Section 9.4: Area of a Surface of Revolution

Consider a continuous function $f$ on the interval $[a, b]$. Revolving the curve $y = f(x)$, $a \leq x \leq b$ about the $x$- or $y$-axis produces a surface known as a surface of revolution. A general formula for the area of such a surface is

$$SA = \int 2\pi r dL,$$

where $L$ denotes the arc length function and $r$ is the distance from the curve to the axis of revolution (the radius).

There are two cases to consider.

1. Revolving about the $x$-axis.

   (a) If the curve $y = f(x)$, $a \leq x \leq b$ is revolved about the $x$-axis, then the area of the resulting surface is given by

   $$SA = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx.$$

   (b) If the curve $x = g(y)$, $c \leq y \leq d$ is revolved about the $x$-axis, then the area of the resulting surface is given by

   $$SA = 2\pi \int_c^d y \sqrt{1 + (g'(y))^2} \, dy.$$

   (c) If the curve defined by $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$ is revolved about the $x$-axis, then the area of the resulting surface is given by

   $$SA = 2\pi \int_\alpha^\beta y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

2. Revolving about the $y$-axis.

   (a) If the curve $y = f(x)$, $a \leq x \leq b$ is revolved about the $y$-axis, then the area of the resulting surface is given by

   $$SA = 2\pi \int_a^b x \sqrt{1 + (f'(x))^2} \, dx.$$

   (b) If the curve $x = g(y)$, $c \leq y \leq d$ is revolved about the $y$-axis, then the area of the resulting surface is given by

   $$SA = 2\pi \int_c^d g(y) \sqrt{1 + (g'(y))^2} \, dy.$$

   (c) If the curve defined by $x = x(t)$, $y = y(t)$, $\alpha \leq t \leq \beta$ is revolved about the $y$-axis, then the area of the resulting surface is given by

   $$SA = 2\pi \int_\alpha^\beta x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$
Example: Find the area of the surface obtained by revolving the curve \( y = x^3, \ 0 \leq x \leq 2 \) about the \( x \)-axis.

Since \( y' = 3x^2 \), the surface area is

\[
SA = 2\pi \int_0^2 x^3\sqrt{1 + 9x^4} \, dx.
\]

Let \( u = 1 + 9x^4, \ du = 36x^3 \, dx \). Then

\[
SA = \frac{\pi}{18} \int_1^{145} \sqrt{u} \, du
= \frac{\pi}{18} \left( \frac{2}{3} u^{3/2} \right) \bigg|_{1}^{145}
= \frac{\pi}{27} (145\sqrt{145} - 1).
\]

Example: Find the area of the surface obtained by revolving the curve \( x = \frac{1}{3}(y^2 + 2)^{3/2}, \ 1 \leq y \leq 2 \) about the \( x \)-axis.

Since \( x' = y\sqrt{y^2 + 2} \), the surface area is

\[
SA = 2\pi \int_1^2 y\sqrt{1 + y^2(y^2 + 2)} \, dy
= 2\pi \int_1^2 y\sqrt{y^4 + 2y^2 + 1} \, dy
= 2\pi \int_1^2 y\sqrt{(y^2 + 1)^2} \, dy
= 2\pi \int_1^2 y(y^2 + 1) \, dy
= 2\pi \int_1^2 (y^3 + y) \, dy
= 2\pi \left( \frac{1}{4}y^4 + \frac{1}{2}y^2 \right) \bigg|_1
= \frac{21\pi}{2}.
\]
Example: Find the area of the surface obtained by revolving the parametric curve defined by \(x(t) = 3t - t^3, y(t) = 3t^2, \ 0 \leq t \leq 1\) about the \(x\)-axis.

Since \(\frac{dx}{dt} = 3 - 3t^2\) and \(\frac{dy}{dt} = 6t\), the surface area is

\[
SA = 2\pi \int_0^1 3t^2 \sqrt{(3 - 3t^2)^2 + (6t)^2} dt
\]

\[
= 6\pi \int_0^1 t^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2 dt}
\]

\[
= 6\pi \int_0^1 t^2 \sqrt{9 + 18t^2 + 9t^4} dt
\]

\[
= 18\pi \int_0^1 (t^2 + t^4) dt
\]

\[
= 18\pi \left( \frac{1}{3}t^3 + \frac{1}{5}t^5 \right) \bigg|_0^1
\]

\[
= \frac{48\pi}{5}.
\]

Example: Find the area of the surface obtained by revolving the curve \(y = 1 - x^2, \ 0 \leq x \leq 1\) about the \(y\)-axis.

Since \(y' = -2x\), the surface area is

\[
SA = 2\pi \int_0^1 x \sqrt{1 + 4x^2} dx
\]

Let \(u = 1 + 4x^2, \ du = 8xdx\). Then

\[
SA = \frac{\pi}{4} \int_1^5 \sqrt{u} du
\]

\[
= \frac{\pi}{4} \left( \frac{2}{3}u^{3/2} \right) \bigg|_1^5
\]

\[
= \frac{\pi}{6} (5\sqrt{5} - 1).
\]
Example: Find the area of the surface obtained by revolving the curve $x = \sqrt{2y-y^2}$, $0 \leq y \leq 1$ about the $y$-axis.

Since $x' = \frac{1-y}{\sqrt{2y-y^2}}$, the surface area is

$$SA = 2\pi \int_0^1 \sqrt{2y-y^2} \sqrt{1 + \left( \frac{1-y}{\sqrt{2y-y^2}} \right)^2} dy$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2} \sqrt{1 + \frac{1-2y+y^2}{2y-y^2}} dy$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2} \frac{1}{2y-y^2} dy$$

$$= 2\pi \int_0^1 dy$$

$$= 2\pi.$$

Example: Find the area of the surface obtained by revolving the parametric curve defined by $x(t) = e^t - t$, $y(t) = 4e^{t/2}$, $0 \leq t \leq 1$ about the $y$-axis.

Since $\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{t/2}$, the surface area is

$$SA = 2\pi \int_0^1 (e^t - t) \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} dt$$

$$= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} - 2e^t + 1 + 4e^t} dt$$

$$= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} + 2e^t + 1} dt$$

$$= 2\pi \int_0^1 (e^t - t) (e^t + 1) dt$$

$$= 2\pi \int_0^1 e^{2t} + e^t - te^t - t dt$$

$$= 2\pi \left[ \frac{1}{2} e^{2t} + e^t - \int_0^1 te^t dt - \frac{1}{2} t^2 \right].$$
Let \( u = t \) and \( dv = e^t \, dt \). Then \( du = dt \) and \( v = e^t \). So

\[
SA = 2\pi \left[ \frac{1}{2} e^{2t} + e^t - te^t + e^t - \frac{1}{2} t^2 \right]_0^1
= \pi e^2 + 2\pi e - 6\pi.
\]

Example: (Gabriel’s Horn)
Consider revolving the region under the graph of \( y = \frac{1}{x} \), \( 1 \leq x \leq \infty \) about the \( x \)-axis.

(a) Find the volume of the resulting solid.

Using the Disk Method,

\[
V = \pi \int_1^\infty \frac{1}{x^2} \, dx
= \pi \lim_{N \to \infty} \int_1^N \frac{1}{x^2} \, dx
= \pi \lim_{N \to \infty} \left( -\frac{1}{x} \right)_1^N
= \pi \lim_{N \to \infty} \left( 1 - \frac{1}{N} \right)
= \pi.
\]

(b) Find the surface area of the solid.

Since \( y' = -\frac{1}{x^2} \), the surface area is

\[
SA = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx
\geq 2\pi \int_1^\infty \frac{1}{x} \, dx
= 2\pi \lim_{N \to \infty} \int_1^N \frac{1}{x} \, dx
= 2\pi \lim_{N \to \infty} \ln |x|_1^N
= 2\pi \lim_{N \to \infty} \ln N
= \infty.
\]

The resulting solid has finite volume, but infinite surface area. Thus, the horn can be filled with a finite amount of paint but the surface can never be completely covered!