Section 13.8: Triple Integrals

Consider a function $f$ of three variables defined on a closed box

$$B = [a, b] \times [c, d] \times [r, s] = \{(x, y, z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$

Partition the intervals $[a, b]$, $[c, d]$, and $[r, s]$ as

$$a = x_0 < x_1 < \cdots < x_l = b,$$
$$c = y_0 < y_1 < \cdots < y_m = d,$$
$$r = z_0 < z_1 < \cdots < z_n = s.$$

This forms a partition, $P$, of the box $B$ into $lmn$ sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k].$$

Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and $\Delta z_k = z_k - z_{k-1}$. Then the area of the box $B_{ijk}$ is

$$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k.$$

Choose a representative point $(x^*_ijk, y^*_ijk, z^*_ijk)$ in each sub-box $B_{ijk}$ and form the triple Riemann sum

$$\sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x^*_ijk, y^*_ijk, z^*_ijk) \Delta V_{ijk}.$$

Let $||P||$ denote the norm of the partition $P$ which is the length of the longest diagonal of all sub-boxes $B_{ijk}$.

**Definition:** The triple integral of $f$ over $B$ is

$$\iiint_B f(x, y, z)dV = \lim_{||P||\to 0} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x^*_ijk, y^*_ijk, z^*_ijk) \Delta V_{ijk}$$

provided that the limit exists. If the limit exists, then $f$ is called integrable.

**Note:** The practical method for evaluating triple integrals is to express them as iterated integrals.

**Theorem:** (Fubini’s Theorem)

If $f$ is continuous on the box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dx dy dz.$$
Note: There are five other possible orders of integration which all give the same result.

Example: Evaluate the triple integral

\[ \iiint_B 8x^2yz^3 \, dV, \]

where \( B = [-1, 2] \times [0, 1] \times [1, 2] \).

By Fubini’s Theorem,

\[
\begin{align*}
\iiint_B 8x^2yz^3 \, dV & = \int_{-1}^{2} \int_{0}^{1} \int_{1}^{2} 8x^2yz^3 \, dz \, dy \, dx \\
& = \int_{-1}^{2} \int_{0}^{1} 2x^2y^2z^4 \bigg|_{1}^{2} \, dy \, dx \\
& = \int_{-1}^{2} \int_{0}^{1} 30x^2y^2 \, dy \, dx \\
& = \int_{-1}^{2} 15x^2y^2 \bigg|_{0}^{1} \, dx \\
& = \int_{-1}^{2} 15x^2 \, dx \\
& = 5x^3 \bigg|_{-1}^{2} \\
& = 45.
\end{align*}
\]

Definition: There are three types of solid regions.

- A solid region \( E \) is said to be of **type 1** if

\[
E = \{(x, y, z) | (x, y) \in D, g_1(x, y) \leq z \leq g_2(x, y) \},
\]

where \( D \) is the projection of \( E \) onto the \( xy \)-plane.

- A solid region \( E \) is said to be of **type 2** if

\[
E = \{(x, y, z) | (y, z) \in D, g_1(y, z) \leq x \leq g_2(y, z) \},
\]

where \( D \) is the projection of \( E \) onto the \( yz \)-plane.

- A solid region \( E \) is said to be of **type 3** if

\[
E = \{(x, y, z) | (x, z) \in D, g_1(x, z) \leq y \leq g_2(x, z) \},
\]

where \( D \) is the projection of \( E \) onto the \( xz \)-plane.
Theorem: Suppose that $f$ is a continuous function over some region $E$.

1. If $E$ is a type 1 region, then
\[
\iiint_E f(x,y,z) \, dV = \iint_D \left[ \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z) \, dz \right] \, dA.
\]

2. If $E$ is a type 2 region, then
\[
\iiint_E f(x,y,z) \, dV = \iint_D \left[ \int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z) \, dx \right] \, dA.
\]

3. If $E$ is a type 3 region, then
\[
\iiint_E f(x,y,z) \, dV = \iint_D \left[ \int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z) \, dy \right] \, dA.
\]

In particular, if $E$ is a type 1 region and its projection $D$ onto the $xy$-plane is a type I region, then
\[
\iiint_E f(x,y,z) \, dV = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y,z) \, dz \, dy \, dx.
\]

Example: Evaluate the triple integral $\iiint_E zdV$, where $E$ is the solid region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 2$.

The solid region $E$ and its projection $D$ onto the $xy$-plane are shown in the figure below. The region $E$ can be expressed as
\[
E = \{(x,y,z) | 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x, 0 \leq z \leq 2 - 2x - y\}.
\]

Then,
\[
\iiint_E zdV = \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} z \, dz \, dy \, dx
\]
\[
= \frac{1}{2} \int_0^1 \int_0^{2-2x} (2-2x-y)^2 \, dy \, dx
\]
\[
= \frac{1}{6} \int_0^1 (2-2x-y)^3 \bigg|_0^{2-2x} \, dx
\]
\[
= \frac{1}{6} \int_0^1 (2-2x)^3 \, dx
\]
\[
= \frac{1}{48} (2-2x)^4 \bigg|_0^1
\]
\[
= \frac{1}{3}.
\]
We could have also projected this region onto the $xz$- or $yz$-planes.

Example: Evaluate the triple integral $\iiint_E \sqrt{y^2 + z^2} dV$, where $E$ is the solid region bounded by the paraboloid $x = y^2 + z^2$ and the plane $x = 9$.

The solid region $E$ and its projection $D$ onto the $yz$-plane are shown in the figure below. The region $E$ can be expressed as

$$E = \{(x, y, z) | (y, z) \in D, y^2 + z^2 \leq x \leq 9\},$$

where $D$ is the disk $y^2 + z^2 \leq 9$. In polar coordinates,

$$D = \{(r, \theta) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3\}.$$

Then,

$$\iiint_E \sqrt{y^2 + z^2} dV = \iint_D \left[ \int_{y^2 + z^2}^9 \sqrt{y^2 + z^2} dx \right] dA$$

$$= \int_0^{2\pi} \int_0^3 r^2 (9 - r^2) dr d\theta$$

$$= 2\pi \int_0^3 (9r^2 - r^4) dr$$

$$= 2\pi \left( 9r^3 - \frac{1}{5} r^4 \right) \bigg|_0^3$$

$$= \frac{1944\pi}{5}.$$

We could have also projected this region onto the $xy$- or $xz$-planes.

Theorem: (Volume as a Triple Integral)
The volume of the solid region $E$ is

$$V = \iiint_E dV.$$

Example: Find the volume of the tetrahedron $T$ bounded by the plane $2x + 2y + z = 12$ and the coordinate planes.

The tetrahedron $T$ can be expressed as

$$T = \{(x, y, z) | 0 \leq x \leq 6, 0 \leq y \leq 6 - x, 0 \leq z \leq 12 - 2x - 2y\}.$$
Thus, the volume of the tetrahedron is

\[ V = \int_0^6 \int_0^{6-x} \int_0^{12-2x-2y} dz dy dx \]

\[ = \int_0^6 \int_0^{6-x} (12 - 2x - 2y) dy dx \]

\[ = \int_0^6 \left[ (12 - 2x)y - y^2 \right]_{y=0}^{y=6-x} dx \]

\[ = \int_0^6 [(12 - 2x)(6 - x) - (6 - x)^2] dx \]

\[ = \int_0^6 (x^2 - 12x + 36) dx \]

\[ = \frac{1}{3} x^3 - 6x^2 + 36 \bigg|_0^6 \]

\[ = 72. \]

Note: The applications of double integrals in Section 13.6 can be extended to triple integrals.

Theorem: (Center of Mass of a Solid)
Suppose a solid occupies some region \( E \) in space and its density at a point \((x, y, z) \in E\) is given by \( \rho(x, y, z) \). Then the mass of the solid is given by

\[ m = \iiint_E \rho(x, y, z) dV. \]

The coordinates \((\bar{x}, \bar{y}, \bar{z})\) of the center of mass of the solid are

\[ \bar{x} = \frac{1}{m} \iiint_E x \rho(x, y, z) dV \quad \bar{y} = \frac{1}{m} \iiint_E y \rho(x, y, z) dV \quad \bar{z} = \frac{1}{m} \iiint_E z \rho(x, y, z) dV. \]

Example: Find the mass and center of mass of the solid \( E \) bounded by the parabolic cylinder \( z = 1 - y^2 \) and the planes \( x + 5z = 5, x = 0, \) and \( z = 0 \) with constant density \( \rho(x, y, z) = 2. \)

The solid region \( E \) can be described as

\[ E = \{(x, y, z)| 0 \leq x \leq 5-5z, 0 \leq z \leq 1 - y^2, -1 \leq y \leq 1 \}. \]
Thus, the mass of the solid is

\[
m = 2 \int_{-1}^{1} \int_{0}^{1-y^2} \int_{0}^{5-5z} dx\,dz\,dy
\]
\[
= 2 \int_{-1}^{1} \int_{0}^{1-y^2} (5-5z)\,dz\,dy
\]
\[
= 10 \int_{-1}^{1} \left( z - \frac{1}{2} z^2 \right) \Big|_{0}^{1-y^2} \, dy
\]
\[
= 10 \int_{-1}^{1} \left[ (1-y^2) - \frac{1}{2}(1-2y^2+y^4) \right] \, dy
\]
\[
= 5 \int_{-1}^{1} (1-y^4)\,dy
\]
\[
= 5y - y^5 \Big|_{-1}^{1}
\]
\[
= 8.
\]

Using the formulas from the previous theorem,

\[
\bar{x} = \frac{1}{m} \iiint_E 2x\,dV
\]
\[
= \frac{1}{4} \int_{-1}^{1} \int_{0}^{1-y^2} \int_{0}^{5-5z} x\,dx\,dz\,dy
\]
\[
= \frac{1}{8} \int_{-1}^{1} \int_{0}^{1-y^2} (5-5z)^2\,dz\,dy
\]
\[
= \frac{25}{8} \int_{-1}^{1} \int_{0}^{1-y^2} (1-z)^2\,dz\,dy
\]
\[
= -\frac{25}{24} \int_{-1}^{1} (1-z)^3 \Big|_{0}^{1-y^2} \, dy
\]
\[
= \frac{25}{24} \int_{-1}^{1} (1-y^6)\,dy
\]
\[
= \frac{25}{24} \left( y - \frac{1}{7} y^7 \right) \Big|_{-1}^{1}
\]
\[
= \frac{25}{14}.
\]

Similarly,

\[
\bar{y} = \frac{1}{4} \iiint_E y\,dV = 0,
\]
and

\[
\bar{z} = \frac{1}{4} \iiint_E z\,dV = \frac{2}{7}.
\]