How does Casimir energy fall? Renormalization of Gravitational and Inertial Masses

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Outline

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   - Parallel plates in zero gravity
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5. Report on current work in progress
   - Definition of force
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In the following the superscript (0) will be used to signify that it is the zeroth order effect of gravity.

Formalism in zero gravity

Let us consider a scalar field $\phi(x)$ in the presence of a background $V^{(0)}(x)$ described by the action

$$W^{(0)} = \int d^4x \, \mathcal{L}^{(0)}(\phi(x))$$

written in terms of the Lagrangian density

$$\mathcal{L}^{(0)}(\phi(x)) = -\frac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} V^{(0)}(x) \phi(x)^2.$$

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The background field $V^{(0)}(x)$ is used as a mathematical device to design the plates. This is achieved by a judicious choice of the background field which provides the appropriate boundary conditions to $\phi(x)$.

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The background field $V^{(0)}(x)$ is used as a mathematical device to design the plates. This is achieved by a judicious choice of the background field which provides the appropriate boundary conditions to $\phi(x)$.

Example: So called semitransparent parallel plates are represented by

$$V^{(0)}(x) = \lambda_a\delta(x) + \lambda_b\delta(x-a)$$

Limit $\lambda_{a,b} \to \infty$ leads to Dirichlet boundary conditions.

Zero gravity: Energy momentum tensor

Equation of motion: (Variation in $\phi(x)$)

$$\left[\partial^2 - V^{(0)}(x)\right] \phi(x) = 0$$

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Energy momentum tensor (stress tensor): A class of arbitrary variation in the field when generated due to infinitesimal general coordinate transformation in space time lets us identify the energy-momentum tensor

$$t^{(0)}_{\alpha\beta}(x) = \partial_\alpha \phi(x) \partial_\beta \phi(x) + g_{\alpha\beta} \mathcal{L}^{(0)}(\phi(x))$$

which is symmetric, $t_{\alpha\beta} = t_{\beta\alpha}$.

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Conformal term: Identification of the stress tensor using this prescription is unique only up to an additive term that satisfies $\partial^\beta t^{(0)}_{\alpha\beta} = 0$.

$$t^{(0)}_{\alpha\beta}(x) = \partial_\alpha \phi(x) \partial_\beta \phi(x) + g_{\alpha\beta} \mathcal{L}^{(0)}(\phi(x)) - \theta \left\{ \partial_\alpha \partial_\beta - g_{\alpha\beta} \partial^2 \right\} \phi(x)^2$$

θ is called the conformal coupling constant. The particular choice, $\theta = \frac{1}{6}$, which renders the stress tensor traceless, is called the conformal choice.

Zero gravity: Force

Force density: The field experiences a force due to the presence of the background field, (the plate, or the source.)

$$
\begin{aligned}
f_\alpha^{(0)}(x) &= \partial^\beta t_{\alpha\beta}^{(0)}(x) \\
&= -\frac{1}{2}\phi(x)^2\,\partial_\alpha V^{(0)}(x)
\end{aligned}
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Force on the apparatus: The force density integrated over the complete space time is interpreted to give the total change in four momentum associated with the field

$$\Delta P_\alpha^{(0)} = \int dt \int d^3x\, f_\alpha^{(0)}(x)$$

which is compatible with the definition $\Delta P_\alpha^{(0)} = \int d\sigma^\beta\, t_{\beta\alpha}^{(0)}(x)$, where σ^β is a time-like four vector, if we use Green's theorem.

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Casimir Force: Integration restricted to one plate.

Zero gravity: Green's function

At the one loop level the Green's function $G^{(0)}(x,x')$ satisfying

$$-\left[\partial^2 - V^{(0)}(x)\right] G^{(0)}(x,x') = \delta^4(x-x')$$

is related to the fields by the correspondence

$$\langle T\phi(x)\phi(x')\rangle = \frac{1}{i}\, G^{(0)}(x,x')$$

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$$\langle T\phi(x)\phi(x')\rangle = \frac{1}{i}\, G^{(0)}(x,x')$$

For the case when $V^{(0)}(x)$ has dependence only on z

$$G^{(0)}(x,x') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{-i\omega(t-t')} e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp)} g^{(0)}(z,z')$$

where $\kappa^2 = \mathbf{k}_\perp^2 - \omega^2$ and $g^{(0)}(z,z')$ satisfies

$$-\left[\frac{\partial^2}{\partial z^2} - \kappa^2 - V^{(0)}(z)\right] g^{(0)}(z,z') = \delta(z-z').$$

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$$-\left[\frac{\partial^2}{\partial z^2} - \kappa^2 - V^{(0)}(z)\right] g^{(0)}(z,z') = \delta(z-z').$$

Force using Green's function:

$$\langle f_\alpha^{(0)}(x)\rangle = -\frac{1}{2i}\, G^{(0)}(x,x)\, \partial_\alpha V^{(0)}(x).$$

Zero gravity: Energy density

The energy density of the field is defined to be the $t_{00}(x)$ component of the stress tensor

$$t_{00}^{(0)}(x) = \frac{1}{2}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t} - \frac{1}{2}\phi\frac{\partial^2}{\partial t^2}\phi + (1-4\theta)\,\frac{1}{4}\,\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\left[\phi^2\right]$$

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Energy density in terms of Green's function:

$$t_{00}^{(0)}(x) = \mathcal{E}_{\text{vol}}^{(0)}(x) + (1-4\theta)\,\mathcal{E}_{\text{sur}}^{(0)}(x)$$

where we have introduced the definitions

$$\begin{aligned} \mathcal{E}_{\text{vol}}^{(0)}(x) &= \frac{1}{2i}\left\{\left(\frac{\partial}{\partial t}\frac{\partial}{\partial t'} - \frac{\partial^2}{\partial t^2}\right)G^{(0)}(x,x')\right\}_{x=x'} \\ \mathcal{E}_{\text{sur}}^{(0)}(x) &= \frac{1}{4i}\,\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\,G^{(0)}(x,x). \end{aligned}$$

Zero gravity: Single plate

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Total force on the plate:

$$\frac{F_a^{(0)}}{A} = -\frac{\lambda_a}{2} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \left\{ \frac{\partial}{\partial z} g^{(0)}(z, z) \right\}_{z=z_a}$$

Total energy of the plate:

$$\frac{E_{\text{tot}}^{(0)}}{A} = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \, 2\zeta^2 \int_{-\infty}^{+\infty} dz \, g^{(0)}(z, z)$$

where

$$g^{(0)}(z, z') = \frac{1}{2\kappa} \, e^{-\kappa|z-z'|} - \frac{\lambda_a}{\lambda_a + 2\kappa} \frac{1}{2\kappa} \, e^{-\kappa|z-z_a|} e^{-\kappa|z'-z_a|}.$$

Zero gravity: Single plate (contd.)

Total force on the plate:

$$\frac{F_a^{(0)}}{A} = -\frac{\lambda_a}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}\left\{\frac{\partial}{\partial z}g^{(0)}(z,z)\right\}_{z=z_a} = 0$$

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where

$$\frac{E_{\text{bulk}}}{A} = -\frac{1}{12\pi^2}\int_0^\infty\kappa^3d\kappa\int_{-\infty}^{+\infty}dz$$

$$\frac{E_a^{(0)}}{A} = \frac{1}{12\pi^2}\int_0^\infty\kappa^2d\kappa\,\frac{\lambda_a}{\lambda_a+2\kappa} = \frac{1}{96\pi^2}\int_0^\infty\frac{dy}{y}\,\frac{y^3}{1+\frac{y}{\lambda_a}}.$$

Zero gravity: Parallel plates

Parallel plates at $z = z_a$ and $z = z_b$, $(z_b - z_a = a)$, is described by

$$V^{(0)}(z) = \lambda_a \, \delta(z - z_a) + \lambda_b \, \delta(z - z_b).$$

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Force on apparatus:

$$\frac{F^{(0)}_{\mathrm{a+b}}}{A} = \frac{F^{(0)}_{\mathrm{a}}}{A} + \frac{F^{(0)}_{\mathrm{b}}}{A} = 0$$

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Casimir force:

$$\frac{F^{(0)}_{\text{b}}}{A} = -\frac{F^{(0)}_{\text{a}}}{A} = -\frac{1}{2\pi^2} \int_0^\infty \kappa^3 \, d\kappa \, \frac{1}{2\kappa} \frac{\partial}{\partial a} \ln \Delta^{(0)}.$$

$$\Delta^{(0)} = 1 + \frac{\lambda_a}{2\kappa} + \frac{\lambda_b}{2\kappa} + \frac{\lambda_a}{2\kappa} \frac{\lambda_b}{2\kappa} \left\{ 1 - e^{-2\kappa a} \right\}$$

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$$\frac{F^{(0)}_{\mathrm{b}}}{A} = -\frac{F^{(0)}_{\mathrm{a}}}{A} \quad = \quad -\frac{1}{2\pi^2} \int_0^\infty \kappa^3 \, d\kappa \, \frac{1}{2\kappa} \frac{\partial}{\partial a} \ln \Delta^{(0)}.$$

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Dirichlet limit $(\lambda_{a,b} \to \infty)$:

$$\frac{F^{(0)}_{\mathrm{b,D}}}{A} = -\frac{F^{(0)}_{\mathrm{a,D}}}{A} = -\frac{\pi^2}{480 \, a^4}$$

Zero gravity: Total energy of parallel plates:

Total energy:

$$\frac{E^{(0)}_{\text{tot}}}{A} - \frac{E_{\text{bulk}}}{A} = \frac{E^{(0)}_{\text{a}}}{A} + \frac{E^{(0)}_{\text{b}}}{A} + \frac{E^{(0)}_{\text{cas}}}{A}$$

where

$$\frac{E^{(0)}_{\text{cas}}}{A} = -\frac{1}{12\pi^2}\int_0^\infty \kappa^2\, d\kappa \left[2\kappa a + \frac{1}{1+\frac{\lambda_a}{2\kappa}} + \frac{1}{1+\frac{\lambda_b}{2\kappa}}\right] \frac{1}{2\kappa}\frac{\partial}{\partial a} \ln \Delta^{(0)}.$$

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Total energy:

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where

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Dirichlet limit ($\lambda_{a,b} \to \infty$):

$$\frac{E_{\rm cas,D}^{(0)}}{A} = -\frac{\pi^2}{1440 a^3}.$$

Zero gravity: Total energy of parallel plates:

Total energy:

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where

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Force and Energy:

$$\frac{F_{\text{b}}^{(0)}}{A} = -\frac{\partial}{\partial a}\frac{E_{\text{tot}}^{(0)}}{A}.$$

Hyperbolic motion

Constant acceleration:

$$v^\mu = \frac{dx^\mu}{d\tau} = \gamma(1, \mathbf{v})$$

$$a^\mu = \frac{dv^\mu}{d\tau} = \gamma^4(\mathbf{v} \cdot \mathbf{a})(1, \mathbf{v}) + \gamma^2(0, \mathbf{a})$$

Hyperbolic motion

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\end{aligned}
$$

If **v** is parallel to **a**

$$
a_\mu a^\mu = \gamma^6 |\mathbf{a}|^2 = \frac{1}{\xi^2}
$$

Hyperbolic motion

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\end{aligned}
$$

If $\mathbf{v}$ is parallel to $\mathbf{a}$

$$
a_\mu a^\mu = \gamma^6 |\mathbf{a}|^2 = \frac{1}{\xi^2}
$$

Equation of motion: for $z(0) = \xi$ and $\mathbf{v}(0) = 0$ is a hyperbola

$$
z(t)^2 - t^2 = \xi^2
$$

Rindler coordinates

Hyperbolic motion:

$$z(t)^2 - t^2 = \xi^2$$

Rindler coordinates

Hyperbolic motion:

$$z(t)^2 - t^2 = \xi^2$$

Rindler coordinates (ξ, τ):

$$z = \xi \cosh \tau \qquad \text{and} \qquad t = \xi \sinh \tau$$

and the metric is

$$-ds^2 = -dt^2 + dz^2 + dx^2 + dy^2 = -\xi^2 d\tau^2 + d\xi^2 + dx^2 + dy^2.$$

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Example: Proper distance between two plates

$$(z_b - z_a)^2 - (t_b - t_a)^2 = (\xi_b - \xi_a)^2 \left[\cosh^2 \tau - \sinh^2 \tau\right] = (\xi_b - \xi_a)^2$$

is preserved. Moves *rigidly*.

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is preserved. Moves *rigidly*.
Weak gravity: $\xi \to \infty$ and $\tau \to 0$, such that constructions, $\xi\tau$, and, differences in ξ, remain finite.
In weak gravity we will have $\Delta z = \Delta\xi$ and $t = \xi\tau$.

Formalism including gravity

In terms of the Rindler coordinates we can thus write the action describing the motion of parallel plates moving with constant proper accelerations ξ_a^{-1} and ξ_b^{-1} as

$$W = \int d^4x \sqrt{-g(x)} \mathcal{L}(\phi(x))$$

where $x \equiv (\tau, x, y, \xi)$ represents the coordinates, $d^4x = d\tau\, d\xi\, dx\, dy$ is the coordinate volume element, $g(x) = \det g_{\mu\nu}(x)$ is the determinant of the metric, $g_{\mu\nu}(x) = \text{diag}(-\xi^2, +1, +1, +1)$ defines the metric, and the Lagrangian density is

$$\mathcal{L}(\phi(x)) = -\frac{1}{2} g_{\mu\nu}(x)(\partial^\mu \phi(x))(\partial^\nu \phi(x)) - \frac{1}{2} V(x)\phi(x)^2$$

where

$$V(x) = \lambda_a \delta(\xi - \xi_a) + \lambda_b \delta(\xi - \xi_b).$$

Gravity: Energy momentum tensor

Equation of motion:(Variation in $\phi(x)$)

$$\left[-\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}+\boldsymbol{\nabla}_\perp^2-V(x)\right]\phi(x)=0$$

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Energy momentum tensor:

$$t_{\alpha\beta}(x)=\partial_\alpha\phi(x)\partial_\beta\phi(x)+g_{\alpha\beta}(x)\mathcal{L}(\phi(x))$$

where have chosen the conformal term to be zero. A couple of components relevant in the present calculation are

$$\begin{aligned} t_{33}(x) &= +\frac{1}{2}\frac{1}{\xi^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2-\frac{1}{2}\left(\boldsymbol{\nabla}_\perp\phi\right)^2-\frac{1}{2}V(x)\phi^2 \\ \frac{1}{\xi^2}\,t_{00}(x) &= +\frac{1}{2}\frac{1}{\xi^2}\left(\frac{\partial\phi}{\partial\tau}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2+\frac{1}{2}\left(\boldsymbol{\nabla}_\perp\phi\right)^2+\frac{1}{2}V(x)\phi^2. \end{aligned}$$

Gravity: Force density

The force density in a non Minkowskian metric is given as the covariant derivative of the stress tensor and explicitly reads

$$\begin{aligned} f_\alpha(x) &= t_\alpha{}^\beta{}_{;\beta} \\ &= \frac{1}{\sqrt{-g(x)}}\,\partial^\beta \left\{ \sqrt{-g(x)}\; t_{\alpha\beta}(x) \right\} - \frac{1}{2} t^{\mu\nu}(x) \partial_\alpha g_{\mu\nu}(x) \end{aligned}$$

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\end{aligned}
$$

Force density in the ξ-direction

$$
\begin{aligned}
f_3(\xi) &= \frac{1}{\xi}\left[\frac{1}{\xi^2}\, t_{00}(x) + t_{33}(x)\right] + \frac{\partial}{\partial\xi} t_{33}(x) \\
&= -\frac{1}{2}\phi(x)^2 \frac{\partial}{\partial\xi} V(\xi) + \frac{1}{\xi^2}\frac{\partial}{\partial\tau}\left[\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial\xi}\right] - \boldsymbol{\nabla}_\perp \cdot \left[(\boldsymbol{\nabla}_\perp\phi)\frac{\partial\phi}{\partial\xi}\right]
\end{aligned}
$$

Gravity: Green's function

Green's function equation:

$$-\left[-\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}+\boldsymbol{\nabla}_\perp^2-V(x)\right]G(x,x') = \frac{\delta(\xi-\xi')}{\xi}\delta(\tau-\tau')\delta(\mathbf{x}_\perp-\mathbf{x}'_\perp)$$

Gravity: Green's function

Green's function equation:

$$-\left[-\frac{1}{\xi^2}\frac{\partial^2}{\partial\tau^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}+\boldsymbol{\nabla}_\perp^2-V(x)\right]G(x,x') = \frac{\delta(\xi-\xi')}{\xi}\delta(\tau-\tau')\delta(\mathbf{x}_\perp-\mathbf{x}'_\perp)$$

When $V(x)$ has only ξ dependence

$$G(x,x')=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}e^{-i\omega(\tau-\tau')}e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp)}g(\xi,\xi')$$

where $g(\xi,\xi')$ satisfies

$$-\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}+\frac{\omega^2}{\xi^2}-k_\perp^2-V(x)\right]g(\xi,\xi')=\frac{\delta(\xi-\xi')}{\xi}$$

Gravitational force

Force density in terms of the Green's function

$$f_3(\xi) = -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}g(\xi,\xi)\frac{\partial}{\partial\xi}V(\xi)$$

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Coordinate force:

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where at this point E is introduced to be the coordinate force defined as the change in momentum per unit coordinate time τ. The coordinate force is intentionally, with foresight, denoted as E (for energy).

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Physical force: The physical force on a body (or field) situated at the coordinate ξ_0 as measured by an observer while at the coordinate ξ_0 is

$$F = \frac{E}{\xi_0} = Eg.$$

Single plate falling in a constant gravitational field

A single plate moving with acceleration ξ_a^{-1} is described by

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Weak gravity:

$$\frac{E_3}{A} = \frac{E_\text{a}^{(0)}}{A} + \mathcal{O}\left(\frac{1}{\xi_0}\right)$$

$$F^3 = E_\text{a}^{(0)} g + \mathcal{O}(g)^2.$$

Parallel plates falling in a constant gravitational field

Two plates with accelerations ξ_a^{-1} and ξ_b^{-1} are described by

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where

$$\frac{E^3_{\rm a,b}}{A} = \frac{\lambda_{a,b}}{2} \int_{-\infty}^{+\infty} \frac{d\zeta}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \left\{ \frac{\partial}{\partial \xi} \Big[\xi \, g(\xi, \xi) \Big] \right\}_{\xi=\xi_{a,b}}$$

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Green's function:

$$g(\xi, \xi') = \begin{cases} I_\zeta(k_\perp \xi_<) K_\zeta(k_\perp \xi_>) - \frac{\Delta_1}{\Delta} \, I_\zeta(k_\perp \xi) I_\zeta(k_\perp \xi') & \text{if } \{\xi, \xi'\} < \xi_a < \xi_b \\ I_\zeta(k_\perp \xi_<) K_\zeta(k_\perp \xi_>) - \frac{\Delta_{ij}}{\Delta} \, I^i_\zeta(k_\perp \xi) I^j_\zeta(k_\perp \xi') & \text{if } \xi_a < \{\xi, \xi'\} < \xi_b \\ I_\zeta(k_\perp \xi_<) K_\zeta(k_\perp \xi_>) - \frac{\Delta_2}{\Delta} \, K_\zeta(k_\perp \xi) K_\zeta(k_\perp \xi') & \text{if } \xi_a < \xi_b < \{\xi, \xi'\} \end{cases}$$

where Δ's are functions of ξ_a and ξ_b in terms of $I_{a,b}$ and $K_{a,b}$.

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where $\Delta = 1 + \lambda_a\xi_a I_a K_a + \lambda_b\xi_b I_b K_b + \lambda_a\xi_a\lambda_b\xi_b\left[(I_a K_a)(I_b K_b) - (I_a K_b)^2\right]$

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Weak gravity: $(\xi \to \infty)$

$$\Delta(\xi_a, \xi_b) = \Delta^{(0)}(\xi_b - \xi_a) + \frac{1}{\xi_0} \Delta^{(1)}(\xi_a, \xi_b) + \mathcal{O}\left(\frac{1}{\xi_0}\right)^2$$

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Coordinate force in weak gravity:

$$\frac{E^3}{A} = \frac{E^{(0)}_{\rm a}}{A} + \frac{E^{(0)}_{\rm b}}{A} + \frac{E^{(0)}_{\rm cas}}{A} + \mathcal{O}\left(\frac{1}{\xi_0}\right)$$

$$F^3 = \left[E^{(0)}_{\rm a} + E^{(0)}_{\rm b} + E^{(0)}_{\rm cas}\right] g + \mathcal{O}(g)^2.$$

Renormalization

Thus, the vacuum energy, including the divergent contributions, falls under a weak gravitational field exactly like conventional mass or energy. Thus we can identify the gravitational force on the Casimir apparatus consisting of two plates to be

$$F = \left[m_a + m_b + E_a^{(0)} + E_b^{(0)} + E_{\text{cas}}^{(0)} \right] g + \mathcal{O}(g)^2.$$

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We identify $m_{a,b}$ as the bare mass of the plates, and interpret the divergent Casimir energy contribution from each plate to renormalize the bare mass. Thus we renormalize the masses by the prescription

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2. A related question: What will be the pseudo force exerted on the Casimir apparatus if we rotate it in a circle? Will the Casimir energy experience a centrifugal force as we expect? Will

   $$F = E^{(0)}_{\text{Cas}}\, \omega^2 r + \mathcal{O}(\omega r)^2$$

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   $\Longrightarrow$ This exercise stemmed out as means of having a cross check for our definition of force.

Local inertial frame associated with Rindler space

Rindler space time:

Coordinates $\xi^\mu \equiv (\tau, x, y, \xi)$ Metric $g_{\mu\nu}(\xi) = (-\xi^2, +1, +1, +1)$

Connections $\Gamma^0_{03} = \frac{1}{\xi}$, $\Gamma^3_{00} = \xi$, Curvature $R_{\mu\nu\alpha\beta} = 0$

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Local inertial frame: A freely falling coordinate system (local inertial frame) will be described by the coordinates $x^a \equiv (t, x, y, z)$ and metric $\eta_{ab} = (-1, +1, +1, +1)$ with connections $\Gamma^a_{bc}(x) = 0$.

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LIF corresponding to Rindler space: x^a's are determined by solving the differential equations

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which has solutions

$$t(\xi, \tau) = \xi \sinh \tau$$
$$z(\xi, \tau) = \xi \cosh \tau$$

Definition of Force

Force in LIF:

$$F_a(t) = \frac{d}{dt} P_a(t)$$

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&= -t \int_0^\infty d\xi \, \frac{1}{\xi} \left[\frac{1}{\xi^2} t_{00}(\xi) + t_{33}(\xi) \right] + \mathcal{O}\left(\frac{t}{\xi} \right)^2
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Force on a single particle in Rindler space

As a first check let us determine the force on a single particle in the Rindler space using

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Energy momentum for a single particle in an arbitrary metric space is

$$t^{\mu\nu}(x) = m \frac{1}{\sqrt{-g(x)}} \int_{-\infty}^{+\infty} ds \, \delta^{(4)}(x - x_a(s)) \, \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

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For a single particle only $t^{00}(x)$ is non zero. Using this we get the correct result

$$\frac{F_3}{A} = -m \frac{1}{\xi} + \mathcal{O}(g)^2$$

Centrifugal force on a single particle moving in a circle

As a second check let us determine the centrifugal force on a single particle moving in a circle. A particle moving in a circle is described by the metric

$$-ds^2 = -dt^2(1 - \omega^2 r^2) + dr^2 + 2\omega r^2 dt d\theta + r^2 d\theta^2 + dz^2$$

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Local inertial coordinates for this space are

$$\begin{aligned} x(r,\theta,t) &= r\cos(\theta+\omega t) \\ y(r,\theta,t) &= r\sin(\theta+\omega t) \end{aligned}$$

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Centrifugal force:

$$\mathbf{F}(t) = m \frac{\omega^2 r}{\sqrt{1-\omega^2 r^2}} \hat{\mathbf{r}}$$

Force on a Casimir apparatus

Let us now calculate the gravitational force on a Casimir apparatus using

$$\frac{F_3(t)}{A} = -\int_0^\infty d\xi \, \frac{1}{\xi} \left[\frac{1}{\xi^2} t_{00}(\xi) + t_{33}(\xi) \right] + \mathcal{O}(g)^2$$

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where all terms except $E_{\text{vol,Cas}}^{(0)}$ and $E_{\text{sur,Cas}}^{(0)}$ are divergent.

Explicit expressions for terms in force

$$F = [m_a + m_b + E^{(0)}_{\text{vol,a}} + E^{(0)}_{\text{vol,b}} + E^{(0)}_{\text{vol,Cas}} + E^{(0)}_{\text{sur,a}} + E^{(0)}_{\text{sur,b}} + E^{(0)}_{\text{sur,Cas}}]g + \mathcal{O}(g)^2$$

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Dirichlet limit $(\lambda_{a,b} \to \infty)$

$$E^{(0)}_{\text{vol,Cas}} = -\frac{\pi^2}{1440\, a^3} \qquad \text{and} \qquad E^{(0)}_{\text{sur,Cas}} = 0$$

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6. To do: Centrifugal force on the Casimir apparatus moving in a circle.

The talk was based on:

- S. A. Fulling, K. A. Milton, P. Parashar, A. Romeo, K. V. Shajesh, J. Wagner, Phys. Rev. D **76**, 025004 (2007), arXiv:hep-th/0702091.
- K. A. Milton, P. Parashar, K. V. Shajesh, J. Wagner, J. Phys. A: Math. Theor. **40**, 1 (2007), arXiv:0705.2611.