewer than p. (There will always be at least 1 and at most p.)

Let's look at some examples in dimension 3:

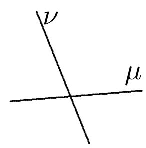
(a) $$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \rho \end{pmatrix}$$



(b) $$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

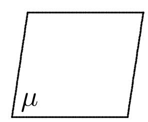


(c) $$\begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$



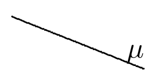
(no basis)

(d) $$\begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$



(no basis)

(e) $$\begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}$$



(no basis)

(f) $$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

(Eigenspace is all $\mathbf{R}^3$.)

These illustrate all the possibilities. In fact, if complex numbers are allowed, every 3×3 matrix can be put into one of these 6 forms by a similarity transformation; this is the 3-dimensional case of the *Jordan canonical form theorem.*

4. If A is (real and) *symmetric* — like

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 0 \end{pmatrix}$$

— then the answer is **YES**! In such a case all the λ_js are real, and the right number of eigenvectors always exists, even if some of the

λ_js coincide. We'll look at this very important case in more detail in the next section. It turns out that the basis of eigenvectors of a symmetric matrix can always be chosen to be orthonormal. A square matrix whose columns are an orthonormal basis is called an *orthogonal matrix* (see the next section for more details). Therefore, if a matrix is symmetric (equals its transpose), then it not only can be diagonalized, but can be diagonalized by an orthogonal matrix. Conversely, if a matrix is diagonalized by an orthogonal matrix, then it is symmetric (Exercise 8.2.17).

EXAMPLES

Example 1. Find the eigenvectors of $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$.

SOLUTION: The eigenvectors are nonzero solutions $\vec{v}$ of the linear system $(A - \lambda)\vec{v} = \vec{0}$, where λ is an eigenvalue. We have

$$A - \lambda = \begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix},$$

so the characteristic equation is

$$0 = \det(A-\lambda) = \begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 1.$$

We could multiply this out and use the quadratic formula, but it is quicker to notice that $(5-\lambda)^2 = 1 \;\Rightarrow\; 5-\lambda = \pm 1$, so that the roots are

$$\lambda_1 = 4, \quad \lambda_2 = 6.$$

Case 1: $\lambda_1 = 4$; let $\vec{v}^{\,(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix}$ be the eigenvector. Set up and solve the appropriate homogeneous system:

$$(\,(A-\lambda_1) \mid \vec{0}\,) = \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \middle| \; \vec{0} \right) \;\Rightarrow\; \begin{cases} v_1^{(1)} + v_2^{(1)} = 0, \\ v_1^{(1)} + v_2^{(1)} = 0. \end{cases}$$

Therefore, $v_1^{(1)} = \alpha$, $v_2^{(1)} = -\alpha$, where α is an arbitrary number.

Case 2: $\lambda_2 = 6$, $\vec{v}^{\,(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix}$;

$$(\,(A-\lambda_2) \mid \vec{0}\,) = \left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \middle| \; \vec{0} \right) \;\Rightarrow\; \begin{cases} -v_1^{(2)} + v_2^{(2)} = 0, \\ \;\;\, v_1^{(2)} - v_2^{(2)} = 0, \end{cases}$$

so $v_1^{(2)} = \beta$, $v_2^{(2)} = \beta$, where β is an arbitrary number.

Summary: The eigenvectors are defined up to arbitrary constant factors and are of the forms

$$\vec{v}^{\,(1)} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}, \qquad \vec{v}^{\,(2)} = \begin{pmatrix} \beta \\ \beta \end{pmatrix}.$$

In other words, one eigenspace is the span of the vector $\binom{1}{-1}$, and the other is the span of $\binom{1}{1}$.

Example 2. Find the characteristic polynomial and the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 1 & -1 \\ -2 & -2 & 1 \end{pmatrix}.$$

SOLUTION: The characteristic polynomial, $P(\lambda) = \det(A - \lambda)$, is

$$\begin{aligned} P(\lambda) &= \begin{vmatrix} 4-\lambda & 2 & 1 \\ -1 & 1-\lambda & -1 \\ -2 & -2 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda)^2 + 4 + 2 + 2(1-\lambda) - 2(4-\lambda) + 2(1-\lambda) \\ &= 4 - \lambda - 8\lambda + 2\lambda^2 + 4\lambda^2 - \lambda^3 + 6 + 2 - 2\lambda - 8 + 2\lambda + 2 - 2\lambda \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6. \end{aligned}$$

To find the eigenvalues one should find the roots of the characteristic equation, $P(\lambda) = 0$. We get (after changing the sign) $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$. Looking among the divisors of 6 for a possible integer root, we find that $\lambda = 1$ works, and hence

$$P(\lambda) = (\lambda - 1)(\lambda^2 - 5\lambda + 6).$$

So it is easy to get the roots: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. These are the eigenvalues.

Case $\lambda_1 = 1$: $\vec{v}^{\,(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{pmatrix}, \qquad (\,(B - \lambda_1) \mid \vec{0}\,) =$

$$\left(\begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \middle| \, \vec{0} \right) \longrightarrow \left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \vec{0} \right).$$

It follows that $v_2^{(1)} = v_3^{(1)} = \alpha$, $v_1^{(1)} = -\alpha$, where α is an arbitrary number.

Case $\lambda_2 = 2$: $\vec{v}^{\,(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{pmatrix}, \qquad (\,(B - \lambda_2) \mid \vec{0}\,) =$

$$\left(\begin{pmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ -2 & -2 & -1 \end{pmatrix} \middle| \, \vec{0} \right) \longrightarrow \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \vec{0} \right).$$

It follows that $v_3^{(2)} = 0$, $v_2^{(2)} = \beta$, $v_1^{(2)} = -\beta$, where β is an arbitrary number.

Case $\lambda_3 = 3$: $\vec{v}^{\,(3)} = \begin{pmatrix} v_1^{(3)} \\ v_2^{(3)} \\ v_3^{(3)} \end{pmatrix}, \qquad (\,(B - \lambda_3) \mid \vec{0}\,) =$

$$\left(\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ -2 & -2 & -2 \end{pmatrix} \middle| \, \vec{0} \right) \longrightarrow \left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \vec{0} \right).$$

It follows that $v_2^{(3)} = 0$, $v_3^{(3)} = \gamma$, $v_1^{(3)} = -\gamma$, where γ is an arbitrary number.

Summary: The simplest basis of eigenvectors is

$$\vec{v}^{\,(1)} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}^{\,(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}^{\,(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Example 3. Let $M = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

(a) Find all eigenvalues and eigenvectors of M.

SOLUTION: The characteristic equation is

$$0 = \begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda) - 2$$
$$= \lambda^2 - 9\lambda + 18 = (\lambda - 6)(\lambda - 3).$$

So the eigenvalues are 6 and 3.

Let us revert to a simpler, if less precise, notation, in which the eigenvectors are $\vec{v}_j = \begin{pmatrix} x \\ y \end{pmatrix}$.

Case $\lambda = 6$*:* Reduce $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$ to $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and conclude that $x = y$. Thus the eigenvectors are multiples of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Case $\lambda = 3$*:* Reduce $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ and conclude that $x = -\frac{1}{2}y$. Thus the eigenvectors are multiples of $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(b) Find the general solution of the system $\begin{cases} \frac{dx}{dt} = 5x + y, \\ \frac{dy}{dt} = 2x + 4y. \end{cases}$

Solution: Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. The system becomes the vectorial differential equation $\dfrac{d\vec{x}}{dt} = M\vec{x}$.

Method 1: The solution is simply

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{3t},$$

where c_1 and c_2 are arbitrary constants.

Method 2: Start from the diagonalized form of M: $D = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$. The matrix $U = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ maps coordinates from the eigenbasis to the natural basis. Then $Ue^{tD}U^{-1}$ maps the initial values $\vec{x}(0)$ into $\vec{x}(t)$. For the problem as stated, we don't care about initial values, so we don't really need to calculate U^{-1}. We can just say (with $\vec{c} = U^{-1}\vec{x}(0)$, whatever it is)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{6t}c_1 - e^{3t}c_2 \\ e^{6t}c_1 + 2e^{3t}c_2 \end{pmatrix}.$$

This is the same as the result of Method 1, written out in components.

Example 4. Raise the matrix A to the 4th power: $A = \begin{pmatrix} -2 & 1 & 2 \\ -6 & 1 & 3 \\ -4 & 2 & 4 \end{pmatrix}$.

Solution:

Step 1: Find the eigenvalues and eigenvectors of the matrix A. We'll leave out some of the details, since the method is the same as for earlier examples.

$$0 = \begin{vmatrix} -2-\lambda & 1 & 2 \\ -6 & 1-\lambda & 3 \\ -4 & 2 & 4-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda-1)(\lambda-2);$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

$$\lambda_1 = 0 : (\, A - \lambda_1 \mid 0\,) = \left(\begin{array}{ccc|c} -2 & 1 & 2 & \\ -6 & 1 & 3 & 0 \\ -4 & 2 & 4 & \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \end{array}\right);$$

so a suitable eigenvector (to use as a basis element) is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}.$$

(We have multiplied by a constant to get a vector without fractional elements.)

$$\lambda_2 = 1\colon\ (\, A - \lambda_2 \mid 0\,) = \left(\begin{array}{ccc|c} -3 & 1 & 2 & \\ -6 & 0 & 3 & 0 \\ -4 & 2 & 3 & \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \end{array}\right);$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.$$

$$\lambda_3 = 2\colon\ (\, A - \lambda_3 \mid 0\,) = \left(\begin{array}{ccc|c} -4 & 1 & 2 & \\ -6 & -1 & 3 & 0 \\ -4 & 2 & 2 & \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right);$$

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Step 2: Find the change-of-basis matrix U from the eigenbasis to the original basis, and find its inverse. Set up U from the column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$U = \begin{pmatrix} 1 & -1 & 1 \\ -6 & 1 & 0 \\ 4 & -2 & 2 \end{pmatrix}.$$

Then

$$(\, U \mid I\,) = \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -6 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -6 & 1 & 3 \\ 0 & 0 & 1 & -4 & 1 & \frac{5}{2} \end{array}\right).$$

Therefore,

$$U^{-1} = \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ -6 & 1 & 3 \\ -4 & 1 & \frac{5}{2} \end{pmatrix}.$$

Step 3: Construct the matrices D and D^4:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

$$D^4 = \begin{pmatrix} \lambda_1^4 & 0 & 0 \\ 0 & \lambda_2^4 & 0 \\ 0 & 0 & \lambda_3^4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

Step 4: Calculate $A^4 = UD^4U^{-1}$:

$$A^4 = \begin{pmatrix} 1 & -1 & 1 \\ -6 & 1 & 0 \\ 4 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ -6 & 1 & 3 \\ -4 & 1 & \frac{5}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 & 16 \\ 0 & 1 & 0 \\ 0 & -2 & 32 \end{pmatrix} \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ -6 & 1 & 3 \\ -4 & 1 & \frac{5}{2} \end{pmatrix} = \begin{pmatrix} -58 & 15 & 37 \\ -6 & 1 & 3 \\ 116 & 30 & 74 \end{pmatrix}.$$

Example 5. Calculate the given functions $f(A)$ of the matrix

$$A = \begin{pmatrix} -2 & 8 & 6 \\ -4 & 10 & 6 \\ 4 & -8 & -4 \end{pmatrix}.$$

(a) $f(A) = e^{tA}$, where t is a real parameter.

(b) $f(A) = \sqrt{A}$ (checking that $A \geq 0$ so that this is defined).

(c) f an arbitrary function.

Solution: Start by finding the eigenvalues and eigenvectors of the matrix A. Then, as in the previous example, build the matrix U, find U^{-1}, and use the formula $f(A) = Uf(D)U^{-1}$, where

$$D = U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

We have

$$0 = \det(A - \lambda) = \begin{vmatrix} -2-\lambda & 8 & 6 \\ -4 & 10-\lambda & 6 \\ 4 & -8 & -4-\lambda \end{vmatrix}$$
$$= (2-\lambda) \begin{vmatrix} -2-\lambda & 2 & 6 \\ -4 & 4-\lambda & 6 \\ 0 & 0 & 1 \end{vmatrix} = (2-\lambda)(\lambda^2 - 2\lambda - 8 + 8)$$
$$= (2-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda - 2)^2.$$

So $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = 2$.

For $\lambda_1 = 0$ we find, in the usual way, that the eigenvectors are precisely the scalar multiples of

$$\vec{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

The calculation for the eigenvalue $\lambda_2 = 2$ is more unusual, since there are two linearly independent eigenvectors:

$$(\, A - \lambda_2 \mid 0\,) = \left(\begin{array}{ccc|c} -4 & 8 & 6 & \\ -4 & 8 & 6 & 0 \\ 4 & -8 & -6 & \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -4 & -3 & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right).$$

Two of the coordinates are arbitrary parameters. To construct a basis, we find one solution with $z = 0$ and y nonzero, and one with the reverse conditions:

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

From the eigenvectors we find

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Now to cases:

(a) $$f(D) = e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix};$$

$$\begin{aligned} e^{tA} &= f(A) = Uf(D)U^{-1} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2e^{2t} & 3e^{2t} \\ 1 & e^{2t} & 0 \\ -1 & 0 & 2e^{2t} \end{pmatrix} \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 - e^{2t} & -4 + 4e^{2t} & -3 + 3e^{2t} \\ 2 - 2e^{2t} & -4 + 5e^{2t} & -3 + 3e^{2t} \\ -2 + 2e^{2t} & 4 - 4e^{2t} & 3 - 2e^{2t} \end{pmatrix}. \end{aligned}$$

(b) Since the eigenvalues are nonnegative, $\sqrt{D}$ is defined. Thus

$$f(D) = \sqrt{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix};$$

$$\begin{aligned}
\sqrt{A} &= f(A) = Uf(D)U^{-1} \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -2\sqrt{2} & 4\sqrt{2} & 3\sqrt{2} \\ -2\sqrt{2} & 5\sqrt{2} & 3\sqrt{2} \\ 2\sqrt{2} & -4\sqrt{2} & -2\sqrt{2} \end{pmatrix}.
\end{aligned}$$

(c) $f(A) = Uf(D)U^{-1} = U \begin{pmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & 0 \\ 0 & 0 & f(\lambda_3) \end{pmatrix} U^{-1}$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} f(0) & 0 & 0 \\ 0 & f(2) & 0 \\ 0 & 0 & f(2) \end{pmatrix} \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} f(0) & 2f(2) & 3f(2) \\ f(0) & f(2) & 0 \\ -f(0) & 0 & 2f(2) \end{pmatrix} \begin{pmatrix} 2 & -4 & -3 \\ -2 & 5 & 3 \\ 1 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2f(0) - f(2) & -4f(0) + 4f(2) & -3f(0) + 3f(2) \\ 2f(0) - 2f(2) & -4f(0) + 5f(2) & -3f(0) + 3f(2) \\ -2f(0) + 2f(2) & 4f(0) - 4f(2) & 3f(0) - 2f(2) \end{pmatrix}.
\end{aligned}$$

Example 6. Solve the linear homogeneous system of ordinary differential equations (ODEs)

$$\frac{dx}{dt} = 2x - 2y, \quad \frac{dy}{dt} = x - z, \quad \frac{dz}{dt} = -y + 2z.$$

SOLUTION: Construct the matrix A of coefficients of the system and find its eigenvalues:

$$A = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix};$$

$$\det(A-\lambda) = \begin{vmatrix} 2-\lambda & -2 & 0 \\ 1 & -\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (-\lambda)(2-\lambda)^2 - (2-\lambda) + 2(2-\lambda)$$
$$= -\lambda(\lambda^2 - 4\lambda + 4) + 2 - \lambda = -\lambda^3 + 4\lambda^2 - 5\lambda + 2.$$

Solve the equation $\det(A-\lambda) = 0$:

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda-2)(\lambda-1)^2 = 0.$$

It follows from this that $\lambda_1 = 2$ and $\lambda_2 = 1$, and the eigenspace for λ_2 may or may not have dimension 2.

For the root $\lambda_1 = 2$, reduce in the usual way

$$\begin{pmatrix} 0 & -2 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the root $\lambda_1 = 2$ is simple (nondegenerate), as expected, and the eigenvectors have $y = 0$, $y = z$. The corresponding solutions of the ODE system are

$$x(t) = C_1 e^{2t}, \quad y(t) = 0, \quad z(t) = C_1 e^{2t},$$

where C_1 is an arbitrary constant. In vectorial notation, we have a basis solution

$$\vec{r}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

For the multiple (degenerate) root $\lambda_2 = 1$, reduce

$$\begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the only eigenvectors are those with $x = 2z$, $y = z$, and the corresponding solutions are

$$x(t) = 2C_2 e^t, \quad y(t) = C_2 e^t, \quad z(t) = C_2 e^t,$$

where C_2 is an arbitrary constant. In vector form the second basis element is

$$\vec{r}_2(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^t.$$

We now have a problem, since to accommodate all initial conditions we need a third type of solution, but there is not a third independent eigenvector. One way of finding the third solution is to use the theory of the Jordan canonical form, but that has been ruled out of the scope of this book. So we will just use a well known trick for this sort of problem (which is actually a Jordan calculation in disguise): Hunt for solutions of the form

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} e^t + t\,\vec{r}_2(t).$$

That is, substitute

$$x(t) = \alpha e^t + 2te^t, \quad y(t) = \beta e^t + te^t, \quad z(t) = \gamma e^t + te^t$$

into the differential equations, getting (after division by the common factor e^t)

$$\begin{aligned} \alpha + 2(1+t) &= 2\alpha + 4t - 2\beta - 2t, \\ \beta + (1+t) &= \alpha + 2t - \gamma - t, \\ \gamma + (1+t) &= -\beta - t + 2\gamma + 2t. \end{aligned}$$

The terms involving t completely cancel (because $\vec{r}_2$ is an eigensolution), leaving a nonhomogeneous system for (α, β, γ) with augmented matrix

$$\begin{pmatrix} 1 & -2 & 0 & 2 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

whose solution is $\beta = -1$, $\alpha = \gamma = 0$. Therefore, the third basis solution is

$$\vec{r}_3(t) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^t + t\,\vec{r}_2(t).$$

In component form, the third type of solution is

$$x(t) = 2C_3te^t, \quad y(t) = C_3(t-1)e^t, \quad z(t) = C_3te^t.$$

The general solution is a sum of the three kinds of solutions we have found, with arbitrary constants C_j; in other words, an arbitrary linear combination of the three basis solutions $\vec{r}_j(t)$.

Exercises

8.1.1 Let $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

(a) Find all the eigenvalues and eigenvectors of A.

(b) Find the coordinate transformation, U, that converts A to a diagonal matrix, D (which you should also exhibit). Be precise about how the three matrices A, D, and U are related!

8.1.2

(a) Find all eigenvalues and eigenvectors of $C = \begin{pmatrix} 5 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$.

(b) What is the rank of C? What is the relevance of this question to the answer to (a)?

8.1.3 Find all eigenvalues and eigenvectors of $M = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 5 \\ 0 & 0 & 4 \end{pmatrix}$, and use the result to define and calculate the matrix called $\sqrt{M}$.

8.1.4 Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}$.

8.1.5 The matrix $M = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, with eigenvectors $\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, respectively.

(a) Find the coordinate transformation, U, that converts M to a diagonal matrix, D.

(b) Find the matrix e^{tM} (for an arbitrary real number t) and use it to solve the differential equation

$$\frac{d\vec{x}}{dt} = M\vec{x} \qquad \text{with initial data} \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

8.1.6

(a) Find all the eigenvectors of $M \equiv \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

(b) Find a matrix U that diagonalizes M.

(c) Use the foregoing to define a matrix that deserves to be called $\sqrt{M+5}$.

8.1.7 Find a basis for $\mathbf{R}^3$ of eigenvectors of $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

8.1.8 Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the eigenvalues of M.

(b) Without solving for the eigenvectors, explain why you can be confident that $\mathbf{R}^3$ has a basis consisting of eigenvectors of B.

8.1.9 Let $M = \begin{pmatrix} 2 & 2 \\ 8 & 2 \end{pmatrix}$.

(a) Find all the eigenvalues of M and the corresponding eigenvectors.

(b) Solve the differential equation

$$\frac{d\vec{x}}{dt} = M\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

8.1.10 Let $A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$.

(a) Show that the eigenvalues of A are 4, 2, and -3.

(b) Find all the eigenvectors of A.

(c) Show how to diagonalize A. (Find the matrices D, U, and U^{-1}, in the notation of this section.)

8.1.11 The matrix $M = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, with eigenvectors $\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, respectively.

(a) Find the coordinate transformation, U, that converts M to a diagonal matrix, D.

(b) Use (a) to calculate the matrix M^8 efficiently.

8.1.12 Let A be an $n \times n$ matrix whose eigenvalues are distinct and nonzero. Prove that A has exactly 2^n different diagonalizable square roots (that is, matrices B such that $B^2 = A$; complex entries are allowed). Do this by considering the action of B^2 on the elements of an eigenbasis in $\mathbf{C}^n$ for B. (You may ignore the possibility of existence of nondiagonalizable square roots.)

8.1.13 Assume that A is a diagonalizable matrix (independent of t) and that all its eigenvalues are positive. Show that the solution of the second-order homogeneous linear differential equation

$$\frac{d^2\vec{x}}{dt^2} = -A\vec{x}\,; \qquad \vec{x}(0) = \vec{x}_0\,, \quad \vec{x}\,'(0) \equiv \frac{d\vec{x}}{dt}\bigg|_{t=0} = \vec{v}_0\,,$$

can be written in terms of trigonometric functions of the matrix $t\sqrt{A}$, acting on the vectors $\vec{x}_0$ and $\vec{v}_0$.

8.1.14 What are the eigenvalues and eigenvectors of the operator $L = d/dt$ acting in the vector space (with complex scalars)

$$\mathcal{V} = \operatorname{span}\{\cos t, \sin t\}\,?$$

In the remaining exercises, find the eigenvalues and eigenvectors of the matrix. Remark upon any case where an eigenbasis of real eigenvectors does not exist.

8.1.15 $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$

8.1.16 $\begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}$

8.1.17 $\begin{pmatrix} 9 & -6 & -2 \\ 18 & -12 & -3 \\ 18 & -9 & -6 \end{pmatrix}$

8.1.18 $\begin{pmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 1 & -2 & -2 \end{pmatrix}$

8.1.19 $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$

8.1.20 $\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$

8.1.21 $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$

8.1.22 $\begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix}$

8.1.23 $\begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$

8.1.24 $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \alpha & 0 & \alpha \end{pmatrix} \quad (\alpha \neq 0)$

8.2 Diagonalization of Real, Symmetric Matrices

A (real) symmetric matrix always has an eigenbasis; moreover, its eigenvectors are orthogonal. This important theorem is the subject of this section. We'll build up to it with several little theorems.

Theorem 1: If A is a real $p \times n$ matrix, then

$$(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^{\mathrm{t}}\vec{y})$$

for all $\vec{x}$ in $\mathbf{R}^n$ and all $\vec{y}$ in $\mathbf{R}^p$. ($A^{\mathrm{t}} \equiv$ transpose of A.)

PROOF: Both sides of the equation equal $\sum_{j=1}^{p}\sum_{k=1}^{n} y_j A_{jk} x_k$.

REMARK: The "complexification" of this theorem is: If A is any $p \times n$ matrix, then $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A^*\vec{y})$ for all $\vec{x}$ in $\mathbf{C}^n$ and all $\vec{y}$ in $\mathbf{C}^p$, where A^* is the *adjoint* matrix (complex conjugate of the transpose): $(A^*)_{jk} \equiv \overline{A_{kj}}$.

Definition and Theorem 2: A real, square matrix U is *orthogonal* if any of the following equivalent conditions holds. (Any one of the conditions implies all the others.)

(a) $U^{-1} = U^{\mathrm{t}}$.

(b) The columns of U are orthonormal (hence form an orthonormal basis for $\mathbf{R}^n$).

(c) The rows of U are orthonormal.

(d) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}$ and $\vec{y}$ in $\mathbf{R}^n$.

(e) The linear function from $\mathbf{R}^n$ onto $\mathbf{R}^n$ represented by U maps the natural basis onto another *orthonormal* basis. (Geometrically, such a transformation of $\mathbf{R}^n$ is a *rotation* if $\det U = 1$, and a rotation composed with a *reflection* if $\det U = -1$. It turns out that these are the only possible values of $\det U$.)

PROOF: See Exercise 8.2.14.

Theorem 3: All eigenvalues of a real, symmetric matrix are real, and its eigenvectors corresponding to distinct eigenvalues are orthogonal.

PROOF: Suppose $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = \mu\vec{y}$, where the coordinates of $\vec{x}$ and $\vec{y}$ may be complex, for the moment. Then, using the complex dot product $\vec{x} \cdot \vec{y} \equiv \sum_j x_j \overline{y_j}$, we have $(A\vec{x}) \cdot \vec{y} = \lambda(\vec{x} \cdot \vec{y})$ and $\vec{x} \cdot (A\vec{y}) = \overline{\mu}(\vec{x} \cdot \vec{y})$. But, by Theorem 1 and the assumed symmetry of A, we have $\vec{x} \cdot (A\vec{y}) = (A^{\mathrm{t}}\vec{x}) \cdot \vec{y} = (A\vec{x}) \cdot \vec{y}$. Therefore,

$$(\lambda - \overline{\mu})(\vec{x} \cdot \vec{y}) = 0. \qquad (*)$$

If $\vec{x} = \vec{y} \neq \vec{0}$ (hence $\lambda = \mu$), $(*)$ says that $\mu - \overline{\mu} = 0$. That is, μ is real. When we return to the general case, $(*)$ now becomes

$$(\lambda - \mu)(\vec{x} \cdot \vec{y}) = 0.$$

So if $\lambda \neq \mu$, necessarily $\vec{x} \cdot \vec{y} = 0$, as claimed.

Now we are ready to consider the problem of finding all the eigenvalues and eigenvectors of an $n \times n$ real, symmetric matrix, A. The characteristic equation for this problem can (in principle) be factored into

$$0 = (-1)^n \, (\lambda - \lambda_1)^{p_1} (\lambda - \lambda_2)^{p_2} \, \cdots \, (\lambda - \lambda_r)^{p_r},$$

where $\sum_{j=1}^{r} p_j = n$. We know that every root of this equation corresponds to at least one eigenvector, and every *real* root corresponds to at least one *real* eigenvector, which may be chosen to have unit length (*normalized*). Putting these two facts together with the first half of Theorem 3, we see that all the roots are real and are eigenvalues of A as a linear operator on $\mathbf{R}^n$ (not just as an operator on $\mathbf{C}^n$). Furthermore, if the roots are distinct (each $p_j = 1$), we know from Sec. 8.1 and Theorem 3 that the normalized eigenvectors form an orthonormal basis for $\mathbf{R}^n$.

Even if some of the p_j are greater than 1, it can be shown (see below) that the dimension of the subspace of $\mathbf{R}^n$ consisting of eigenvectors with eigenvalue λ_j actually does have dimension p_j (rather than something smaller). Thus A does possess an eigenvector basis. The basis within each eigensubspace can be chosen to be orthonormal (by the Gram–Schmidt process), and the eigenvectors corresponding to different λ_j's are automatically orthogonal by Theorem 3. So:

Theorem 4: If A is an $n \times n$ real symmetric matrix, there is an orthonormal basis for $\mathbf{R}^n$ consisting of eigenvectors of A. In other words, A can be diagonalized by an *orthogonal* matrix U:

$$A = UDU^{-1}, \qquad D \text{ diagonal and real.}$$

(The columns of U are the elements of the orthonormal eigenvector basis.)

PROOF: All that is left to prove is the claim that a full set of eigenvectors exists even when some of the eigenvalues are multiple. Let $\mathcal{W}$ be the subspace of $\mathbf{R}^n$ spanned by the eigenvectors of A. Assume (in order to get a contradiction) that $\mathcal{W}$ is not all of $\mathbf{R}^n$. Let $\mathcal{U} \equiv \mathcal{W}^\perp$ be the subspace of vectors orthogonal to $\mathcal{W}$. By the projection theorem (Sec. 6.2), $\mathcal{W}$ and $\mathcal{U}$ together span the whole space $\mathbf{R}^n$. Since $A = A^{\mathrm{t}}$, Theorem 1 shows that

$$(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y}) = 0$$

for all $\vec{x} \in \mathcal{U}$ and $\vec{y} \in \mathcal{W}$ (hence $A\vec{y} \in \mathcal{W}$). This shows that A maps $\mathcal{U}$ into itself. We know that every linear operator from a nontrivial finite-dimensional space into itself has at least one eigenvector, so A must have an eigenvector in $\mathcal{U}$. This is a contradiction, since $\mathcal{W}$ was supposed to contain all the eigenvectors, and a nonzero vector in $\mathcal{W}$ can't also be in $\mathcal{W}^\perp$. The only possible explanation is that $\mathcal{U} = \{\vec{0}\}$; that is, $\mathcal{W} = \mathbf{R}^n$, as we wanted to prove.

Remarks and Warnings: Let's get clear when and how the eigenvectors are orthogonal:

1. If A is symmetric, the eigenvectors corresponding to *different* eigenvalues are *automatically* orthogonal.
2. Any basis may be used for the subspace of eigenvectors corresponding to *one particular* eigenvalue. This basis may be *chosen* to be orthogonal. If you construct a basis by parametrizing in the usual way

the solutions of the row-reduced version of the eigenvector equations, the result will *not* automatically be orthogonal; in general, you must perform a Gram–Schmidt calculation to replace it by an orthogonal basis.

3. If A is *not symmetric*, the eigenvectors corresponding to different eigenvalues are usually *not* orthogonal. Applying Gram–Schmidt to vectors belonging to *different* eigenvalues is a *mistake*; the resulting vectors will no longer be eigenvectors!

As a corollary to Theorem 4 we see:

Theorem 5: If A is a real symmetric matrix and $L\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear operator represented by A, then:

(1) Every vector in the kernel of L is orthogonal to every vector in the range of L.

(2) Every vector in $\mathbf{R}^n$ can be written (uniquely) as the sum of a vector in the kernel and a vector in the range. That is, $\mathbf{R}^n = \ker L \oplus \operatorname{ran} L$.

(The restriction to $\mathbf{R}^n$ is not essential. The theorem holds for any vector space with an inner product, as long as the linear operator in question is represented with respect to an ON basis by a real symmetric matrix.)

Proof: The kernel is the span of the eigenvectors (if any) with eigenvalue 0. The range is the span of the eigenvectors with nonzero eigenvalues — i.e., the vectors represented by the nonzero columns of the diagonalized matrix, which looks something like this:

$$D = U^{-1}AU = \begin{array}{c} \text{kernel} \\ \hline \text{range} \end{array} \overset{\begin{array}{cc|ccc} \text{kernel} & & & \text{range} & \end{array}}{\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right)}$$

These two subspaces are orthogonal (claim 1), and they span the whole space (claim 2). The uniqueness statement in claim 2 is true because the two types of eigenvectors are independent.

Remark: Theorem 5 tells us that (for real, symmetric A) the set of vectors $\vec{y}$ in $\mathbf{R}^n$ for which $A\vec{x} = \vec{y}$ is solvable is precisely those vectors which are orthogonal to the solutions $\vec{z}$ of the homogeneous system $A\vec{z} = \vec{0}$. This makes our previous discussion (in Sec. 5.4) of the solvability of homogeneous and nonhomogeneous systems simpler and more explicit. The corresponding theorem for a real, possibly nonsymmetric or even nonsquare, A is:

$$A\vec{x} = \vec{y} \textit{ is solvable if and only if } \vec{y}\cdot\vec{z} = 0 \textit{ for all solutions of } A^{\mathrm{t}}\vec{z} = \vec{0}. \qquad (\dagger)$$

(For complex A, replace the transpose by the adjoint.) This theorem is actually fairly easy to prove without any reference to eigenvalues or diagonalization (see Theorem 18.3 of R. M. Bowen and C.-C. Wang, *Introduction to Vectors and Tensors*, Vol. 1 (Plenum, 1976).) The theorem is the finite-dimensional version of a general principle called the **Fredholm alternative**.

Quadratic forms

Symmetric matrices play another role in mathematics besides representing certain linear functions. They appear as the coefficients in the second-order terms of Taylor series for functions of several variables. Their diagonalization leads directly to the extension of the *second-derivative test for maxima and minima* to such functions.

Definition: A *quadratic form* is a function of n variables of the type

$$Q = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk}x_j x_k\,.$$

If we regard the xs as elements of a column vector in $\mathbf{R}^n$, this can be written as

$$Q(\vec{x}) = \vec{x}^{\,\mathrm{t}}A\vec{x} \qquad \text{or} \qquad Q(\vec{x}) = \vec{x}\cdot(A\vec{x}).$$

Without loss of generality we may assume that the matrix A is symmetric, since the function Q is unchanged if we replace A_{jk} by $\frac{1}{2}(A_{jk} + A_{kj})$.

Example: $Q(\vec{x}) = 2x_1^2 - x_2^2 + 2x_1x_2$ has the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$. Note that each off-diagonal element of A equals *half* the coefficient of the corresponding "cross term" in Q.

Consider the linear change of variables in $\mathbf{R}^n$ described by the equation $\vec{x} = U\vec{y}$. One has $Q = (U\vec{y})^{\mathrm{t}}AU\vec{y} = \vec{y}^{\mathrm{t}}\,U^{\mathrm{t}}AU\vec{y}$. That is, the matrix representing a quadratic form transforms under change of basis to $U^{\mathrm{t}}AU$, *not* to $U^{-1}AU$ as the matrix representing a linear function does.

However, if U is *orthogonal*, then $U^{-1} = U^{\mathrm{t}}$, so there's no difference after all in that case. Hence our knowledge of diagonalization of symmetric matrices can be applied, yielding —

Theorem Q: Given a quadratic form $Q(\vec{x}) = \vec{x}^{\,\mathrm{t}} A\vec{x}$, one can introduce new coordinates, $\vec{y} \equiv U^{-1}\vec{x}$, with respect to which it takes the diagonal form

$$Q = \tilde{Q}(\vec{y}) \equiv Q(U\vec{y}) = \sum_{j=1}^{n} \lambda_j y_j^2 .$$

The λ_j are the eigenvalues of A.

EXAMPLE 1. Let $Q(x, y) = x^2 + xy + y^2$ and consider the linear change of variables

$$x = u - v, \quad y = u + v.$$

We obtain the new quadratic form (or, rather, the *same* function Q, written in terms of new variables)

$$\tilde{Q}(u, v) = (u - v)^2 + (u - v)(u + v) + (u + v)^2 = 3u^2 + v^2 .$$

If we do the inverse transformation

$$u = \frac{x + y}{2}, \quad v = \frac{y - x}{2}$$

on $\tilde{Q}$, we get back the original quadratic form:

$$3\left(\frac{x + y}{2}\right)^2 + \left(\frac{y - x}{2}\right)^2 = x^2 + xy + y^2 .$$

EXAMPLE 2. Consider the quadratic form

$$2{x_1}^2 + 5{x_2}^2 + 7{x_3}^2 - 6x_1x_2 + 2x_1x_3 - 8x_2x_3 \,.$$

In the usual way, one finds that

$$U = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{diagonalizes} \quad A = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & -4 \\ 1 & -4 & 7 \end{pmatrix} .$$

Therefore, the diagonalizing transformation of variables, $\vec{x} = U\vec{y}$, is

$$\begin{aligned} x_1 &= y_1 + y_2 + 2y_3 \,, \\ x_2 &= y_1 + 2y_2 + 2y_3 \,, \\ x_3 &= y_1 + y_2 + y_3 \,. \end{aligned}$$

The matrix of the quadratic form with respect to the new coordinates is

$$\tilde{A} = U^{\mathrm{t}} A U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & -4 \\ 1 & -4 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 4 \\ -3 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The new quadratic form itself is

$$2{y_1}^2 + 3{y_2}^2 - {y_3}^2\,.$$

Corollary to Theorem Q: $Q(\vec{x})$ is positive for all nonzero $\vec{x}$ if and only if all the λ_j are positive. (Likewise with "positive" replaced in both places by "negative", "nonnegative", or "nonpositive".)

The most important feature of a quadratic form is the *signs* of its eigenvalues. The numbers of positive, negative, and zero eigenvalues of the matrix representing Q can be shown to be invariant even under nonorthogonal coordinate changes.* In dimension 2, these numbers determine whether the curve with equation

$$Q(\vec{x}) = \vec{v}\cdot\vec{x} - F$$

is an ellipse, hyperbola, or parabola. Indeed, you have probably already been taught how to diagonalize 2×2 quadratic forms under the heading of "conic sections". Given an equation

$$Ax^2 + Bxy + Cy^2 = -Dx - Ey - F, \qquad (\ddagger)$$

one can tell which kind of conic section it represents by knowing that the quantities

$$d \equiv -\frac{1}{4}(B^2 - 4AC) \qquad \text{and} \qquad t \equiv A + C$$

* Note that the *magnitudes* $|\lambda_j|$ are certainly not invariant, since the nonorthogonal coordinate transformation $z_j \equiv \sqrt{|\lambda_j|}\,y_j$ (for every j such that $\lambda_j \neq 0$) converts each eigenvalue to either 1, -1, or 0. Thus every quadratic form can be reduced to a standard diagonal form whose nonvanishing coefficients have absolute value 1.

are "invariant under rotations of axes" — that is, under coordinate transformations

$$\begin{pmatrix} x \\ y \end{pmatrix} = U \begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad U \text{ orthogonal.}$$

There is some coordinate system in which the equation has no cross term:

$$A'(x')^2 + C'(y')^2 = -D'x' - E'y' - F.$$

(This is guaranteed by Theorem Q.) One can find A' and C' by solving the equations

$$d = A'C', \qquad t = A' + C'.$$

The signs of A' and C' determine the type of curve; those signs in turn can be found from the signs of d and t:

- $d > 0 \;\Rightarrow\; A'$ and C' have the same sign $\;\Rightarrow\;$ ellipse. (In this case the sign of t determines the overall sign of Q. This is important in testing for a maximum or minimum of a function of two variables.)
- $d < 0 \;\Rightarrow\; A'$ and C' have opposite signs $\;\Rightarrow\;$ hyperbola.
- $d = 0,\; t \neq 0 \;\Rightarrow\;$ either A' or $C' = 0 \;\Rightarrow\;$ parabola.
- $d = 0,\; t = 0 \;\Rightarrow\; A' = 0 = C' \;\Rightarrow\;$ straight line.

(Here we are not distinguishing the "degenerate cases", such as the hyperbola's collapsing to two intersecting lines, which occur for certain values of the coefficients of the linear and constant terms ($\vec{v} \equiv (D, E)$ and F).)

Why does this analysis work? Obviously, A' and C' are the eigenvalues of the matrix of the quadratic form $Q \equiv Ax^2 + Bxy + Cy^2$, but what are d and t? The answer lies in some general theorems:

Definition: The *trace* of a square matrix is the sum of its diagonal elements:

$$\operatorname{tr} A = \sum_{j=1}^{n} A_{jj}\,.$$

Theorem T: $\operatorname{tr} AB = \operatorname{tr} BA$ for any matrices A and B (square and of the same size).

PROOF: $\operatorname{tr} AB = \sum_j \sum_k A_{jk} B_{kj} = \sum_k \sum_j B_{kj} A_{jk} = \operatorname{tr} BA.$

Theorem D: The determinant and trace of a matrix are invariant under similarity transformations:

$$\det\, A = \det\,(UAU^{-1}), \qquad \operatorname{tr} A = \operatorname{tr}\,(UAU^{-1}).$$

(U is any invertible matrix of the same size as A — not necessarily orthogonal.)

PROOF:

$$\begin{aligned}\det\,(UAU^{-1}) &= (\det\,U)(\det\,A)(\det\,U^{-1})\\ &= (\det\,U)(\det\,A)(\det\,U)^{-1} = \det\,A.\end{aligned}$$

By Theorem T, $\operatorname{tr}\,(UAU^{-1}) = \operatorname{tr}\,(U^{-1}UA) = \operatorname{tr}\,A$.

Corollary: If A is diagonalizable, then $\det A$ equals the product of the eigenvalues, and $\operatorname{tr} A$ equals the sum of the eigenvalues.

The matrix of the quadratic form on the left side of equation (‡) is

$$\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix},$$

which has d as determinant and t as trace. So it's clear from Theorem D and its corollary why d and t are fundamental numbers describing the shape of the conic section, which are invariant under rotations of axes (orthogonal changes of coordinates). Another way of seeing the special significance of d and t is to write out the characteristic polynomial for our matrix:

$$\begin{vmatrix} A-\lambda & B/2 \\ B/2 & C-\lambda \end{vmatrix} = \lambda^2 - (A+C)\lambda + AC - \frac{1}{4}B^2 = \lambda^2 - t\lambda + d.$$

The rightmost member of this equation is a correct formula for the characteristic polynomial of *any* 2×2 matrix in terms of its determinant and trace.

This last observation gives a clue for analyzing quadratic forms in n variables. The coefficients of the characteristic polynomial of a matrix are invariant under similarity transformations (Exercise 8.2.15). They are called "the invariants" of the matrix or quadratic form. For an $n \times n$ matrix there are n such coefficients (not counting the coefficient of λ^n, which is always $(-1)^n$). For example, for a 3×3 matrix one finds

$$\det\,(A-\lambda) = -\lambda^3 + (\operatorname{tr}\,A)\lambda^2 - c\lambda + \det\,A,$$

where

$$c \equiv \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$

is the sum of the cofactors of the diagonal elements of A. The coefficients in this polynomial must be equal to the corresponding quantities for the diagonalized matrix $D = U^{-1}AU$:

$$t = \operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3 \,,$$
$$d = \det\ A = \lambda_1\lambda_2\lambda_3 \,,$$
$$c = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 \,.$$

In principle one could solve these equations for the eigenvalues, given t, d, and c; but this is not a practical way of solving the eigenvalue problem. In fact, even determining the *signs* of the eigenvalues from a knowledge of the invariants is much harder here than in dimension 2.

The most important question is whether all the eigenvalues have the same sign. Fortunately, there is a theorem which enables us to answer that question by straightforward calculation.

Theorem P: If A is a real, symmetric matrix, all its eigenvalues are positive if and only if all the upper left-hand corner minor determinants

$$A_{11}, \qquad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}, \qquad \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}, \qquad \ldots,$$

are positive. All the eigenvalues are negative if and only if these minors alternate in sign, starting with "$-$".

The proof, a massive exercise in completing the square, is omitted; see, for instance, J. Heading, *Matrix Theory for Physicists* (Wiley, 1958), Sec. 2.6. (The proof of the second-derivative test for two variables in Sec. 12.7 of J. Stewart, *Calculus*, 3rd ed. (Brooks–Cole, 1995) is essentially a special case of this proof of Theorem P.) The negative case follows from the positive case immediately by changing the sign of the matrix. (Note, incidentally, that the theorem is obviously true if the matrix is already diagonal; this may help one to remember the theorem. Since the minors are not invariant under similarity transformations, the validity of the theorem in the general case is far from obvious.)

Application of the application: **The "second-derivative test" for functions of many variables.** Suppose that all first-order partial derivatives of $w = f(x_1, \ldots, x_n) \equiv f(\vec{x})$ vanish at some point $\vec{x}_0$. Then that point is a candidate for the location of a maximum or minimum. For notational ease assume $\vec{x}_0 = \vec{0}$ and $f(\vec{x}_0) = 0$. A smooth function of several variables can be expanded near a point in a Taylor series in all the variables simultaneously.† The Taylor series of f about $\vec{x}_0$ is

$$w = \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 f(\vec{x}_0)}{\partial x_j \partial x_k} x_j x_k + \text{higher-order terms.}$$

The leading term is a quadratic form, $Q(\vec{x})$, whose matrix A is built out of the second derivatives of f at $\vec{x}_0$. (Q or A is called the *Hessian* of f. Notice, incidentally, that its trace is the Laplacian of f.) Its eigenvalues determine the behavior of w near $\vec{x}_0$:

λ *signs*	*behavior*
all positive	minimum
all negative	maximum
mixed	saddle point
some zero, others all one sign	more information needed

The signs can be determined in principle by solving the eigenvalue problem. Or, with luck, they can be deduced from the invariants, as we have seen for dimension 2, or from Theorem P.

Warning: What is important is the signs of the *eigenvalues*, not of the individual partial derivatives in the matrix

$$A_{jk} \equiv \frac{\partial^2 f(\vec{x}_0)}{\partial x_j \partial x_k} \,.$$

For example, by looking at the determinants we see that

$$A = \begin{pmatrix} 1 & 40 \\ 40 & 2 \end{pmatrix}$$

has a negative eigenvalue (so it doesn't mark a minimum), while

$$A = \begin{pmatrix} 100 & -1 \\ -1 & 40 \end{pmatrix}$$

† Unfortunately, most calculus textbooks don't teach this fact, except possibly for the case of only two variables. But see L. H. Loomis and S. Sternberg, *Advanced Calculus* (Addison–Wesley, 1968), Secs. 3.16–17; G. B. Folland, Remainder estimates in Taylor's theorem, *Amer. Math. Monthly* **97**, 233–235 (1990); J. A. Facenda Aguirre, A note on Taylor's theorem, *Amer. Math. Monthly* **96**, 244–247 (1990).

has only positive eigenvalues (so it would indicate a minimum). On the other hand, by the corollary to Theorem Q, if one of the *diagonal* elements of A is negative, then the eigenvalues can't be all positive, because $A_{jj} = Q(\hat{e}_j)$.

Example. Investigate the function

$$u(x, y, z) = x^3 + y^3 + z^2 - 3xy$$

for extrema.

SOLUTION: Examine the gradient to find the critical points:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y = 0 \iff y = x^2.$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3x = 0 \iff x = y^2.$$

$$\frac{\partial u}{\partial z} = 2z = 0 \iff z = 0.$$

There are two such points: $(1, 1, 0)$ and $(-1, -1, 0)$.

The Hessian matrix is

$$\begin{pmatrix} 6x & -3 & 0 \\ -3 & 6y & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

At $(1, 1, 0)$ this is

$$\begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Because $\begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0$, all the signs in Theorem P are positive and this point marks a minimum. At $(-1, -1, 0)$ we have

$$\begin{pmatrix} -6 & -3 & 0 \\ -3 & -6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Without even calculating the 2×2 minor, we can see that this is a saddle point, because the first minor, A_{11}, is negative, but the third eigenvalue is obviously positive.

In hindsight, the fact that there is no local maximum is obvious from the behavior of u in the z variable.

These conclusions can be checked by calculating the eigenvalues. This is effectively a 2×2 problem, since the third eigenvalue, 2, is already known. For the case $(1,1,0)$, the matrix $\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ has determinant 27 and trace 12, both positive, so the two nontrivial eigenvalues are positive. For $(-1,-1,0)$, the matrix $\begin{pmatrix} -6 & -3 \\ -3 & -6 \end{pmatrix}$ has determinant -45, negative, so the two eigenvalues are of opposite sign.

Exercises

8.2.1 The only eigenvalues of the matrix $B = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = -3$. Construct an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors of B.

8.2.2 Let $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$. Find an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors of A.

8.2.3 Consider the matrix $N = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$.

(a) Does there exist an orthonormal basis consisting of eigenvectors of N? Explain.

(b) Suppose that a function f satisfies $\nabla f(\vec{x}_0) = 0$ and the matrix of second-order partial derivatives of f at $\vec{x}_0$ equals N. Does $\vec{x}_0$ mark a maximum of f, a minimum, or neither? Explain.

8.2.4 Find the eigenvalues and eigenvectors of $M = \begin{pmatrix} 1 & b & b \\ b & 1 & b \\ b & b & 1 \end{pmatrix}$, where b is a parameter in $\mathbf{R}$.

8.2.5 Find an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors of

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

8.2.6 The real-valued function $y = g(x_1, x_2)$ has a critical point (i.e., both first-order partial derivatives are zero) at $x_1 = 0$, $x_2 = -5$. Let M be the matrix of its second derivatives, evaluated at that point $(M_{jk} = \partial^2 y / \partial x_j \, \partial x_k)$. Is the point the location of a maximum of g, a minimum, or a saddle point? Explain.

(a) $M = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

(b) $M = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$

8.2.7 Suppose that at a point $\vec{x}_0$ in $\mathbf{R}^2$, both first-order partial derivatives of a function $f(\vec{x})$ are zero, and the matrix of second-order partial derivatives is one of those given below. In each case, tell whether $\vec{x}_0$ is the location of a maximum, a minimum, or a saddle point.

(a) $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 8 & -10 \\ -10 & 8 \end{pmatrix}$

8.2.8 Roger Rapidrudder measured the gradient vector of the air temperature to be exactly zero, and the matrix of second derivatives of the temperature to be

$$\begin{pmatrix} \frac{\partial^2 T}{\partial x^2} & \frac{\partial^2 T}{\partial x \, \partial y} & \cdots \\ & \cdots & \\ & \cdots & \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

At that moment, was Roger at a maximum of temperature, a minimum, a saddle point, or none of these? Explain.

8.2.9 The real-valued function $y = g(x_1, x_2)$ has vanishing gradient at the point $x_1 = 0$, $x_2 = -5$. At that point the matrix of second derivatives of g is

$$M \equiv \left\{ \frac{\partial^2 y}{\partial x_j \, \partial x_k} \right\} = \begin{pmatrix} 1 & 5 \\ 5 & 2 \end{pmatrix}.$$

Is that point the location of a maximum of g, a minimum, or a saddle point? Explain.

8.2.10 Suppose that at a point $\vec{x}_0$ in $\mathbf{R}^2$, both first-order partial derivatives of a function $f(\vec{x})$ are zero, and the matrix of second-order partial derivatives is

$$Q_{\vec{x}_0} f = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

(a) Is $\vec{x}_0$ the location of a maximum, a minimum, or neither? Explain.

(b) Let C be a constant close to (but not equal to) $f(\vec{x}_0)$. What kind of conic section (ellipse, hyperbola, or parabola) would you expect the graph of the relation $f(\vec{x}) = C$ to resemble? HINT: Think of using $Q_{\vec{x}_0} f$ to approximate f by a quadratic polynomial.

8.2.11 Find all the local extrema and saddle points of $f(u, v) = 2 \cosh u \cos v$. Explain your reasoning thoroughly.

8.2.12 Let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Explain why M is guaranteed to possess an orthonormal eigenbasis.

(b) Find the eigenvalues and normalized eigenvectors of M.

(c) Find the matrix called e^{tM} (for each $t \in \mathbf{R}$), and show how to use it to solve the system of differential equations

$$\begin{aligned} u' &= v, \qquad & u(0) &= 0, \\ v' &= u, \qquad & v(0) &= 1. \end{aligned}$$

8.2.13 Show that the Fredholm theorem (†) implies that the dimension of the space spanned by the rows of a matrix equals that of the space spanned by the columns. (This is Theorem 2 of Sec. 5.4.) HINT: Apply Theorem 1 of Sec. 5.4 (the equation relating rank to nullity) to the linear functions represented by A and A^{t}.

8.2.14 Prove that the 5 proposed definitions of *orthogonal matrix* are equivalent.

8.2.15 Prove that all the coefficients of the characteristic polynomial of a matrix are invariant under similarity transformations. HINT: Observe that $UAU^{-1} - \lambda = U(A - \lambda)U^{-1}$.

8.2.16 Suppose that at a point $\vec{x}_0$ in $\mathbf{R}^3$, both first-order partial derivatives of a function $f(\vec{x})$ are zero, and the matrix of second-order partial derivatives is one of those given below. In each case, tell whether $\vec{x}_0$ is the location of a maximum, a minimum, or a saddle point.

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 3 \\ 0 & 3 & 5 \end{pmatrix}$

8.2.17 Prove the assertion in Sec. 8.1 that the only matrices that are diagonalizable by orthogonal matrices are the symmetric matrices.

8.2.18 Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$
Little or no calculation is needed to answer the following questions.

(a) Which matrix can be diagonalized by an orthogonal matrix?

(b) Which can be diagonalized, but only by a nonorthogonal matrix?

(c) Which can't be diagonalized at all?

8.2.19 Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$
Little or no calculation is needed to answer the following questions.

(a) Which matrix can be diagonalized by an orthogonal matrix?

(b) Which can be diagonalized, but only by a nonorthogonal matrix?

(c) Which can't be diagonalized at all?

8.3 A Note on History

The history of matematics should be more important to the nonmathematician than to the mathematician. The math major studies mathematics for its own sake, and may be willing to accept the concepts and concerns of the subject as if engraved on stone tablets handed down through the professors. The technical student from outside, who has been told to learn some mathematical subject as a useful tool, is more likely to wonder *why* anyone would come up with those concepts and problems. Mathematical ideas develop for a reason; they grow out of real problems (not necessarily "applied" problems, but natural questions that arise at a less abstract level), and understanding where they came from can help one to see how they can be used in new, but similar, circumstances today. Furthermore, seeing the gradual way in which understanding has developed — often cluttered for some time by confusion or irrelevant side issues — can help the student to appreciate his or her own learning struggles as natural and healthy.

I therefore took it for granted that if I ever wrote a textbook, it would be grounded in the history of mathematical ideas. When I sat down to write this book, I realized with a shock that I didn't know any history of linear algebra. When I studied linear algebra in college in the 1960s, the tablets had already been written. But more than that, unlike textbooks in calculus, Fourier analysis, probability, number theory, or differential equations, linear algebra books seem to be peculiarly devoid of history, lacking even those sterile footnotes listing the nationalities and birth and death dates of mathematicians.*

In reality, despite its static nature in the curriculum since 1950, linear algebra as we know it today is a rather recent consensus. Its serious development began around 1800; it reached its present form in the 1920s; and *only then* did it attain its central, fundamental place in pure and applied mathematics.

As reflected in the first two chapters of this book, the concept of a vector emerged out of two distinct areas of mathematics, geometry and algebra. "Algebra" in the nineteenth century meant largely the study of the transformation (with luck, simplification) of equations by various changes of variables, but also the study of "systems" of "quantities" satisfying various rules for multiplication, etc. (compare our Sec. 2.1 and Sec. 3.1). Near the beginning of the nineteenth century, several people independently recognized that complex numbers could be represented by line segments (characterized by length and direction) in a plane, and vice versa. It was natural to ask whether three-dimensional space also had such an algebraic structure.

The Irish mathematician (and theoretical physicist) William Rowan Hamilton devoted the last two-thirds of his career to this question — first to finding an answer, then to developing and promoting it. Progress was

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* An exception is J. B. Fraleigh and R. A. Beauregard, *Linear Algebra* (Addison–Wesley, 1987), with historical notes by Victor Katz.

hampered by the fact that nineteenth-century mathematicians were trying to think of vectors as a new kind of "number", much as the negative integers had grown out of the positive ones and had in turn been extended to the rational numbers and then to the complex numbers.† It was expected, therefore, that two vectors could be multiplied or divided to yield a third vector. Hamilton discovered that multiplication of three-dimensional vectors was possible, at the cost of abandoning the commutative law of multiplication; thereby he invented the vector cross product — almost. However, he also maintained the associative law of multiplication and the possibility of division (contrast Sec. 2.5!) at the cost of introducing a fourth dimension. He called his new hypercomplex numbers *quaternions*, and he and his followers proceeded to reformulate physics in terms of them. Quaternions did indeed provide a way of thinking about three-dimensional geometry and physics that was independent of any particular choice of coordinate axes; velocities and forces could be comprehended as single objects, rather than triples of numbers. Unfortunately, Hamilton had been somewhat misled by the coincidence that the dimension of the space of 3×3 antisymmetric matrices (which are related to rotations in three-dimensional space, as remarked in Secs. 6.3 and 7.2) is also equal to 3. From a modern point of view, in the quaternion formalism the vectors and the rotation operators that act upon them are jumbled together. (In fact, Hamilton consistently made an error of a factor of 2 in rotation angles as a result of trying to force the vectors and the rotations to be the same thing. The relation between vectors and rotations was more accurately understood in that era by the less famous Olinde Rodrigues. See the fascinating article by S. L. Altmann, Hamilton, Rodrigues, and the quaternion scandal, *Math. Mag.* **62**, 291–308 (1989).)

Simultaneously, a different algebraic approach to geometry was developed by Hermann Grassmann, a German schoolteacher. His theory, called *Ausdehnungslehre* ("theory of extension"), had the great advantage of applying in all dimensions, not just 3. He was largely concerned with the relation between p-dimensional affine subspaces and antisymmetric algebraic combinations of p vectors, something we have touched on in Sec. 7.2. Grassmann's best ideas were almost a century ahead of their time, and they were buried in other material that turned out to be irrelevant or misguided. His books

† Logically, the real numbers should come between the rationals and the complexes, but historically a solid understanding of the reals did not come until later in the century.

were considered unreadable, even by other Germans, so for a long time he was essentially ignored.

The era of Hamilton and Grassmann occupied the middle of the nineteenth century; it was truly the Mesozoic Era of our subject. The last two decades of that century saw the emergence of vectors and vector spaces in their modern forms.

On the one hand, J. Willard Gibbs (American physicist) and Oliver Heaviside (British engineer) distilled from the quaternion theory its useful essence, the three-dimensional vectors we see in physics and engineering textbooks today, with their dot and cross products and nabla operators. This quickly became the standard language of physics and applied mathematics.

On the other hand, several mathematicians in Italy, especially Giuseppe Peano, were separating the wheat from the chaff in Grassmann's work. The result was the modern definition of a vector space. However, this development did not attract much attention at the time.

The triumph of the abstract vector concept did not happen until the study of *infinite-dimensional* vectors made it necessary. A number of mathematicians, notably Stefan Banach, independently reinvented the concept of a vector space (equipped with an inner product or a norm) around 1920 in order to deal with infinite-dimensional problems, in which the vectors were functions or infinite sequences. (We have made the point throughout this book that the abstract concepts, which may be elegant and mildly helpful in elementary low-dimensional problems, reach their full power and become indispensible in more complicated, especially infinite-dimensional, problems.) It is striking that this development roughly coincided with the rise of quantum mechanics in physics. Quantum theory would be almost impossible without the idea of an infinite-dimensional vector space, but it was not quantum theory that sparked the creation of the mathematical theory. Functional analysis, or infinite-dimensional linear algebra, developed just a few years earlier, largely in response to purely classical physical problems involving integral and differential equations.

In more recent years both Grassmann and Hamilton have been partially vindicated. Grassmann's geometrical ideas evolved into the modern theory of differential forms (Sec. 7.4). Hamilton's strange four-dimensional space, with a few sign changes, turned out to be related to the space-time of special relativity, in a formalism well adapted to the study of rotations acting on the fields and wave functions that represent electrons and other half-integralspin particles (*fermions*). At the same time that Gibbs was cleaning up the vector half of the quaternion formalism, mathematicians such as Felix Klein and Sophus Lie were, in effect, rescuing its rotational half, by creating the modern theory of the groups of rotations in n-dimensional space, and the more general theory of Lie groups and Lie algebras. In short, the *Pauli matrices* of quantum mechanics are quaternions in disguise — or, better, in their true form with their nineteenth-century disguise removed. The quaternions in their original form retain some followers among applied researchers whose work heavily involves rotations, as evidenced by the recent book of J. B. Kuipers, *Quaternions and Rotation Sequences, A Primer with Applications to Orbits, Aerospace, and Virtual Reality* (Princeton University Press, 1999). They have always been of interest to pure mathematicians as a very special kind of algebraic system.

In writing this addendum I found the following resources helpful. Two books:

- M. J. Crowe, *A History of Vector Analysis* (University of Notre Dame Press, 1967; Dover Publications, 1994).
- A. F. Monna, *Functional Analysis in Historical Perspective* (Halsted Press, 1973).

Two scholarly historical articles:

- J.-L. Dorier, A general outline of the genesis of vector space theory, *Historia Mathematica* **22**, 227–261 (1995).
- G. H. Moore, The axiomatization of linear algebra: 1875–1940, *Hist. Math.* **22**, 262–305 (1995).

(See also the article by Altmann cited above and the one by Katz cited in Sec. 7.5.) Three short articles (by V. J. Katz, O. B. Becken, and K. Reich) in the collection

- *Learn from the Masters!*, ed. by F. Swetz *et al.* (Mathematical Association of America, 1995).

And one World Wide Web site:

- `http://www-groups.dcs.st-and.ac.uk/ history/`, The MacTutor History of Mathematics Archive, especially the pages on "Matrices and Determinants" and "Abstract Linear Spaces".

This view of history is necessarily incomplete and oversimplified. The seriously interested reader should consult the sources listed above and seek out others.