Lecture Notes for Math 309
(Fourier Series and Bessel Functions)

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August 20, 2013
revised 12/18/13

Contents

CHAPTER 1

Boundary Value Problems

Preliminaries

We will be looking at *partial differential equations* (PDE) for the rest of the course. Recall from 308: an ordinary differential equation is something like: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$ on $[0, 1]$, with perhaps *initial conditions* $y(0) = 1, y'(0) = 0$. A solution to this is a function (so, solving a regular equation might give you a number, but solving a differential equation gives you a function). Both e^x and e^{-3x} solve this differential equation. The general solution to this differential equation (ignoring the initial conditions) is $C_1e^x + C_2e^{-3x}$ and the linear combination which satisfies the initial conditions is $\frac{3}{4}e^x + \frac{1}{4}e^{-3x}$. This actually fits in pretty well with linear transformations: if we define $L : C^2[0, 1] \to C[0, 1]$ by $L[y] = y'' + 2y' - 3y$, it's not hard to see that L is linear. Looking for solutions to the equation is looking for the kernel of L. It turns out that a basis for the kernel is $\{e^x, e^{-3x}\}$, and then we look for the linear combination which satisfies the initial conditions.

Reminder from 308: solving linear homogeneous ODE's with constant coefficients. A first order linear homogeneous ODE with constant coefficients looks like $y' = ky$. The solution to this is $y = y_0e^{kt}$, which you saw in calculus.

To solve the second order linear homogeneous ODE with constant coefficients $ay'' + by' + cy = 0$, plug in $y = e^{rt}$. Then we want r so that $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$, i.e., $ar^2 + br + c = 0$. This is called the *auxiliary equation* for the ODE. The following can happen:

- (Real and unequal roots): If the auxiliary equation has two real roots r_1, r_2, with $r_1 \neq r_2$, then both e^{r_1t} and e^{r_2t} are solutions to the ODE, and the general solution is $C_1e^{r_1t} + C_2e^{r_2t}$.
- (Real and equal roots): If the auxiliary equation has a double root r_1 (i.e., the auxiliary equation is some constant times $(r - r_1)^2$), then both e^{r_1t} and te^{r_1t} are solutions, and the general solution is $C_1e^{r_1t} + C_2te^{r_1t}$.
- (Complex roots): If the auxiliary equation has the pair of complex roots $\alpha + i\beta$ and $\alpha - i\beta$ (assuming that a, b, and c are real, complex roots must come in conjugate pairs), then both $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are solutions, and the general solution is $C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t)$.

For our purposes, the most important cases of second order linear ODE's are:

- $y'' = -\lambda^2 y$. The general solution to this can be written as $y(t) = a\cos\lambda t + b\sin\lambda t$.
- $y'' = \lambda^2 y$. The general solution to this may be written as $y(t) = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$. If we introduce two (possibly) new functions $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, this can be written as $y(t) = a\cosh(\lambda t) + b\sinh(\lambda t)$. We'll see more about these hyperbolic trig functions later, when we need them.

Partial differential equations involve partial derivatives. Recall that, if u is a function of x and y, for example, that $\frac{\partial u}{\partial x}$ means to treat y as a constant and take the derivative with respect to x. You should be aware that there is another notation for partial derivatives which is in common use: we can write the partial of u with respect to x as u_x instead of $\frac{\partial u}{\partial x}$. I will be using both notations. A typical PDE is something like Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. (Which can also be written as $u_{xx} + u_{yy} = 0$.) We're looking for a function $u(x,y)$ now, so we aren't on an interval anymore. Instead, u would have some domain in the x, y plane, so that's where we'd solve the equation. There's still the problem of what the analog of initial conditions will be, but we'll get to that. A *general solution* often has arbitrary constants, and then we might look for a *particular solution,* by choosing the constants to satisfy some additional conditions. Sometimes (we'll see examples later) general solutions even have arbitrary functions, not just arbitrary constants.

Things will be more complicated than ODE's: for example, there are many solutions to Laplace's equation in the plane. Here are a few: $u(x,y) = e^x \cos y$, $u(x,y) = x^2 - y^2$, $u(x,y) = x^3 - 3xy^2$. It's not nearly as simple as the ODEs we were looking at. In fact, if we define $L(u)$ to be $u_{xx} + u_{yy}$, then the kernel of L turns out to be infinite dimensional, in contrast to the ODE case we were looking at before.

An example of a **boundary value problem** is something like: find a function $u(x,y)$ satisfying $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the interior of the square $0 < x < 1$, $0 < y < 1$, continuous on the closed square $0 \le x \le 1$, $0 \le y \le 1$, and equal to given functions on the boundary of the square: say $u(x,0) = x^2$, $u(x,1) = 1$, $u(0,y) = y$, $u(1,y) = 1$.

The order of a PDE is the order of the highest derivative in it. For example, the order of Laplace's equation is 2, since we're taking second derivatives. We will generally be looking at second order linear PDE's. In two variables, these look like

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} + D(x,y)\,u_x + E(x,y)\,u_y + F(x,y)\,u = G(x,y),$$

although the variables could be other than x and y: x and t is also quite common. If G is identically zero, this is a homogeneous equation, if not it's inhomogeneous. So, what's linear about this? The functions A, B, and so on could be rather wild.

The point is that

$$L(u) = A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} + D(x,y)\,u_x + E(x,y)\,u_y + F(x,y)\,u$$

is a linear transformation: $L(\alpha u_1 + \beta u_2) = \alpha L(u_1) + \beta L(u_2)$. Having things like u^2 or $\sin(u_x)$ would make the equation non-linear. Also, something like $u_y u_{xy} - u = 0$ is non-linear because of the product.

Principle of Superposition

If our equation is a homogeneous, linear PDE, i.e., $L(u) = 0$, then if u_1 and u_2 are solutions, so is their sum. As an example, since $e^x \cos y$ and $x^2 - y^2$ both satisfy $u_{xx} + u_{yy} = 0$, so does their sum. We will be using this a lot, in constructing series solutions to PDE's.

Classification of second order linear PDE's

A second order linear PDE is *hyperbolic, elliptic,* or *parabolic* at a point if $B^2 - 4AC$ is positive, negative, or zero (depends only on the highest order term). Generally, we'll be considering cases where A through F are constant, but that's not required in the definition. It's possible for an equation to be one type at one point and another somewhere else: $L(u) = xu_{xx} + u_{yy} = \cos x$ is elliptic on $x > 0$, hyperbolic on $x < 0$, and parabolic on $x = 0$. Notice that the cosine had nothing to do with the classification.

Basic Examples of PDE's

Here are the three most important PDE's. Notice that they all have constant coefficients.

- The *wave equation*: $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$. Here u is a function of x and t. One physical interpretation is that we pluck a string, and $u(x,t)$ is the displacement of the point on the string x units from the end at time t (picture). So, if t is fixed, you're taking a snapshot of the string. The constant a is related to the tension in the string and the density (mass per unit length) of the string. This is a hyperbolic equation (using t instead of y): $A = a^2$, $C = -1$, and B is zero, so $B^2 - 4AC$ is positive. If a string of length L has initial position $f(x)$ and is given initial velocity $g(x)$, and if the ends of the string are fixed, then the boundary value problem is

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= 0, \quad t > 0 \\ u(L,t) &= 0, \quad t > 0 \\ \frac{\partial^2 u}{\partial t^2} &= a^2 \frac{\partial^2 u}{\partial x^2} \end{aligned}$$

  If you're banging a drum rather than plucking a string (so it's a two dimensional membrane), the equation that the displacement $u(x,y,t)$ must satisfy is $\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$.
- The *heat equation*: Suppose that we have a metal bar of length L. Suppose that the temperature at location x at time t is $u(x,t)$. Then the temperature satisfies $\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$. Here κ is a physical constant called the

diffusivity, which depends on the material of the bar. This is a parabolic equation. Typical boundary conditions are to hold the ends of the bar at temperature 0, so that $u(0,t) = u(L,t) = 0$ for $t > 0$, or to have the ends insulated, which gives $u_t(0,t) = u_t(L,t) = 0$. A boundary value problem will often assign an initial temperature $u(x,0)$. Unlike the wave equation, it wouldn't make sense to also assign $u_t(x,0)$, since u_t is already forced on you by the initial temperature and the differential equation. If the object that we're taking the temperature of is two or three dimensional (a plate or a solid), the equation becomes $\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ and $\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ respectively.

- *Laplace's equation*: If the temperature in the previous example has reached a steady state in, say, a two dimensional plate, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The right hand side is called the Laplacian, and is often written as $\nabla^2 u$ or Δu. (In three dimensions, the Laplacian adds a u_{zz}.) Engineers tend to use the first notation, mathematicians maybe both. I prefer to use $\nabla^2 u$ to avoid confusing Δu with the change in u. So, how does a steady state temperature avoid ending up as a constant? We can include boundary conditions, say keeping one part of the boundary at temperature u_1 and another at temperature u_2. Laplace's equation is elliptic. There are other applications of Laplace's equations besides steady state heat.

Notice that the Laplacian appears in all three of these: $u_{tt} = a^2\nabla^2 u$ is the wave equation, $u_t = \kappa\nabla^2 u$ is the heat equation, and of course $\nabla^2 u = 0$ is Laplace's equation.

Examples of Explicit Solutions

There are a few PDE's that we can solve explicitly. The simplest type is a second order partial equal to a function of the independent variables. You can often solve these by integration. Here's an example.

EXAMPLE 1. *Solve the PDE $u_{yy} = e^y$ with boundary conditions $u(x,0) = e^x$ and $u_y(x,0) = x^3$.*

Solution: Since $u_{yy} = e^y$, we can integrate both sides with respect to y to get $u_y(x,y) = \int e^y\,dy = e^y + f(x)$, for some function f depending only on x. The point is that the "$+C$" in the integration is only a constant as far as y is concerned, i.e., it can depend on x. From the initial condition on u_y, we must have $x^3 = e^0 + f(x)$, so $f(x) = x^3 - 1$, therefore $u_y(x,y) = e^y + x^3 - 1$. Integrate again: $u(x,y) = e^y + yx^3 - y + g(x)$ for some function $g(x)$. Plug in $y = 0$ to use the initial condition: $e^x = 1 + g(x)$, so $g(x) = e^x - 1$, so the answer is that $u(x,y) = e^y + yx^3 - y + e^x - 1$.

EXAMPLE 2. *For what values of a will $u(x,y) = F(x+ay)$ solve the equation $u_{xx} + 4u_{xy} - 5u_{yy} = 0$, where F is an arbitrary twice differentiable function?*

Most of the time we won't be able to find an explicit solution except by separation of variables, and I won't spend any more time on these.

Problems for chapter 1

(1) For what value(s) of a will $u\left(x,y\right)=F\left(x+ay\right)$ solve the PDE $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x \partial y}-2\frac{\partial^2 u}{\partial y^2}=0$, where F is an arbitrary twice differentiable function of one variable?

(2) Classify the following PDEs as hyperbolic, elliptic, or parabolic:
  (a) $x^2 u_{xx}+2xyu_{xy}+y^2 u_{yy}+x^3 u_x-y^3 u_y=0$.
  (b) $\frac{\partial^2 f}{\partial x^2}-2\frac{\partial^2 f}{\partial x \partial t}+2\frac{\partial^2 f}{\partial^2 t}=0$
  (c) $u_{tt}=tu_x$

(3) Find a solution to the PDE $u_{xy}=x$, satisfying $u(x,0)=e^x$ and $u(0,y)=\cos y$.

(4) Solve the initial value problem $u_{yy}=xe^y$, $u(x,0)=3x$, $u_y(x,0)=e^x$.

(5) For what number(s) α will $u(x,y)=e^{2x+\alpha y}$ solve the PDE $u_{xx}-u_{yy}-2u_x+u=0$?

CHAPTER 2

Method of separation of variables

Separation of variables will be our most powerful tool in solving PDE's, which will lead to series solutions later on. I'll illustrate with an example.

EXAMPLE 3. *Solve the first order PDE*

$$u_x = 6u_y,$$

with initial condition $u(0, y) = e^y$ *by separation of variables.*

What this means is that we seek a solution of the form $u(x, y) = X(x)Y(y)$. If there is such a solution, we must have

$$X'(x)Y(y) = 6X(x)Y'(y),$$

so that

$$\frac{X'(x)}{X(x)} = 6\frac{Y'(y)}{Y(y)}.$$

The key observation is that the left side doesn't depend on y, and the right side doesn't depend on x. Since they're equal, they can't depend on x or y. What sort of function doesn't depend on anything? A constant! In other words, there is a constant λ called the separation constant, so that

$$\frac{X'(x)}{X(x)} = 6\frac{Y'(y)}{Y(y)} = \lambda.$$

Therefore, the functions X and Y must satisfy $X'(x) = \lambda X(x)$ and $Y'(y) = \frac{\lambda}{6}Y(y)$. These are ODE's that we know how to solve: they're exponentials. So, $X(x) = Ae^{\lambda x}$ and $Y(y) = Be^{\frac{\lambda}{6}y}$, so $u = ABe^{\lambda x}e^{\frac{\lambda}{6}y} = Ke^{\lambda x}e^{\frac{\lambda}{6}y}$. Now, the initial condition that $u(0, y) = e^y$ gives us that $Ke^{\frac{\lambda}{6}y} = e^y$, thus $K = 1$ and $\lambda = 6$, and the solution is $e^{6x}e^y$, which of course you can write as e^{6x+y}.

EXAMPLE 4. *Solve*

$$u_t = 4u_{xx}$$

on $0 < x < 2$, $t > 0$, *with initial conditions* $u(x, 0) = \sin \pi x - 2 \sin 2\pi x$, $u(0, t) = u(2, t) = 0$ *for all* t, *under additional assumption that* $u(x, t)$ *is bounded on the region we're looking at. (This is the heat equation for one specific choice of the diffusivity.)*

Solution: Let's seek functions of the form $u(x, t) = X(x)T(t)$ We must have

$$XT' = 4X''T,$$

so

$$\frac{T'}{4T} = \frac{X''}{X} = c$$

for some separation constant c. I claim that we have to have $c \leq 0$. The reason is that the equation for T is $T' = 4cT$, with solution $T = Ae^{4ct}$ for some constant A. If $c > 0$ this is unbounded as t tends to ∞. Thus, I can put in $-\lambda^2$ for c:

$$\frac{T'}{4T} = \frac{X''}{X} = -\lambda^2.$$

The solution for T is $Ae^{-4\lambda^2 t}$, as noted before. Now for X, we get $X'' = -\lambda^2 X$, which we recognize from 308 as having the general solution $B\cos(\lambda x) + C\sin(\lambda x)$. Therefore, we're getting that if the solution separates, it must be

$$e^{-4\lambda^2 t}\left(B\cos(\lambda x) + C\sin(\lambda x)\right)$$

for some values B, C, and λ. There's no way to choose these to get the given initial conditions. Instead, find two solutions, one for each initial condition, and then superimpose them. To find u_1 satisfying $u_1(x,0) = \sin \pi x$, we need $B = 0$, $C = 1$, and $\lambda = \pi$. Thus $u_1(x,t) = e^{-4\pi^2 t}\sin(\pi x)$. Similarly, to find u_2 satisfying $u_2(x,0) = -2\sin 2\pi x$, we need $B = 0$, $C = -2$, and $\lambda = 2\pi$. Thus $u_2 = -2e^{-16\pi^2 t}\sin(2\pi x)$. Therefore the solution is the sum:

$$u(x,t) = e^{-4\pi^2 t}\sin(\pi x) - 2e^{-16\pi^2 t}\sin(2\pi x).$$

Note that this solution doesn't separate, but it is the sum of separated solutions. The physical problem is that of a metal bar stretching from $x = 0$ to $x = 2$, starting at the given initial temperature, with the ends being held at temperature zero as time goes along.

Some examples on separating variables:

EXAMPLE 5. *Does $u_{xx} + u_{yy} + u_y = 0$ separate? If so, find the resulting ODE's.*

Solution:

EXAMPLE 6. *Here's one that doesn't separate: $(x+y)u_{xx} + u_{yy} = 0$. Plug in $u = X(x)Y(y)$, and you get $(x+y)X''Y + XY'' = 0$. Try as you might, there's no way to separate the x's and the y's.*

EXAMPLE 7. *(Here's an example for three variables, which gets a little more involved.) Separate $yu_{xx} + xu_{yy} + xyu_{zz} = 0$, finding the resulting ODE's.*

Now u is a function of x, y, and z. Write $u = X(x)Y(y)Z(z)$. Then $yX''YZ + xXY''Z + xyXYZ'' = 0$. Divide through by $xyXYZ$ to get

$$\frac{X''}{xX} + \frac{Y''}{yY} + \frac{Z''}{Z} = 0.$$

So, now what do we do? Since

$$\frac{X''}{xX} + \frac{Y''}{yY} = -\frac{Z''}{Z} = \lambda,$$

where λ can't depend on any variable, we must have $Z'' = -\lambda Z$ and $\frac{X''}{xX} + \frac{Y''}{yY} = \lambda$. The last equation separates between x and y:

$$\frac{X''}{xX} = \lambda - \frac{Y''}{yY} = \mu,$$

so the equations that X and Y must satisfy are $X'' = \mu x X$ and $Y'' = (\lambda - \mu) y Y$.

Problems for chapter 2

(1) The variables in the following PDE's separate. Write (but do not solve) the resulting ODE's:
  (a) $u_{xy} = u$.
  (b) $u_{xx} - u_{yy} + 2u_x - 2u_y + u = 0$.
  (c) $t^2 u_{tt} - x^2 u_{xx} = 0$.
  (d) $u_{xx} + u_{yy} = u_t$. (Note: since there are three variables, there will be two constants of separation.)

(2) Find the general separated solution to the PDE $u_{tt} = 4u_{xx}$ under the assumption that
  (a) the constant of separation is negative (call it $-\lambda^2$).
  (b) the constant of separation is postive (call it λ^2).

(3) Solve $u_t = u_{xx}$, $u(x, 0) = \cos 3x + e^{2x}$ by separation of variables.

(4) Solve $u_t + u = u_x$, $u\,(x, 0) = e^{3x} - 2e^{4x}$ by separation of variables.

CHAPTER 3

Fourier Series and Applications

PDE's and Trigonometric Polynomials

Let's look at a slight generalization of example 4.

EXAMPLE 8. *Find a solution to the heat equation* $u_t = 4u_{xx}$ *on* $[0, L]$, *with* $u(0,t) = u(L,t) = 0$, $u(x,t)$ *bounded, and initial conditions* $u(x,0) = \sum_{n=1}^{m} b_n \sin\left(\frac{n\pi x}{L}\right)$.

The above initial conditions are an example of a *trigonometric polynomial.* It's going to be particularly good to fit in with the separation of variables approach. The factors of $\frac{n\pi}{L}$ were chosen so that it's automatically zero at the endpoints. Using what we already did in example 4, set $u_n(x,t) = b_n e^{-4\lambda^2 t} \sin(\lambda x)$, where the choice of λ must be $\frac{n\pi}{L}$. So, we're getting that

$$u(x,t) = \sum_{n=1}^{m} b_n e^{-4(n^2\pi^2/L^2)t} \sin\left(\frac{n\pi x}{L}\right).$$

Very pretty, but having the initial conditions a trigonometric polynomial seems pretty specialized. In fact, it's not: we will be approximating functions with trigonometric polynomials, or to put it another way, we will be writing functions as *Fourier series.* We'll then be able to apply this to similar initial value problems.

Approximation by Trigonometric Polynomials

Suppose that we have a function $f(x)$ defined on the interval $(-L, L)$, and that, for fixed m, we want the trigonometric polynomial of the form

$$t_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

which approximates it best in some sense. Recall that we had an inner product and norm defined by integrals. Define $\langle f, g\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\, g(x)\, dx$, and $\|f\| = \sqrt{\langle f, f\rangle} = \frac{1}{\sqrt{L}}\sqrt{\int_{-L}^{L} f^2(x)\, dx}$. Let's look for the trigonometric polynomial which approximates it best in this norm on $(-L, L)$, in other words we want to find a trigonometric polynomial $t_m(x)$ so that $\|t_m - f\|$ is as small as possible. (We actually saw this example with $L = \pi$ in section 5.5 of Leon, but it won't hurt to look at it in a little more detail.) The set of trigonometric polynomials we're looking at are spanned by $\left\{\frac{1}{\sqrt{2}}, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$, n, k from 1 to m. We saw in homework in Leon, section 5.4, that for $L = \pi$, this is an orthonormal set with respect to the inner product $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$. The same holds true for general $L > 0$.

The closest vector in the span of $\left\{\frac{1}{\sqrt{2}}, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$ to $f(x)$ is given by inner products:

$$\alpha_0 \frac{1}{\sqrt{2}} + \sum_{n=1}^{m}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

where

$$\begin{aligned}
\alpha_0 &= \left\langle f(x), \frac{1}{\sqrt{2}}\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\frac{1}{\sqrt{2}}\,dx, \\
a_n &= \left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \\
b_n &= \left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.
\end{aligned}$$

Notice that when $n = 0$ for the formula for a_n, you get almost the formula for α_0; it's off by a factor of $\sqrt{2}$. We saw this before, and we can include that into the constant term to get that the closest vector in the span of $\left\{\frac{1}{\sqrt{2}}, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$, n, k from 1 to m, is

$$\frac{a_0}{2} + \sum_{n=1}^{m}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx,$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

What happens when m tends to infinity? We get a Fourier series.

Fourier Series

Definition: Suppose that $f(x)$ is defined on the interval $(-L, L)$ and that it is defined on the rest of the real line by extending it periodically, i.e., by $f(x + 2L) = f(x)$. The *Fourier series* corresponding to $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx,$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

A natural question is when does a function equal its Fourier series. For this we need a definition:

Definition: A function $f(x)$ is *piecewise continuous* on an interval $[a, b]$ if there are a finite number (possibly zero) of x_i's so that $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$, such that f is continuous on (x_{j-1}, x_j) for all $j = 1, \cdots, k$, and the one-sided limits at each x_j exist and are finite.

The result about when a function equals its Fourier series is this:

Theorem: (Dirichlet conditions)

- Suppose that f is defined except for possibly a finite number of points in $(-L, L)$.
- $f(x)$ is periodic of period $2L$.
- f and f' are piecewise continuous in $(-L, L)$.

Then, the Fourier series corresponding to f converges at every x. If $f(x)$ is continuous at a, then the Fourier series at a converges to $f(a)$. If f has a jump discontinuity at a, then the Fourier series at a converges to $\frac{1}{2}\left(\lim_{x\to a^-} f(x) + \lim_{x\to a^+} f(x)\right)$. So, for such functions,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

at every point of continuity of f. These are the sorts of functions which generally arise in engineering applications. (Although $f(x) = x^{1/3}$ on, say, $[-1, 1]$, does not satisfy the Dirichlet conditions, since its derivative is unbounded at 0. The Fourier series still seems to converge as above, though.)

Half-range Fourier series

Note (even and odd functions): A function $g(x)$ is *even* if $g(-x) = g(x)$ (picture). A function $g(x)$ is *odd* if $g(-x) = -g(x)$. The way to remember this is even power and odd powers: $g(x) = x^4$ is an even function, $g(x) = x^3$ is an odd function. Most functions are neither even nor odd: $f(x) = x^2 + x$ for example, since $f(-1) = 0$ is not $f(1)$ or $-f(1)$. For our purposes, it's important to note that $\sin x$ and in fact $\sin\left(\frac{n\pi x}{L}\right)$ is an odd function, and $\cos\left(\frac{n\pi x}{L}\right)$ is an even function. Also, the product of an odd function with an odd function is even, the product of an even function with an even function is even, and the product of an odd and an even function is odd.

The reason that we care is that for functions which are even or odd, you can sometimes simplify integrals on intervals whose midpoint is 0. If $g(x)$ is odd, then $\int_{-L}^{L} g(x)\,dx = 0$, since it's not hard to believe that the integral from 0 to L is cancelled out by the integral from $-L$ to 0 (picture). If g is even, then $\int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx$, since the integrals on the two halves of the interval are equal. (picture). So, if f is an odd function, so is $f(x)\cos\left(\frac{n\pi x}{L}\right)$. This means that all the a_n's in the Fourier series are zero, and there are just sine terms. In fact, if f is odd,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned}$$

This is called a *half-range Fourier sine series*: all we need are the values of f on $[0, L]$ and the fact that f is odd and periodic. If f is even you similarly get a *half-range Fourier cosine series*. In that case, all the b_n's are zero since now $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd, and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

EXAMPLE 9. *Suppose you expand $f(x) = x$, $0 \le x \le 1$ in a half-range Fourier sine series. Sketch what the series converges to. (i.e., what does the series $\sum_{k=1}^{\infty} b_k \sin(k\pi x)$ converge to on $\mathbb{R}$, where $b_k = 2\int_0^1 x\sin(k\pi x)\,dx$.) Same question for a half-range cosine series for $f(x) = x$.*

EXAMPLE 10. *Find the Fourier series for the square wave: $f(x) = -1$, for $-1 \le x < 0$, and $f(x) = 1$, $0 \le x < 1$, and f is periodic of period 2.*

This is an odd function, so that there will only be sines. In fact,

$$\begin{aligned} b_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \\ &= 2\int_0^1 \sin(n\pi x)\,dx \\ &= 2\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 \\ &= \begin{cases} 0, & n \text{ even}, \\ \frac{4}{n\pi}, & n \text{ odd}. \end{cases} \end{aligned}$$

So, the Fourier series for the square wave is

$$\frac{4}{\pi}\sin\pi x + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \cdots = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin((2k+1)\pi x).$$

By the Dirichlet conditions, this series converges to the square wave at every x which is not an integer. For those x's, it converges to the average of the left and right limits, i.e., 0.

Double Fourier Series

Suppose that $f(x,y)$ is a nice function (say, continuous, with continuous derivatives) defined on the rectangle $-K < x < K$, $-L < y < L$. If we hold y constant for the moment, we can write $f(x,y)$ as a Fourier series in x:

$$f(x,y) = \frac{a_0(y)}{2} + \sum_{m=1}^{\infty} a_m(y)\cos\left(\frac{m\pi x}{K}\right) + b_m(y)\sin\left(\frac{m\pi x}{L}\right),$$

where the familiar formulas for a_i and b_i now have y's in them. But then we can expand $a_i(y)$ and $b_i(y)$ in Fourier series in y, leading to a double Fourier series for $f(x,y)$. The formulas are sufficiently nasty that I won't do the general case, but let's look at the special case of a double sine series.

Suppose now that $f(x,y)$ is defined on $0 < x < K$ and $0 < y < L$. If we do a half-range sine series for f, then

$$f(x,y) = \sum_{m=1}^{\infty} b_m(y)\sin\left(\frac{m\pi x}{K}\right),$$

where $b_m(y) = \frac{2}{K}\int_0^K f(x,y)\sin\left(\frac{m\pi x}{K}\right)\,dx$. Now take a half-range sine series for $b_m(y)$:

$$b_m(y) = \sum_{n=1}^{\infty} b_{mn}\sin\left(\frac{n\pi y}{L}\right),$$

where

$$\begin{aligned} b_{mn} &= \frac{2}{L}\int_0^L b_m(y)\sin\left(\frac{n\pi y}{L}\right)\,dy \\ &= \frac{4}{KL}\int_0^L\int_0^K f(x,y)\sin\left(\frac{m\pi x}{K}\right)\sin\left(\frac{n\pi y}{L}\right)\,dx\,dy, \end{aligned}$$

where these are the coefficients in the double sine series for $f(x,y)$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{mn}\sin\left(\frac{m\pi x}{K}\right)\sin\left(\frac{n\pi y}{L}\right),$$

where this equals $f(x,y)$ where f and the partials of f are continuous.

EXAMPLE 11. *Find the double sine series for* $f(x,y) = 1$, *on* $0 < x < 2$, $0 < y < 1$.

We have that

$$b_{mn} = \frac{4}{2 \cdot 1} \int_0^1 \int_0^2 1 \cdot \sin\left(\frac{m\pi x}{2}\right) \sin(m\pi y)\, dx\, dy.$$

The double integral is easy to evaluate, and recalling that $\cos k\pi = (-1)^k$, we get that the series is

$$4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1-(-1)^m)(1-(-1)^n)}{mn\pi^2} \sin\left(\frac{m\pi x}{2}\right) \sin(m\pi y).$$

If you plot the sum over the x, y plane, the graph is a checkerboard of $+1$'s and -1's.

Problems for chapter 3

(1) $f(x)$ equals 2 for $0 < x < 2$, equals -2 for $2 < x < 4$ and has period 4. Find its Fourier series.

(2) Define $f(x)$ by

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x < 2. \end{cases}$$

Expand f in a sine series. For x not in the open interval $(0, 2)$, what will the sine series converge to? (Sketch your answer,)

(3) (a) Prove that $x^2 = \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \cdots$ for $-\pi \le x \le \pi$.

(b) Use the result from part a to evaluate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$.

(c) Use the result from part a to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

(4) Expand $f(x, y) = xy$, $0 < x < 1$, $0 < y < 2$ in a double sine series.

CHAPTER 4

Series solutions to PDE's

Now to apply this sort of thing to a boundary value problem.

EXAMPLE 12. *Suppose that we start with a bar of metal of length 1, with constant temperature 1, except at the ends of the bar we keep the temperature at zero. What happens to the temperature as a function of time?*

Naturally we expect it to eventually drop to zero, but we'd like to come up with the exact formula for the temperature. The problem is this: Find a solution to the heat equation $u_t = \kappa u_{xx}$, $u(x, 0) = 1$ for $0 < x < 1$, and $u(0, t) = u(1, t) = 0$ for $t > 0$. (There's a discontinuity at the corners of the domain, but that won't be a serious problem.) Let's separate variables first. We get $XT' = \kappa X''T$, so $\frac{1}{\kappa}\frac{T'}{T} = \frac{X''}{X} = \mu$, so $T' = \mu\kappa T$ and $X'' = \mu X$. Now, we expect the temperature to be bounded as t tends to infinity. The solution to $T' = \mu\kappa T$ is $T(t) = T_0 e^{\mu\kappa t}$, so for this to stay bounded, we need the exponent to be negative. So, set μ to be $-\lambda^2$, which will work nicely with the X equation. The solutions to $X'' = -\lambda^2 X$ are spanned by $\sin\lambda x$ and $\cos\lambda x$. To ensure that we're zero at $x = 0$ and $x = 1$, we're looking at $\sin(n\pi x)$. Here's the idea: expand the initial conditions in a half-range sine series, find the solution for initial conditions for each term of the series, and then superimpose them.

The half-range sine series which equals 1 from 0 to 1 is precisely the Fourier series for the square wave which we just saw. So, we solve the boundary value problem for $u_t = \kappa u_{xx}$, $u(x, 0) = \frac{4}{(2k+1)\pi}\sin((2k+1)\pi x)$, $u(0, t) = u(1, t) = 0$ for $t > 0$ for each k, and then add them up. For this, we take λ to be $(2k+1)\pi$, so we get

$$\frac{4}{(2k+1)\pi} e^{-\kappa\pi^2(2k+1)^2 t}\sin((2k+1)\pi x)$$

(check that this does in fact satisfy $u_t = \kappa u_{xx}$). Superpose all of them to get the solution

$$u(x, t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} e^{-\kappa\pi^2(2k+1)^2 t}\sin((2k+1)\pi x).$$

We're glossing over some fine points here, about whether we can take the derivative of an infinite sum by taking the sum of the derivatives. These are okay in the case we're looking at by general theory, but I won't spend any time worrying about this.

Note: We've seen one type of boundary condition for the heat equation: just holding the end of the bar at a constant temperature. There are two others which make sense: If the ends are insulated, then the heat flow through the ends is zero, which says that $u_x(0, t) = 0$, $u_x(1, t) = 0$. If we looked at the previous example with these boundary conditions, we'd get that the temperature just stays constantly 1, which makes sense. The third one is based on Newton's law of cooling, that the heat flow (i.e., u_x) leaving the rod is proportional to the difference between the heat of the bar at the end and a prescribed external temperature. I would write any boundary conditions out explicitly when I posed a problem.

EXAMPLE 13. *Not everything is the heat equation, though. Let's solve Laplace's equation* $u_{xx}+u_{yy}=0$ *on the square* $0<x<1$, $0<y<1$, *with boundary conditions* $u(0,y)=u(1,y)=u(x,0)=0$, $u(x,1)=u_1$. *(A problem which specifies values of* u *on the boundary is called a* Dirichlet *problem.)*

Separate variables: $X''Y+XY''=0$, so $\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda^2$. Okay, so why is this assumed to be negative? The answer is that we expect X to be a trig function to be zero at each endpoint, i.e., at $x=0$ and $x=1$. We don't expect the same for Y. We're getting $X''+\lambda^2X=0$, $Y''-\lambda^2Y=0$. The solution to the first we already know: it's of the form $a_1\cos\lambda x+b_1\sin\lambda x$. For the second, we could write it as a linear combination of $e^{\lambda y}$ and $e^{-\lambda y}$. Instead, I'll introduce two (possibly) new functions: $\sinh t=\frac{e^t-e^{-t}}{2}$ and $\cosh t=\frac{e^t+e^{-t}}{2}$. These are called *hyperbolic* functions and have properties similar to regular trig functions, except that about half the time the sign switches: $(\sinh t)'=\cosh t$, $(\cosh t)'=\sinh t$, $\cosh^2 t-\sinh^2 t=1$ for example. Anyway, we can write the general solution to $Y''-\lambda^2Y=0$ as $a_2\cosh\lambda y+b_2\sinh\lambda y$. The general separated solution is therefore

$$(a_1\cos\lambda x+b_1\sin\lambda x)(a_2\cosh\lambda y+b_2\sinh\lambda y).$$

The plan is to find those guys which are zero on the three sides that we're asking for, and then get the fourth side right by taking a series. Okay, so when $y=0$, we get $(a_1\cos\lambda x+b_1\sin\lambda x)a_2$. We don't want the X function always zero, or else we won't have anything, so we'll require that $a_2=0$. When $x=0$, we therefore have $a_1b_2\sinh\lambda y$, so we want a_1 to be zero (if b_2 is zero, we get Y identically zero, and again we have nothing). So, we've got $b_1b_2\sin\lambda x\sinh\lambda y$ at this point. Finally, to get this zero along $x=1$, we need $\lambda=m\pi$. So, a separated solution which is 0 on the three sides we're asking for is $B\sin m\pi x\sinh m\pi y$ (might as well call b_1b_2 B).

A series formed of these functions must look like

$$\sum_{m=1}^{\infty}B_m\sin m\pi x\sinh m\pi y.$$

Now we need to find B_m so that the boundary value on the top edge of the square is u_1. Thus we need to find B_m so that $\sum_{m=1}^{\infty}B_m\sinh m\pi\sin m\pi x=u_1$. But this is writing a constant as a Fourier sine series, which is our friend the square wave (times u_1). We need

$$B_m\sinh m\pi=\begin{cases}0, & m \text{ even},\\ \frac{4u_1}{m\pi}, & m \text{ odd}.\end{cases}$$

This says that our solution is

$$u(x,y)=\sum_{k=0}^{\infty}\frac{4u_1}{(2k+1)\pi\sinh(2k+1)\pi}\sin((2k+1)\pi x)\sinh((2k+1)\pi y).$$

EXAMPLE 14. *Solve the heat equation* $u_t=\kappa u_{xx}$ *on* $[0,L]$, *with initial temperature* $u(x,0)=f(x)$, *and with ends insulated, so that* $u_x(0,t)=0$ *and* $u_x(L,t)=0$

for all $t > 0$ (the solution will be in terms of the function $f(x)$, which I'll assume is reasonably nice, i.e., it and its derivative are continuous).

Two new things here: first, I'm not specifying $f(x)$, and second, the boundary conditions (i.e., at $x = 0$ and $x = L$) are different that what we've been seeing.

Solution: First we separate variables. Set $u(x,t) = X(x)T(t)$, so that $XT' = \kappa X''T$, so that $\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda^2$. The choice of the separation constant is as we've seen before: it must be negative for u to be bounded, and the λ^2 will work well with the trig functions which will arise for X. The general separated solution will be $e^{-\lambda^2\kappa t}(A\cos\lambda x + B\sin\lambda x)$. The solution to the example will be the sum of a series of terms which look like this. To ensure that the sum has zero x derivatives at the endpoints, we'll make sure that each term has zero derivative at the endpoints. So, take the partial of the general separated solution with respect to x to get $e^{-\lambda^2\kappa t}(-A\lambda\sin\lambda x + B\lambda\cos\lambda x)$. At $x = 0$ this is $e^{-\lambda^2\kappa t}(B\lambda)$. For this to always be zero, either $B = 0$ or $\lambda = 0$. If $\lambda = 0$ we'll get a constant term that we'll have to remember. Otherwise we've got $B = 0$, and the sine terms won't appear. At this point we know the terms in the series look like $Ae^{-\lambda^2\kappa t}\cos\lambda x$. For the partial with respect to x to be zero at $x = L$ we need $e^{-\lambda^2\kappa t}(-A\lambda\sin\lambda L) = 0$, which occurs for $\lambda = \frac{n\pi}{L}$. Therefore each term in the series be of the form $Ae^{-\frac{n^2\pi^2}{L^2}\kappa t}\cos\left(\frac{n\pi}{L}x\right)$, so that we're looking for a solution which looks like

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}\kappa t}\cos\left(\frac{n\pi}{L}x\right),$$

where I put in the $\frac{a_0}{2}$ to agree with the formula for Fourier series. When $t = 0$, this should be $f(x)$. But when $t = 0$ we have a half-range cosine series. Thus we have

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\,dx.$$

This is as far as we can go with this problem since we aren't given a specific f. On the other hand, since we've solved it for general f, we can write a Maple sheet which has the integrals evaluated for given f, so we can solve this for any reasonable starting f.

Here's an example in three variables:

EXAMPLE 15. *Solve the heat equation in the square $0 < x < 1$, $0 < y < 1$, if the sides of the square are kept at temperature 0, and the initial temperature distribution is $u(x, y, 0) = f(x, y)$, where $f(x, y)$ is a smooth function defined on the square.*

The heat equation is now

$$u_t = \kappa(u_{xx} + u_{yy}).$$

In separating variables, there will be two constants of separation, since there are three variables. First, we have

$$\frac{T'}{\kappa T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

It turns out to be more important to have the second order fractions equal to nice things, so before we set things equal to a constant of separation, write it as

$$\frac{T'}{\kappa T} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda^2,$$

where I take a negative constant since the boundary conditions are going to lead us to have trig functions for X. The ODE for X is $X'' = -\lambda^2$, with solution $A_1 \cos \lambda x + B_1 \sin \lambda x$. The equation for Y and T separates again:

$$\frac{Y''}{Y} = \frac{T'}{\kappa T} + \lambda^2 = -\mu^2,$$

where we also expect trig functions for Y. The ODE for Y is $Y'' = -\mu^2 Y$, with solution $A_2 \cos \mu y + B_2 \sin \mu y$, and the ODE for T is $T' = -\kappa(\lambda^2 + \mu^2)T$, with solution $T = T_0 e^{-\kappa(\lambda^2+\mu^2)t}$. Thus (absorbing the constant T_0 as usual) the general separated solution to the heat equation can be written as

$$e^{-\kappa(\lambda^2+\mu^2)t} \left(A_1 \cos \lambda x + B_1 \sin \lambda x\right) \left(A_2 \cos \mu y + B_2 \sin \mu y\right).$$

If this is to be zero on $x = 0$, we must have $A_1 = 0$, and if it is to be zero on $y = 0$, we must have $A_2 = 0$, so we will be looking at a double series of sines. To make this zero on $x = 1$, we need $\lambda = m\pi$, and to make this zero on $y = 1$ we need $\mu = n\pi$. The solution will be a series of these terms, a solution to the heat equation on the square which is zero on all sides of the square must be of the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{mn} e^{-\kappa\pi^2(m^2+n^2)t} \sin(m\pi x) \sin(n\pi y).$$

To find the specific series which also satisfies the initial conditions, we need $u(x, y, 0)$ to equal $f(x, y)$. But plugging in $t = 0$ just gives us the double sine series we saw in Chapter 3, so (taking $K = L = 1$),

$$b_{mn} = 4 \int_0^1 \int_0^1 f(x, y) \sin mx \sin ny \, dy \, dx$$

as cofficients will give us the solution.

Problems for chapter 4

(1) Solve the boundary value problem $u_t = 2u_{xx}$, $u(0, t) = u(2, t) = 0$ for $t > 0$, $u(x, 0) = x$ on $0 < x < 2$. Give a physical interpretation of this boundary value problem.

(2) Solve the boundary value problem $u_t = 2u_{xx}$, $u_x(0, t) = u_x(2, t) = 0$ for $t > 0$, $u(x, 0) = x$ on $0 < x < 2$. Give a physical interpretation of this boundary value problem.

(3) Solve the wave equation $u_{tt} = a^2 u_{xx}$ $u(0, t) = u(L, t) = 0$ for $t > 0$, $u(x, 0) = f(x)$, $u_t(x, 0) = 0$, by constructing a series solution. Since you aren't told what f is, you will have to leave the coefficients in terms of f as in example 14.

(4) Solve the wave equation $u_{tt} = a^2 u_{xx}$ $u(0, t) = u(L, t) = 0$ for $t > 0$, $u(x, 0) = 0$, $u_t(x, 0) = g(x)$, by constructing a series solution. Since you aren't told what g is, you will have to leave the coefficients in terms of g.

(5) Solve the wave equation in the square $0 < x < 1$, $0 < y < 1$

$$u_{tt} = a^2(u_{xx} + u_{yy})$$

with boundary conditions $u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0$, and initial conditions $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = 0$. (Physically, this is a vibrating drum with a square head.)

CHAPTER 5

Bessel Functions

Bessel's equation of order $\nu \geq 0$ is this ODE:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0. \tag{1}$$

(Note that this is not great terminology: it's a second order ODE no matter what ν is.) We'll see where these come from a little later, and their connection to PDE's. First, though, a few facts. In general, you expect a second order ODE to have two linearly independent solutions. That's the case here, although when $x = 0$, the coefficients of the ODE are zero and some singular stuff can happen. The general solution to the equation is $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$, where $J_\nu(x)$ is the Bessel function of the first kind of order ν and $Y_\nu(x)$ is the Bessel function of the second kind of order ν. So, what are these? It turns out that there is only one solution (up to a constant) which is bounded at $x = 0$: that's J_ν. The other guy blows up like natural log of x or a negative power of x (depending on ν) at $x = 0$: that's Y_ν. By assuming that J_ν has a power series expression and plugging that into Bessel's equation, you can figure out what the power series must be. This is the method of Frobenius. We won't go through that, though, and instead just take these as new functions. Both J_ν and Y_ν have an infinite number of roots.

The power series expansion for J_ν is

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \left\{ 1 - \frac{x^2}{2(2\nu+2)} + \frac{x^4}{2 \cdot 4 \cdot (2\nu+2)(2\nu+4)} + \cdots \right\}, \tag{2}$$

where Γ is the gamma function, which is a generalization of factorial. In particular, if ν is a positive integer, $\Gamma(\nu+1) = \nu!$. (However, the gamma function is defined for other numbers as well, but we won't get into that.) Expansions for Y_ν exist but are a bit messier, and I'll skip that.

A slight variation of Bessel's equation is

$$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right) y = 0. \tag{3}$$

This turns out to have general solution $c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$. You will show this in problem 2.

So, where do these come from in practice? Basically, they arise from putting PDE's which involve the Laplacian into polar or cylindrical coordinates. It's not worth getting into the derivation, but in polar coordinates, $\nabla^2 u = \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r + u_{rr}$. (If you want to try this at home, use $x = r\cos\theta$, $y = r\sin\theta$ and the chain rule:

$u_r = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} = u_x \cos\theta + u_y \sin\theta$, find u_{rr} and $u_{\theta\theta}$ similarly, plug in to the right side and it eventually simplifies to $u_{xx} + u_{yy}$.)

EXAMPLE 16. *A circular plate of radius 1 has initial temperature $F(r)$ (so that there's no theta dependence), and the rim is kept at temperature zero. Find the temperature at time t. (Assume that the faces are insulated, so that it's strictly a problem in the plane.) We won't be able to finish this problem yet, but it will guide us in a useful direction.*

The temperature is independent of θ, so the boundary value problem for determining the temperature $u(r,t)$ is

$$u_t = \kappa \left(u_{rr} + \frac{1}{r} u_r \right),$$

with $u(r,0) = F(r)$ and $u(1,t) = 0$. This separates: let $u = R(r)T(t)$ to get $RT' = \kappa \left(R''T + \frac{1}{r} R'T \right)$, so that

$$\frac{T'}{\kappa T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2.$$

The constant of separation is chosen to be negative so that the temperature remains bounded (otherwise it would grow exponentially). Also, $-\lambda^2$ goes well with Bessel functions. From this, $T' = -\lambda^2 \kappa T$ and $R'' + \frac{1}{r} R' + \lambda^2 R = 0$, where the second equation can be written as $r^2 R'' + rR + (\lambda^2 r^2 - 0^2)R = 0$. These have general solutions $T = C_1 e^{-\kappa \lambda^2 t}$ and $R = A_1 J_0(\lambda r) + B_1 Y_0(\lambda r)$. Since we require the solution to be bounded at $r = 0$, we must have $B_1 = 0$, so that a general separated solution must be $u(r,t) = Ae^{-\kappa\lambda^2 t} J_0(\lambda r)$.

Now, we would like to construct a series solution, and require each term of the series to always be zero on the edge of the plate at $r = 1$ (so that their sum will automatically be zero at $r = 1$). To have $u(1,t) = Ae^{-\kappa\lambda^2 t} J_0(\lambda)$ to always be zero, we need λ such that $J_0(\lambda) = 0$. Call the positive roots of J_0 λ_1, λ_2, and so on, we get that a separated solution which is always zero on the edge of the plate must be $Ae^{-\kappa\lambda_m^2 t} J_0(\lambda_m r)$. Therefore the series solution must be

$$u(r,t) = \sum_{m=1}^{\infty} A_m e^{-\kappa\lambda_m^2 t} J_0(\lambda_m r).$$

To satisfy the initial condition, we need to have

$$\sum_{m=1}^{\infty} A_m J_0(\lambda_m r) = F(r). \tag{4}$$

This is an example of a *Fourier-Bessel* series. To come up with a formula for the coefficients A_m we will have to develop the theory of these series a bit.

More generally, we will want to write a function $f(r)$ on an interval $0 < r < b$ as a series $\sum_{m=1}^{\infty} A_m J_n(\lambda_m r)$, where λ_m will be the zeroes of Bessel function J_n divided by b. At least for now, n will be fixed (in the plate example, it was 0). Thinking back to Fourier series, we were able to find the coefficients by using the inner product $\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x)g(x)\,dx$, and the fact that $\left\{ \frac{1}{\sqrt{2}}, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \right\}$ is an orthonormal set. What we need is the correct inner product and orthonormal basis for the set of functions $\{J_n(\lambda_m r)\}$.

Orthogonality of Bessel functions

To make things specific, I'll look at the interval $0 < r < 1$; for the interval $0 < r < b$ we can just scale. The result is that for $\lambda \neq \mu$,

$$\int_0^1 rJ_n(\lambda r)J_n(\mu r)\,dr = \frac{\mu J_n(\lambda)J_n'(\mu) - \lambda J_n(\mu)J_n'(\lambda)}{\lambda^2 - \mu^2}. \tag{5}$$

The reason is this: let $y_1 = J_n(\lambda x)$ and $y_2 = J_n(\mu x)$. From (3), we know that

$$r^2 y_1'' + ry_1' + (\lambda^2 r^2 - n^2)y_1 = 0$$

and

$$r^2 y_2'' + ry_2' + (\mu^2 r^2 - n^2)y_2 = 0.$$

Multiply the first equation by y_2, the second equation by y_1 and subtract to get

$$r^2(y_2 y_1'' - y_1 y_2'') + r(y_2 y_1' - y_1 y_2') = (\mu^2 - \lambda^2)r^2 y_1 y_2,$$

which, if you divide by r, can be written as

$$\frac{d}{dr}\left(r(y_2 y_1' - y_1 y_2')\right) = (\mu^2 - \lambda^2) r y_1 y_2.$$

Integrate both sides from 0 to 1:

$$1\cdot(y_2(1)y_1'(1) - y_1(1) - y_2'(1)) - 0\cdot(y_2(0)y_1'(0) - y_1(0) - y_2'(0)) = (\mu^2 - \lambda^2)\int_0^1 ry_1(r)y_2(r)\,dr.$$

Plug in what y_1 and y_2 and the result follows.

From (5): if λ and μ are both zeroes of J_n, then

$$\int_0^1 rJ_n(\lambda r)J_n(\mu r)\,dr = 0,$$

i.e., $J_n(\lambda r)$ and $J_n(\mu r)$ are orthogonal with respect to the inner product $\langle f, g\rangle = \int_0^1 rfg\,dr$.

We'll also need the inner product of $J_n(\lambda r)$ with itself. Taking the limit of (5) as μ approaches a fixed λ, and using l'Hôpital's rule, we get

$$\int_0^1 rJ_n^2(\lambda r)\,dr = \frac{1}{2}\left({J_n'}^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right)J_n^2(\lambda)\right), \tag{6}$$

so that if λ is a root of J_n,

$$\int_0^1 rJ_n^2(\lambda r)\,dr = \frac{{J_n'}^2(\lambda)}{2}. \tag{7}$$

It turns out that by looking at the power series for Bessel functions (in a process similar to problem 1 in the homework) that one can show the following identity:

$$rJ_n'(r) = nJ_n(r) - rJ_{n+1}(r),$$

so that if λ is a root of J_n, we have $J_n'(\lambda) = -J_{n+1}(\lambda)$, so that we can write (7) as

$$\int_0^1 rJ_n^2(\lambda r)\,dr = \frac{J_{n+1}^2(\lambda)}{2}.,$$

which is what one generally sees in references, and is a little more convenient, since software generally has Bessel functions easily available, and not their derivatives.

The point of all of this is to find coefficients in a Fourier-Bessel series such as (4). Suppose that

$$F(r) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p r)$$

on $(0,1)$, where λ_p is the p^{th} root of J_n. Then formally, i.e., ignoring any questions about convergence, if λ_k is the k^{th} root of J_n,

$$\int_0^1 rF(r)J_n(\lambda_k r)\,dr = \sum_{p=1}^{\infty} A_p \int_0^1 rJ_n(\lambda_k r)J_n(\lambda_p r)\,dr = A_k \int_0^1 rJ_n^2(\lambda_k r)\,dr = A_k \frac{J_{n+1}^2(\lambda_k)}{2},$$

since all other terms are zero. We can solve this for the coefficients:

$$A_k = 2\frac{\int_0^1 rF(r)J_n(\lambda_k r)\,dr}{J_{n+1}^2(\lambda_k)}. \tag{8}$$

The actual theorem is parallel to the theorem for Fourier series: if f and f' are piecewise continuous on $(0,1)$, then the Fourier-Bessel series with coefficients given by (8) will converge to $f(x)$ at every x for which f is continuous, and at jump discontinuities it will converge to the average of the left and right hand limits.

Back to example 16. Given an initial temperature $F(r)$, we need to write F as the Fourier-Bessel series $\sum_{m=1}^{\infty} A_m J_0(\lambda_m r)$, using (8), and the solution will be

$$u(r,t) = \sum_{m=1}^{\infty} A_m e^{-\kappa\lambda_m^2 t} J_0(\lambda_m r).$$

Unfortunately, unlike Fourier series, there's really no way to evaluate the integrals in (8) in general. However, we can certainly do these numerically.

Vibrating drum example

EXAMPLE 17. *Suppose that I have a drum with a circular head, I push the head out a little so that it has an initial displacement (**not** assumed to be radially symmetric), and then let it go (starting with zero as initial velocity). What happens?*

Let's be a little more precise. Put polar coordinates on the drum head, assume that it has radius 1. Then the wave equation becomes

$$u_{tt} = a^2\left(\frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + u_{rr}\right),$$

where $u(r,\theta,t)$ is the displacement of the drum head at location r, θ, and time t. The boundary conditions are that $u(1,\theta,t) = 0$ for all t. The initial conditions are that $u(r,\theta,0)$ is some specified $F(r,\theta)$, and that $u_t(r,\theta,0) = 0$. We also assume that the displacement remains bounded as t goes to infinity. Let's try to separate variables. Write u as $R(r)\,\Theta(\theta)\,T(t)$ and get

$$R\Theta T'' = a^2\left(\frac{R\Theta'' T}{r^2} + \frac{R'\Theta T}{r} + R''\Theta T\right).$$

Divide through by $a^2 R\Theta T$ to get

$$\frac{T''}{a^2 T} = \frac{\Theta''}{r^2\Theta} + \frac{R'}{rR} + \frac{R''}{R}.$$

The left side depends only on t, the right on r and θ, so they must be a constant. Moreover, the constant must be negative, or otherwise T will in general grow exponentially: you get sinh's and cosh's. Thus

$$\frac{T''}{a^2T} = -\lambda^2$$

and

$$\frac{\Theta''}{r^2\Theta} + \frac{R'}{rR} + \frac{R''}{R} = -\lambda^2.$$

The second equation will separate again, since

$$-\frac{\Theta''}{\Theta} = \frac{r^2R''}{R} + \frac{rR'}{R} + \lambda^2r^2 = \mu^2.$$

Here the constant of separation is positive so that we'll get Θ as a linear combination of sines and cosines, so that $\Theta(\theta)$ can be periodic in θ. After all, if we increase θ by 2π we get back to the same point, and therefore expect the same displacement. The equations for Θ and R are:

$$\Theta'' = -\mu^2\Theta$$

and

$$r^2R'' + rR' + \left(\lambda^2r^2 - \mu^2\right)R = 0.$$

The general solutions are $T = A_1\cos\lambda at + B_1\sin\lambda at$, $\Theta = A_2\cos\mu\theta + B_2\sin\mu\theta$, and $R = A_3J_\mu(\lambda r) + B_3Y_\mu(\lambda r)$, and the separated solution is a product of these. As I said above, we expect Θ to be periodic with period 2π. That doesn't have to be its least period, but certainly the least period is $\frac{2\pi}{m}$ for some positive integer m. This gives us that $\mu = m$, where $m = 0, 1, \cdots$. ($m = 0$ is a special case: then Θ is constant, which can certainly happen. This corresponds to no θ dependence in the solution, which means it is the same in every direction). Also, B_3 must be zero, since Y_μ is unbounded at the origin, and we'd poke a hole through the drum if that term is in there. To get u_t to be zero at $t = 0$, we want B_1 to be zero. Finally, to get $J_m(\lambda r)$ to be zero at $r = 1$, we need λ to be a root of J_m. We define λ_{mk} to be the k^{th} positive root of J_m, so $\lambda = \lambda_{mk}$.

Thus a separated solution must be

$$J_m(\lambda_{mk}r)\cos\lambda_{mk}at\,(A\cos m\theta + B\sin m\theta).$$

These are the various modes of vibration of the drumhead. The frequency of vibration for each mode is $f_{mk} = \frac{\lambda_{mk}}{2\pi}a$ (this is the value of t which increases the angle by 2π, so that you repeat in time by this). Since these aren't in general integral multiples of the lowest frequency, a drum won't sound as musical as, say, a violin. A general solution will be a sum of these, chosen so that the initial conditions are correct.

The general solution will be

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} J_m(\lambda_{mk}r)\cos\lambda_{mk}at\,(A_{mk}\cos m\theta + B_{mk}\sin m\theta),$$

Where we have to choose A_{mk} and B_{mk} so that, at $t = 0$, we start with the given initial conditions, i.e., we need

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} J_m(\lambda_{mk}r)(A_{mk}\cos m\theta + B_{mk}\sin m\theta) = F(r,\theta).$$

(we already have $u_t = 0$ at $t = 0$ automatically, by choosing the value of B_1 in the formula for $T(t)$ to be zero.) Write the sum as

$$\begin{aligned}F(r,\theta) &= \sum_{m=0}^{\infty}\left(\sum_{k=1}^{\infty} A_{mk} J_m(\lambda_{mk} r)\right)\cos(m\theta) + \left(\sum_{k=1}^{\infty} B_{mk} J_m(\lambda_{mk} r)\right)\sin(m\theta).\\ &= \sum_{m=0}^{\infty} C_m \cos(m\theta) + D_m \sin(m\theta)\end{aligned}$$

First, think of this as a Fourier series (i.e., focus on the m sum and not the k sum). From the theory of Fourier series, (holding r as a constant for the moment)

$$C_m(r) = \begin{cases} \frac{1}{\pi}\int_{-\pi}^{\pi} F(r,\theta)\cos m\theta\, d\theta, & m = 1, 2, \cdots \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} F(r,\theta)\, d\theta & m = 0 \end{cases}$$

and

$$D_m(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(r,\theta)\sin m\theta\, d\theta.$$

Now use the results for Fourier-Bessel series:

$$\begin{aligned}A_{mk} &= \frac{2}{J_{m+1}(\lambda_{mk})^2}\int_0^1 r J_m(\lambda_{mk} r) C_m(r)\, dr \\ &= \begin{cases} \frac{2}{\pi(J_{m+1}(\lambda_{mk})^2})\int_0^1\int_{-\pi}^{\pi} rF(r,\theta)J_m(\lambda_{mk} r)\cos m\theta\, dr\, d\theta, & m = 1, 2, \cdots \\ \frac{1}{\pi(J_{m+1}(\lambda_{0k})^2})\int_0^1\int_{-\pi}^{\pi} rF(r,\theta)J_0(\lambda_{0k} r)\, dr\, d\theta, & m = 0 \end{cases}\end{aligned}$$

and

$$\begin{aligned}B_{mk} &= \frac{2}{J_{m+1}(\lambda_{mk})^2}\int_0^1 r J_m(\lambda_{mk} r) D_m(r)\, dr \\ &= \frac{2}{\pi(J_{m+1}(\lambda_{mk}))^2}\int_0^1\int_{-\pi}^{\pi} rF(r,\theta)J_m(\lambda_{mk} r)\sin m\theta\, dr\, d\theta.\end{aligned}$$

This is animated in "Drum.mws".

Problems for chapter 5

(1) Use equation (2) to show that, for positive integers n,

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x).$$

(2) Verify that $c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$ solves equation (3).

(3) Determine the solution $u(r,t)$ to the boundary value problem $u_t = u_{rr} + \frac{1}{r}u_r$, $u(1,t) = 0$ for all $t > 0$, $u(r,t)$ is bounded as $r \to 0$, and $u(r,0) = J_0(\lambda_1 r) + 2J_0(\lambda_2 r)$. Here J_0 is the zeroth order Bessel function of the first kind, and λ_n is its n^{th} positive root.

(4) Determine the solution $u(r,t)$ to the boundary value problem $u_{tt} = u_{rr} + \frac{1}{r}u_r$, $u(1,t) = 0$ for all $t > 0$, $u(r,t)$ is bounded as $r \to 0$, $u(r,0) = J_0(\lambda_1 r) + 4J_0(\lambda_2 r)$, and $u_t(r,0) = 0$.

(5) Suppose that the initial temperature of the plate in example 16 is $F(r,\theta)$, in other words, it depends on θ as well as r. Find a series solution to this problem, giving formulas for the coefficients to the series as in example 17.

(6) A solid cylinder of height and radius 1 and with diffusivity κ is initially at temperature $f(r, z)$ (i.e., no θ dependence). The entire surface (both circular ends and the cylindrical side) is suddenly lowered to temperature zero and kept at that temperature. Determine the temperature $u(r, z, t)$ of the cylinder for all $t \geq 0$.

