

“Exact” and Numerical Methods in Casimir Calculations

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Introduction

Recently, there has been a flurry of papers concerning “exact” methods of calculating Casimir energies or forces between arbitrary distinct bodies. Most notable is the recent paper by **Emig, Graham, Jaffe, and Kardar**, “Casimir forces between arbitrary compact objects,” [arXiv:0707.1862](https://arxiv.org/abs/0707.1862) [cond-mat.stat-mech].



This paper has spawned responses noting that the methods are not so novel: **Duplantier**, stating that the idea was explicit in his famous papers with Balian (1977); **Barton**, pointing out precursor in Sommerfeld (1909); and most explicitly, the appearance of a earlier drafted paper by **Kenneth and Klich**, “Casimir forces in a T operator approach.” [arXiv:0707.4017](https://arxiv.org/abs/0707.4017) [quant-ph]

Green's Function Approach

We agree with these critiques, as to novelty of the formulation, and note that indeed, the derivation of the chief result of Emig et al. is much more general than that given in their paper. In fact, it is a consequence of the general formula for Casimir energies (for simplicity here we restrict attention to a massless scalar field)(τ is the “infinite” time that the configuration exists)[Schwinger, 1975]

$$E = \frac{i}{2\tau} \text{Tr} \ln G,$$

where G is the Green's function satisfying (matrix notation)

$$(-\partial^2 + V)G = 1,$$

subject to some boundary conditions at infinity.

Derivation

Start from the vacuum amplitude in terms of sources,

$$\langle 0_+ | 0_- \rangle^K = e^{iW[K]},$$

$$W[K] = \frac{1}{2} \int (dx)(dx') K(x) G(x, x') K(x').$$

From this the effective field is

$$\phi(x) = \int (dx') G(x, x') K(x').$$

If the geometry of the region is altered slightly, as through moving one of the bounding surfaces, the vacuum amplitude is altered:

$$\begin{aligned}\delta W[K] &= \frac{1}{2} \int (dx)(dx') K(x) \delta G(x, x') K(x') \\ &= -\frac{1}{2} \int (dx)(dx') \phi(x) \delta G^{-1}(x, x') \phi(x'),\end{aligned}$$

$$GG^{-1} = 1.$$

Effective two-particle source

Upon comparison with the two particle term in

$$\begin{aligned} e^{iW[K]} &= e^{i \int (dx) K(x) \phi(x) + i \int (dx) \mathcal{L}} \\ &= \cdots + \frac{1}{2} \left[i \int (dx) K(x) \phi(x) \right]^2, \end{aligned}$$

we deduce

$$iK(x)K(x') \Big|_{\text{eff}} = -\delta G^{-1}(x, x').$$

Thus the change in the generating functional is

$$\begin{aligned}\delta W &= \frac{i}{2} \int (dx)(dx') G(x, x') \delta G^{-1}(x', x) \\ &= -\frac{i}{2} \int (dx)(dx') \delta G(x, x') G^{-1}(x', x).\end{aligned}$$

From this, in matrix notation

$$\delta W = -\frac{i}{2} \text{Tr} \ln G \Rightarrow E = \frac{i}{2\tau} \text{Tr} \ln G.$$

Ambiguity in formula

The above formula for the Casimir energy is defined up to an infinite constant, which can be at least partially compensated by inserting a factor as do Kenneth and Klich:

$$E = \frac{i}{2\tau} \text{Tr} \ln GG_0^{-1}.$$

Here G_0 satisfies, with the same boundary conditions as G , the free equation

$$-\partial^2 G_0 = 1.$$

T-matrix

Now we define the *T*-matrix,

$$T = S - 1 = V(1 + G_0V)^{-1}.$$

- The following is just standard scattering theory *a la* Lippmann-Schwinger (1950).
- There seem to be some sign and ordering errors in Kenneth & Klich.

Expression for Green's function

The Green's function can be alternatively written as

$$\begin{aligned} G &= G_0 - G_0 T G_0 \\ &= \frac{1}{1 + G_0 V} G_0 = V^{-1} T G_0, \end{aligned}$$

which results in two formulæ for the Casimir energy

$$E = \frac{i}{2\tau} \text{Tr} \ln \frac{1}{1 + G_0 V} = \frac{i}{2\tau} \text{Tr} \ln V^{-1} T.$$

Interaction between two potentials

If the potential has two disjoint parts,

$$V = V_1 + V_2$$

it is easy to show that

$$T = (V_1 + V_2)(1 - G_0 T_1)(1 - G_0 T_1 G_0 T_2)^{-1}(1 - G_0 T_2),$$

where

$$T_i = V_i(1 + G_0 V_i)^{-1}, \quad i = 1, 2.$$


Interaction in terms of T_i or G_i

Thus, we can write the general expression for the interaction between the two bodies (potentials) in two alternative forms:

$$\begin{aligned} E_{12} &= -\frac{i}{2\tau} \text{Tr} \ln(1 - G_0 T_1 G_0 T_2) \\ &= -\frac{i}{2\tau} \text{Tr} \ln(1 - V_1 G_1 V_2 G_2), \end{aligned}$$

where

$$G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.$$



The first form is exactly that given by Emig et al., and by Kenneth and Klich, while the latter is actually easily used if we know the individual Green's functions.

Green's functions for δ -plates

We now use the second formula above to calculate the Casimir energy between two parallel semitransparent plates, with potential

$$V = \lambda\delta(z - z_1) + \lambda\delta(z - z_2).$$

The free reduced Green's function is

$$g(z, z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|}, \quad \kappa^2 = \zeta^2 + k^2.$$

Here $\mathbf{k} = \mathbf{k}_\perp$ and $\zeta = -i\omega$ is the Euclidean frequency.

The Green's function associated with a single potential is

$$g_i(z, z') = \frac{1}{2\kappa} \left(e^{-\kappa|z-z'|} - \frac{\lambda}{\lambda + 2\kappa} e^{-\kappa|z-z_i|} e^{-\kappa|z'-z_i|} \right).$$

Casimir interaction between δ -plate

Then the energy/area is ($a = |z_2 - z_1|$)

$$E_1 = \frac{1}{16\pi^3} \int d\zeta \int d^2k \int dz \ln(1 - A)(z, z),$$

$$A(z, z') = \frac{\lambda^2}{4\kappa^2} \delta(z - z_1) \left(1 - \frac{\lambda}{\lambda + 2\kappa} \right) e^{-\kappa|z_1 - z_2|}$$

$$\times \left(1 - \frac{\lambda}{\lambda + 2\kappa} \right) e^{-\kappa|z' - z_2|}$$

$$= \left(\frac{\lambda}{\lambda + 2\kappa} \right)^2 e^{-\kappa a} e^{-\kappa|z' - z_2|} \delta(z - z_1).$$

Casimir Energy

We expand the logarithm according to

$$\ln(1 - A) = - \sum_{n=1}^{\infty} \frac{A^n}{n}.$$

For example the leading term is easily seen to be ($a = |z_2 - z_1|$)

$$E^{(2)} = - \frac{\lambda^2}{16\pi^3} \int \frac{d\zeta d^2k}{4\kappa^2} e^{-2\kappa a} = - \frac{\lambda^2}{32\pi^2 a},$$

which uses the change to polar coordinates,

$$d\zeta d^2k = d\kappa \kappa^2 d\Omega.$$

General λ

In general, it is easy to check that, because $A(z, z')$ factorizes here, $A(z, z') = B(z)C(z')$,

$$\text{Tr} \ln(1 - A) = \ln(1 - \text{Tr} A),$$

so the Casimir interaction between the two semitransparent plates is

$$E = \frac{1}{4\pi^2} \int_0^\infty d\kappa \kappa^2 \ln \left(1 - \left[\frac{\lambda}{\lambda + 2\kappa} e^{-\kappa a} \right]^2 \right),$$

which is exactly the well-known result.

Multipole expansion

To proceed to apply this method to general bodies, Emig et al. revert to an even older technique, the multipole expansion. Let's illustrate this with a $2 + 1$ dimensional version, which allows us to describe cylinders with parallel axes. We seek an expansion of the free Green's function

$$\begin{aligned} G_0(\mathbf{R} + \mathbf{r}' - \mathbf{r}) &= \frac{e^{i|\omega||\mathbf{r} - \mathbf{R} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{R} - \mathbf{r}'|} \\ &= \int \frac{dk_z}{2\pi} e^{ik_z(z - Z - z')} g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp), \end{aligned}$$

Reduced Green's function

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \int \frac{(d^2 k_\perp)}{(2\pi)^2} \frac{e^{-i\mathbf{k}_\perp \cdot \mathbf{R}_\perp} e^{i\mathbf{k}_\perp \cdot (\mathbf{r}_\perp - \mathbf{r}'_\perp)}}{k_\perp^2 + k_z^2 + \zeta^2}.$$

As long as the two potentials do not overlap, so that we have $\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp \neq 0$, we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I'_m(\kappa r') e^{-im'\phi'} \\ \times \tilde{g}_{m,m'}^0(\kappa R), \quad \kappa^2 = k_z^2 + \zeta^2.$$

Expression for $g_{m,m'}^0$

By Fourier transforming, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} d\phi e^{-im\phi} e^{ikr \cos \phi},$$

we easily find

$$\tilde{g}_{m,m'}^0(\kappa R) = \frac{1}{2\pi} \int \frac{dk k}{k^2 + \kappa^2} J_{m-m'}(kR) \frac{J_m(kr) J_m(kr')}{I_m(\kappa r) I_m(\kappa r')},$$

which is in fact independent of r, r' .

Discrete matrix realization

Thus we can derive an expression for the interaction between two bodies, in terms of discrete matrices,

$$\frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2} \int d\zeta dk_z \ln \det \left(1 - \tilde{g}^0 \tilde{T}_1 \tilde{g}^0 \tilde{T}_2 \right),$$

where the \tilde{T} matrix elements are given by

$$\begin{aligned} \tilde{T}_{mm'} = & \int dr r d\phi \int dr' r' d\phi' I_m(\kappa r) e^{-im\phi} I_{m'}(\kappa r') e^{im'\phi'} \\ & \times T(r, \phi; r', \phi'). \end{aligned}$$

Numerics of multipole expansion

- This method can be used, by successive truncation of the matrices, to develop a systematic multipole expansion of the Casimir interaction between the bodies, which is quite rapidly convergent.
- Precisely the analogous expansion can be developed in 3 dimensions, in terms of spherical Bessel functions. This was what was used in the Emig et al., and Kenneth and Klich papers.
- We are further exploring this technique.

Capasso Method

- A. Rodrigues, M. Ibanescu, D. Iannuzzi, F. Capasso, J. D. Joannopoulos, and S. G. Johnson, “Computation and visualization of Casimir forces in arbitrary geometries: non-monotonic lateral forces and failure proximity force approximations,” arXiv:0704.1890v2.
- Calculate force from stress tensor from

$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}') \rangle = \frac{\omega^2}{c^2} \mathbf{G}(\mathbf{r} - \mathbf{r}', i\omega)$$

$$\langle \mathbf{H}(\mathbf{r})\mathbf{H}(\mathbf{r}') \rangle = \nabla \times \mathbf{G}(\mathbf{r} - \mathbf{r}', i\omega) \times \overleftarrow{\nabla'};$$

- the Green's dyadic satisfies the equation

$$\left(\nabla \times \nabla \times + \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, i\omega) \right) \mathbf{G}(\mathbf{r} - \mathbf{r}', i\omega) = \mathbf{1} \delta(\mathbf{r} - \mathbf{r}').$$

- This method is completely standard, indeed Schwingerian, in the cases where G is known exactly.
- However, Capasso et al. use a numerical engineering method: finite-difference frequency-domain methods are employed in 2-dimensions to obtain forces between metal squares and plates to 3% accuracy, “using reasonable computational resources.”
- Problem: Scalability?

Earlier Emig method

- R. Büscher and T. Emig, “Geometry and Spectrum of Casimir Forces,” Phys. Rev. Lett. **94**, 133901 (2005); T. Emig, R. L. Jaffe, M. Kardar, and A. Scardicchio, “Casimir Interaction between a Plate and a Cylinder, Phys. Rev. Lett. **96**, 080403 (2005)
- Then from the change in the density of states,

$$E = \frac{\hbar c}{2} \int_0^\infty dq q \delta\rho(q),$$
$$\delta\rho(q) = -\frac{1}{\pi} \frac{\partial}{\partial q} \text{Tr} \ln \mathcal{M} \mathcal{M}_\infty^{-1},$$

where the matrix operator \mathcal{M} is given by the Euclidean Green's function

$$G_0(\mathbf{x}, \mathbf{x}', q) = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} e^{-q|\mathbf{x} - \mathbf{x}'|}$$

evaluated on the (Dirichlet, for example) surfaces. \mathcal{M}_∞ is defined at infinite surface separation.

Applications

- Emig first used this method to calculate the force between corrugated surfaces.
- He later used the technique to calculate the exact force between a cylinder and a plate. The determinant is obtained by a truncation on partial waves; $l = 25$ is sufficient even for $a/R = 0.1$. Strong deviation from the PFA is seen for $a/R \geq 1$. (As Geis and Klingmüller note, 1% deviations from the PFA occur when $a/R \sim 0.01$.) (a is the distance between the cylinder and plate, R is the radius of the cylinder.)

Bulgac et al. method

- A. Bulgac, P. Marierski, and A. Wirzba, “Scalar Casimir effect between Dirichlet spheres or a plate and a sphere,” Phys. Rev. D 73, 025007 (2006)
- They use a modified Krein formula for the change in the density of states due to the presence of N scatterers,

$$\delta g(\epsilon) = \frac{1}{2\pi i} \frac{d}{d\epsilon} \ln \det S_N(\epsilon),$$

where S_N is the scattering matrix for N point scatterers.

- This leads to

$$E = \frac{\hbar c}{2\pi} \int_0^{\infty} dk_4 \ln \det M(ik_4),$$

where M is the multiple-scattering matrix, an energy integral over the multiple-scattering phase shift.


Applications

- They obtain results for the interaction of two spheres, or a sphere and a plate (Dirichlet).
- “The exact results . . . are easy to calculate and definitely simpler to evaluate than in a path integral approach.”
- “Proximity formula and the semiclassical/orbit approaches are limited to small separations only, typically much smaller than the curvature radii of the two surfaces.”

Bordag method

- M. Bordag, “Casimir effect for a sphere and a cylinder in front of a plane and corrections to the proximity force theorem,” Phys. Rev. D **73**, 125018 (2006); “Generalized Lifshitz formula for a cylindrical plasma sheet in front of a plane beyond proximity force approximation,” Phys. Rev. D **75**, 065003 (2007).
- He rederives the representation for the Casimir energy found by Bulgac et al., and by Emig et al., using a path integral approach:

$$E = \frac{1}{2T} \text{Tr} \ln K,$$


$$K(z, z') = \int dx dy H(x, z) D(x, y) H(y, z'),$$

where H is the surface profile function,

$$H(x, z) = \delta(x - f(z)).$$

Applications

- He obtains an exact expression for the interaction between a cylinder and a plane, with thereby corrections to the PFA, for both TE and TM modes, agreeing with Emig.
- For a sphere and a plane he only obtains the large-separation limit, agreeing with Bulgac.
- He uses the same method to calculate the first correction to the PFA for a cylindrical graphene sheet in front of a flat graphene sheet or dielectric plate.

Dalvit method

- D. A. R. Dalvit, F. C. Lombardo, F. D. Mazzitelli, and R. Onofrio, “Exact Casimir interaction between eccentric cylinders,” Phys. Rev. A 74, 020101(R) (2006); “Exact zero-point interaction energy between cylinders,” New J. Phys. , 240 (2006)
- They use the argument principle to calculate the interaction between conducting cylinders with parallel axes,

$$E_{12} = \frac{\hbar c L}{4\pi} \int_0^{\infty} dy y \ln M(iy),$$



where M is a function which vanishes at the eigenvalues.

- As we've seen above, this is an exactly solvable problem.


Multiple-Scattering Approach

- R. Balian and B. Duplantier, “Electromagnetic waves near perfect conductors. I. Multiple scattering expansions and distributions of modes,” *Ann. Phys. (N.Y.)* **104**, 300 (1977); “Electromagnetic waves near perfect conductors. II. Casimir effect,” *ibid.* **112**, 165 (1978); “Geometry of the Casimir effect, [quant-ph/0408124](https://arxiv.org/abs/quant-ph/0408124).
- They famously confirmed Boyer’s result for the Casimir self-repulsion of a perfectly conducting spherical shell.

- Most importantly, they derive a multiple scattering formalism that is completely equivalent to the recent Emig et al. approach, at least for the case of perfect conductors.

Comments and Prognosis

- The methods proposed are in fact not particularly novel, and illustrate the ability of physicists to continually rediscover old methods.
- What is new is the ability, largely due to enhancement in computing power and flexibility, to evaluate continuum determinants (or infinitely dimensional discrete ones) accurately numerically.
- This will make it possible to compute Casimir forces for geometries previously inaccessible.

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- Whether this will lead to improved conceptual understanding, or to better comparison with experiment, remains to be seen.