Section 10.6: Representations of Functions as Power Series

In this section, we learn how to represent certain types of functions as power series by manipulating geometric series.

Recall that by the Geometric Series Test, if $|r| < 1$, then
\[ \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}. \]
Therefore, if $|x| < 1$, then the power series
\[ \sum_{n=0}^{\infty} ax^n = \frac{a}{1 - x}. \]

**Example:** Find the sum of the series $\sum_{n=0}^{\infty} (3x)^n$ and the radius of convergence.

By the Geometric Series Test, if $|3x| < 1$, then
\[ \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1 - 3x}. \]
To find the radius of convergence, set
\[ |3x| < 1 \]
\[ |x| < \frac{1}{3}. \]
The radius of convergence is $R = 1/3$.

**Example:** Find the sum of the series $\sum_{n=0}^{\infty} (2x + 1)(27x^3)^n$ and the radius of convergence.

By the Geometric Series Test, if $|27x^3| < 1$, then
\[ \sum_{n=0}^{\infty} (2x + 1)(27x^3)^n = \frac{2x + 1}{1 - 27x^3}. \]
To find the radius of convergence, set
\[ |27x^3| < 1 \]
\[ |x^3| < \frac{1}{27} \]
\[ |x| < \frac{1}{3}. \]
The radius of convergence is $R = 1/3$.

Example: Find a power series representation for the given function and determine the radius of convergence.

(a) $f(x) = \frac{x}{1 - 5x}$

The function can be represented as

$$\frac{x}{1 - 5x} = x \left[ \frac{1}{1 - (5x)} \right] = x \sum_{n=0}^{\infty} (5x)^n = \sum_{n=0}^{\infty} 5^n x^{n+1}.$$  

The series converges for $|5x| < 1$. Therefore, the radius of convergence is $R = \frac{1}{5}$.

(b) $f(x) = \frac{1}{1 + 4x^2}$

The function can be represented as

$$\frac{1}{1 + 4x^2} = \frac{1}{1 - (-4x^2)} = \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-4)^n x^{2n}.$$  

To find the radius of convergence, set

$$|-4x^2| < 1$$

$$|x|^2 < \frac{1}{4}$$

$$|x| < \frac{1}{2}.$$  

Therefore, the radius of convergence is $R = \frac{1}{2}$.

(c) $f(x) = \frac{x}{x + 3}$

The function can be represented as

$$\frac{x}{3 + x} = \frac{x}{3 \left( 1 + \frac{x}{3} \right)} = \frac{x}{3} \left[ \frac{1}{1 - \left( \frac{x}{3} \right)} \right] = \frac{x}{3} \sum_{n=0}^{\infty} \left( \frac{-x}{3} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{n+1}.$$  

The series converges for $\left| -\frac{x}{3} \right| < 1$. Therefore, the radius of convergence is $R = 3$. 
Theorem: (Differentiation and Integration of Power Series)

If the power series \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) has radius of convergence \( R > 0 \), then

1. \( f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1} \) and has radius of convergence \( R \)

2. \( \int f(x) \, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} \) and has radius of convergence \( R \).

Example: Find a power series representation for the given function and determine the radius of convergence.

(a) \( f(x) = \frac{1}{(3 + x)^2} \)

Integrating the function gives

\[
\int f(x) \, dx = \int \frac{1}{(3 + x)^2} \, dx
= C - \frac{1}{3 + x}
= C - \frac{1}{3} \left( \frac{1}{1 + \frac{x}{3}} \right)
= C - \frac{1}{3} \left[ \frac{1}{1 - \left( -\frac{x}{3} \right) } \right]
= C + \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right) \left( -\frac{x}{3} \right)^n
= C + \sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^{n+1} x^n.
\]

Therefore,

\[
f(x) = \frac{d}{dx} \left[ C + \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^{n+1} x^n \right] = \sum_{n=1}^{\infty} \left( -\frac{1}{3} \right)^{n+1} n x^{n-1}.
\]

The series converges for \( \left| -\frac{x}{3} \right| < 1 \). Thus, the radius of convergence is \( R = 3 \).
(b) \( f(x) = \frac{x^2}{(1 - 3x)^2} \)

Let \( g(x) = \frac{1}{(1 - 3x)^2} \). Then

\[
\int g(x) \, dx = \int \frac{1}{(1 - 3x)^2} \, dx
\]
\[
= -\frac{1}{3} \int u^2 \, du
\]
\[
= C + \frac{1}{3u}
\]
\[
= C + \frac{1}{3(1 - 3x)}
\]
\[
= C + \sum_{n=0}^{\infty} \frac{1}{3}(3x)^n
\]
\[
= C + \sum_{n=0}^{\infty} 3^{n-1} x^n.
\]

Therefore,

\[
g(x) = \frac{d}{dx} \left[ C + \sum_{n=0}^{\infty} 3^{n-1} x^n \right] = \sum_{n=1}^{\infty} 3^{n-1} n x^{n-1}.
\]

So a power series for \( f(x) \) is,

\[
f(x) = x^2 g(x) = \sum_{n=1}^{\infty} 3^{n-1} n x^{n+1}.
\]

The series converges for \(|3x| < 1\). Thus, the radius of convergence is \( R = \frac{1}{3} \).

(c) \( f(x) = \ln(2 + x) \)

Differentiating the function gives

\[
f'(x) = \frac{1}{2 + x} = \frac{1}{2} \left( \frac{1}{1 + \frac{x}{2}} \right) = \frac{1}{2} \left[ \frac{1}{1 - \left( -\frac{x}{2} \right)^2} \right] = \sum_{n=0}^{\infty} \frac{1}{2^n} \left( -\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n.
\]
Therefore,

\[ f(x) = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1} \, dx \]

\[ = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \]

Now \( f(0) = C = \ln 2 \) and so

\[ f(x) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left( \frac{x}{2} \right)^{n+1}. \]

The series converges for \( \left| \frac{x}{2} \right| < 1 \). Thus, the radius of convergence is \( R = 2 \).

(d) \( f(x) = \tan^{-1} x \)

Differentiating the function gives

\[ f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \]

Therefore,

\[ f(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx \]

\[ = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \]

Now \( f(0) = C = 0 \) and so

\[ f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \]

This series converges for \( |x| < 1 \). That is, for \( |x| < 1 \). So the radius of convergence is \( R = 1 \). This series is known as Gregory’s series.
Note: Using the power series for $\tan^{-1} x$, we have
\[
\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.
\]
This result is known as the Leibniz formula for $\pi$.

**Example:** Evaluate $\int \frac{x}{1 + x^5} \, dx$ as a power series.

The integrand can be expressed as
\[
\frac{x}{1 + x^5} = \frac{x}{1 - (-x^5)} = \sum_{n=0}^{\infty} x(-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n+1}.
\]
Therefore,
\[
\int \frac{x}{1 + x^5} \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{5n+1} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{5n+2} x^{5n+2}.
\]

**Example:** Use a power series to approximate $\int_{0}^{0.2} \frac{1}{1 + x^4} \, dx$ to six decimal places.

The integrand can be expressed as
\[
\frac{1}{1 + x^4} = \frac{1}{1 - (-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}.
\]
Therefore,
\[
\int_{0}^{0.2} \frac{1}{1 + x^4} \, dx = \int_{0}^{0.2} \sum_{n=0}^{\infty} (-1)^n x^{4n} \, dx = \sum_{n=0}^{\infty} \left. (-1)^n x^{4n+1} \right|_{0}^{0.2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1} \cdot \frac{1}{4n+1}.
\]

By the Remainder Estimate for Alternating Series,
\[
|R_n| \leq a_{n+1} = \frac{1}{5^{4n+1}(4n+1)}.
\]
If we stop adding after the term with $n = 1$, the error is smaller than the term with $n = 2$:
\[
\frac{1}{5^9} \cdot \frac{1}{9} \approx 5.7 \times 10^{-8}.
\]
So an approximation is
\[
\int_{0}^{0.2} \frac{1}{1 + x^4} \, dx \approx \frac{1}{5} - \frac{1}{5^5} \cdot \frac{1}{5} \approx 0.1999360000.
\]