Section 11.8: Motion in Space: Velocity and Acceleration

In this section, we apply the concepts of tangent and normal vectors to study the motion of an object along a curve in space.

Definition: Suppose a particle moves through space so that its position at time \( t \) is \( \vec{R}(t) \). Then the **velocity vector** of the particle is

\[
\vec{V}(t) = \vec{R}'(t),
\]

and the **speed** of the particle at time \( t \) is \( ||\vec{V}(t)|| \). Similarly, the **acceleration** of the particle is given by

\[
\vec{A}(t) = \vec{V}'(t) = \vec{R}''(t).
\]

Example: The position vector of an object is given by \( \vec{R}(t) = \langle t^3, t^2 + 1, t^3 - 1 \rangle \), \( t \geq 0 \). Find its velocity, acceleration, and speed at time \( t = 1 \).

The velocity and acceleration at time \( t \) are

\[
\vec{V}(t) = \langle 3t^2, 2t, 3t^2 \rangle \\
\vec{A}(t) = \langle 6t, 2, 6t \rangle.
\]

The speed at time \( t \) is

\[
||\vec{V}(t)|| = \sqrt{6t^4 + 4t^2 + 6t^4} = \sqrt{12t^4 + 4t^2}.
\]

At time \( t = 1 \), \( \vec{V}(1) = \langle 3, 2, 3 \rangle \), \( \vec{A}(1) = \langle 6, 2, 6 \rangle \), and \( ||\vec{V}(1)|| = \sqrt{16} = 4 \).

Example: A moving particle starts at initial position \( \vec{R}(0) = \langle 0, 1, 0 \rangle \) with initial velocity \( \vec{V}(0) = \langle 1, 0, 1 \rangle \). Its acceleration is \( \vec{A}(t) = \langle t, t^2, \cos 2t \rangle \). Find its velocity and position at time \( t \).

Since \( \vec{A}(t) = \vec{V}'(t) \),

\[
\vec{V}(t) = \int \vec{A}(t)\,dt \\
= \int \langle t, t^2, \cos 2t \rangle\,dt \\
= \left\langle \frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{2} \sin 2t \right\rangle + \vec{C}.
\]
Using the initial velocity, \( \vec{V}(0) = \vec{C} = (1, 0, 1) \). Then

\[
\vec{V}(t) = \left\langle \frac{1}{2}t^2 + 1, \frac{1}{3}t^3, \frac{1}{2}\sin(2t) + 1 \right\rangle.
\]

Since \( \vec{V}(t) = \vec{R}'(t) \),

\[
\vec{R}(t) = \int \vec{V}(t)dt = \int \left\langle \frac{1}{2}t^2 + 1, \frac{1}{3}t^3, \frac{1}{2}\sin(2t) + 1 \right\rangle dt = \left\langle \frac{1}{6}t^3 + t, \frac{1}{12}t^4, -\frac{1}{4}\cos(2t) + t \right\rangle + \vec{C}.
\]

Using the initial position,

\[
\vec{R}(0) = \left\langle 0, 0, -\frac{1}{4} \right\rangle + \vec{C} = (0, 1, 0).
\]

Thus, \( \vec{C} = (0, 1, 1/4) \) and

\[
\vec{R}(t) = \left\langle \frac{1}{6}t^3 + t, \frac{1}{12}t^4 + 1 - \frac{1}{4}\cos(2t) + t + \frac{1}{4} \right\rangle.
\]

Example: A force of 20 N acts directly upward from the xy-plane on an object with mass 4 kg. The object starts at the origin with initial velocity \( \vec{V}(0) = (1, -1, 0) \). Find its position function and its speed at time \( t \).

By Newton’s Second Law of Motion,

\[
\vec{F}(t) = m\vec{A}(t)
\]

\[
(0, 0, 20) = 4\vec{A}(t)
\]

\[
\vec{A}(t) = (0, 0, 5).
\]

Since \( \vec{A}(t) = \vec{V}'(t) \),

\[
\vec{V}(t) = \int \vec{A}(t)dt = \int \left(0, 0, 5\right)dt = \left(0, 0, 5t\right) + \vec{C}.
\]

Using the initial velocity, \( \vec{V}(0) = \vec{C} = (1, -1, 0) \). Thus,

\[
\vec{V}(t) = (1, -1, 5t).
\]

Therefore, the speed of the object is

\[
||\vec{V}(t)|| = \sqrt{2 + 25t^2}.
\]
Since $\vec{V}(t) = \vec{R}'(t)$,

$$\vec{R}(t) = \int \vec{V}(t) \, dt = \int \langle 1, -1, 5t \rangle \, dt = \left\langle t, -t, \frac{5}{2}t^2 \right\rangle + \vec{C}.$$  

Using the initial position, $\vec{R}(0) = \vec{C} = \vec{0}$. Thus,

$$\vec{R}(t) = \left\langle t, -t, \frac{5}{2}t^2 \right\rangle.$$

Note: When studying the motion of a particle in space, it is often useful to break the acceleration down into two components: one in the direction of the unit tangent and one in the direction of the unit normal. These are known as the **tangential and normal components of acceleration**.

![Illustration of the tangential and normal components of acceleration.](image)

**Figure 1:** Illustration of the tangential and normal components of acceleration.

**Theorem:** (Tangential and Normal Components of Acceleration) 

Suppose that the position of a particle at time $t$ is given by $\vec{R}(t)$. Then the tangential and normal components of the particle’s acceleration are

$$A_T = \frac{\vec{R}'(t) \cdot \vec{R}''(t)}{||\vec{R}'(t)||} \quad \text{and} \quad A_N = \frac{||\vec{R}'(t) \times \vec{R}''(t)||}{||\vec{R}'(t)||}.$$
Example: A particle moves with position function \( \vec{R}(t) = \langle t^3, t^2, t \rangle \). Find the tangential and normal components of acceleration at a general position and at the origin.

First,

\[
\vec{R}'(t) = \langle 3t^2, 2t, 1 \rangle
\]

\[
\vec{R}''(t) = \langle 6t, 2, 0 \rangle
\]

\[
||\vec{R}'(t)|| = \sqrt{9t^4 + 4t^2 + 1}
\]

\[
\vec{R}'(t) \cdot \vec{R}''(t) = 18t^3 + 4t
\]

\[
\vec{R}'(t) \times \vec{R}''(t) = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
3t^2 & 2t & 1 \\
6t & 2 & 0
\end{vmatrix} = \langle -2, 6t, -6t^2 \rangle
\]

\[
||\vec{R}'(t) \times \vec{R}''(t)|| = \sqrt{4 + 36t^2 + 36t^4} = 2\sqrt{1 + 9t^2 + 9t^4}.
\]

Then

\[
A_T = \frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}} \quad \text{and} \quad A_N = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{\sqrt{9t^4 + 4t^2 + 1}}.
\]

At the origin \((t = 0)\), \(A_T = 0\) and \(A_N = 2\).