Section 11.7: Arc Length and Curvature

Theorem: (Arc Length)
Consider the curve $C$ defined by $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. If $C$ is traversed exactly once as $t$ increases from $a$ to $b$, then its length is

$$L = \int_a^b ||\vec{R}'(t)|| \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt.$$  

Example: Find the length of the curve defined by $\vec{R}(t) = \langle 6t, 3\sqrt{2}t^2, 2t^3 \rangle$, $0 \leq t \leq 1$.

Since $\vec{R}'(t) = \langle 6, 6\sqrt{2}t, 6t^2 \rangle$, we have

$$||\vec{R}'(t)|| = \sqrt{36 + 72t^2 + 36t^4} = \sqrt{36(1 + 2t^2 + t^4)} = \sqrt{36(1 + t^2)^2} = 6(1 + t^2).$$

Then

$$L = \int_0^1 6(1 + t^2) \, dt = (6t + 2t^3)|_0^1 = 4.$$ 

Definition: Suppose $C$ is a space curve defined by $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. Then its **arc length function** is defined by

$$s(t) = \int_a^t ||\vec{R}'(u)|| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du.$$ 

Thus, $s(t)$ is the length of the part of $C$ between $\vec{R}(a)$ and $\vec{R}(t)$.

Note: A single curve $C$ can be represented by more than one vector function. Different representations of curves are called **parameterizations** of the curve $C$. It is often useful to parameterize a curve with respect to arc length.

Example: Reparameterize the helix $\vec{R}(t) = \langle 3\sin t, 4t, 3\cos t \rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing $t$.

Since $\vec{R}'(t) = \langle 3\cos t, 4, -3\sin t \rangle$, we have

$$||\vec{R}'(t)|| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{9(\sin^2 t + \cos^2 t) + 16} = \sqrt{25} = 5.$$
The arc length function is

\[ s = s(t) = \int_0^t 5du = 5t. \]

Thus, \( t = s/5 \) and the reparameterization is given by

\[ \vec{R}(s) = \left\langle 3 \sin \left( \frac{s}{5} \right) , \frac{4s}{5} , 3 \cos \left( \frac{s}{5} \right) \right\rangle. \]

**Definition:** The **curvature** of a curve defined by \( \vec{R}(t) \) is

\[ \kappa(t) = \frac{||\vec{T}'(t)||}{||\vec{R}'(t)||} = \frac{||\vec{R}'(t) \times \vec{R}''(t)||}{||\vec{R}'(t)||^2}, \]

where \( \vec{T} \) is the unit tangent vector.

**Note:** The curvature of a curve \( C \) at a given point is a measure of how quickly the curve changes direction at that point. Specifically, it measures the rate of change of the unit tangent vector (which indicates the direction of \( C \)) with respect to arc length.

![Figure 1: Unit tangent vectors for a space curve \( C \).](image-url)
Example: Find the curvature of the twisted cubic $\tilde{R}(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0, 0, 0)$.

First,

\[ \tilde{R}'(t) = \langle 1, 2t, 3t^2 \rangle \]
\[ \tilde{R}''(t) = \langle 0, 2, 6t \rangle \]
\[ ||\tilde{R}'(t)|| = \sqrt{1 + 4t^2 + 9t^4} \]
\[ \tilde{R}'(t) \times \tilde{R}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle. \]

Then

\[ ||\tilde{R}'(t) \times \tilde{R}''(t)|| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}. \]

Thus, the curvature is

\[ \kappa(t) = \frac{||\tilde{R}'(t) \times \tilde{R}''(t)||}{||\tilde{R}'(t)||^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}. \]

At the origin the curvature is $\kappa(0) = 2$.

Theorem: (Curvature of a Plane Curve)

The curvature of the plane curve $y = f(x)$ is given by

\[ \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}. \]

Example: Find the curvature of $y = \sin x$ at a general point and at $\left( \frac{\pi}{2}, 0 \right)$.

Since $y' = \cos x$ and $y'' = -\sin x$,

\[ \kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}. \]

At $\left( \frac{\pi}{2}, 0 \right)$ the curvature is $\kappa(1) = 1$.

Definition: Consider the curve defined by $\tilde{R}(t)$. The **principal unit normal vector** is given by

\[ \tilde{N}(t) = \frac{\tilde{T}'(t)}{||\tilde{T}'(t)||}, \]
and the **binormal vector** is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

**Note:** The vectors $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ form a set of orthogonal vectors, called a frame, that moves along the curve as $t$ varies. This frame has applications in differential geometry and motion of spacecraft.

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**Figure 2:** Illustration of the frame formed by the unit tangent, unit normal, and binormal vectors.

**Example:** Find the unit normal and binormal vectors for the helix $\vec{R}(t) = (\sin t, \cos t, t)$.

Since $\vec{R}'(t) = (\cos t, -\sin t, 1)$,

$$||\vec{R}'(t)|| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.$$ 

Thus, the unit tangent vector is

$$\vec{T}(t) = \frac{\vec{R}'(t)}{||\vec{R}'(t)||} = \frac{1}{\sqrt{2}}(\cos t, -\sin t, 1).$$

Therefore,

$$\vec{T}'(t) = \frac{1}{\sqrt{2}}(-\sin t, -\cos t, 0), \quad \text{and} \quad ||\vec{T}'(t)|| = \frac{1}{\sqrt{2}}.$$ 

The unit normal vector is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = (-\sin t, -\cos t, 0).$$
Now the binormal vector is
\[ \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\cos t, -\sin t, -1). \]

**Definition:** The plane determined by the unit normal and binormal vectors \( \vec{N} \) and \( \vec{B} \) at a point \( P \) on the curve \( C \) is called the **normal plane** of \( C \) at \( P \). The unit tangent vector is orthogonal to the normal plane. Similarly, the plane determined by the unit tangent and unit normal vectors \( \vec{T} \) and \( \vec{N} \) is called the **osculating plane** of \( C \) at \( P \). The binormal vector is orthogonal to the osculating plane.

**Example:** Find equations for the normal and osculating planes of the helix \( \vec{R}(t) = \langle \sin t, \cos t, t \rangle \) at the point \( P = (0, -1, \pi) \).

The tangent vector \( \vec{R}'(t) = \langle \cos t, -\sin t, 1 \rangle \) is normal to the normal plane. At the point \( P \), \( \vec{R}'(\pi) = \langle -1, 0, 1 \rangle \). Thus, an equation of the normal plane is
\[
-1(x - 0) + 0(y + 1) + 1(z - \pi) = 0
\]
\[
-x + z = \pi.
\]

The binormal vector is normal to the osculating plane. From the previous example,
\[
\vec{B}(t) = \frac{1}{\sqrt{2}}(\cos t, -\sin t, -1)
\]
and so \( \vec{B}(\pi) = \frac{1}{\sqrt{2}}(-1, 0, -1) \).

An easier normal vector is \( \langle -1, 0, 1 \rangle \). So an equation of the osculating plane is
\[
-1(x - 0) + 0(y + 1) - 1(z - \pi) = 0
\]
\[
z - x = -\pi
\]
\[
x - z = \pi.
\]