Section 13.9: Cylindrical and Spherical Coordinates

In the **cylindrical coordinate system**, a point P in space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

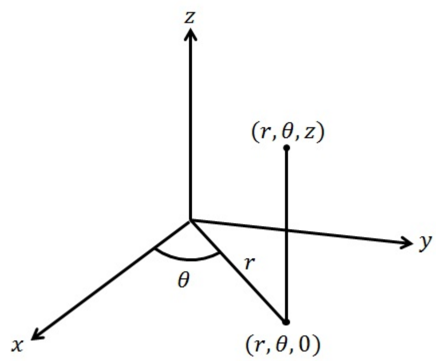

Figure 1: A point expressed in cylindrical coordinates.

To convert from cylindrical to rectangular coordinates we use the relations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

To convert from rectangular to cylindrical coordinates we use the relations

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \quad z = z.$$

Example: Convert the point $\left(2, \frac{4\pi}{3}, 8\right)$ from cylindrical to rectangular coordinates.

Since $r = 2$, $\theta = 4\pi/3$, and $z = 8$,

$$x = r\cos\theta = 2\cos\left(\frac{4\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1,$$
$$y = r\sin\theta = 2\sin\left(\frac{4\pi}{3}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3},$$
$$z = z = 8.$$

Thus, the point is $(-1, -\sqrt{3}, 8)$ in rectangular coordinates.

Example: Convert the point $(\sqrt{3}, 1, 4)$ from rectangular to cylindrical coordinates.

Since $x = \sqrt{3}$, $y = 1$, and $z = 4$,

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2,$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6},$$
$$z = z = 4.$$

Thus, the point is $\left(2, \frac{\pi}{6}, 4\right)$ in cylindrical coordinates.

Example: Find an equation in cylindrical coordinates for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.

Since $r^2 = x^2 + y^2$, it follows that

$$\begin{aligned} 4x^2 + 4y^2 + z^2 &= 1 \\ 4r^2 + z^2 &= 1 \\ z^2 &= 1 - 4r^2. \end{aligned}$$

In the **spherical coordinate system**, a point P in space is represented by the ordered triple (ρ, θ, ϕ), where $\rho \geq 0$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and $0 \leq \phi \leq \pi$ is the angle between the positive z-axis and the line segment OP.

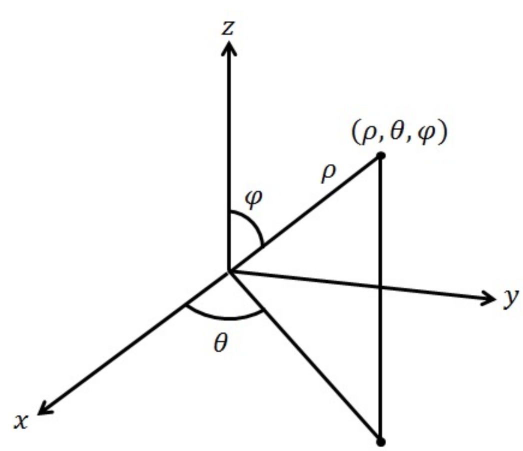

Figure 2: A point expressed in spherical coordinates.

The connection between rectangular and spherical coordinates can be seen in Figure 3. If the point P has rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ), then

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

and

$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

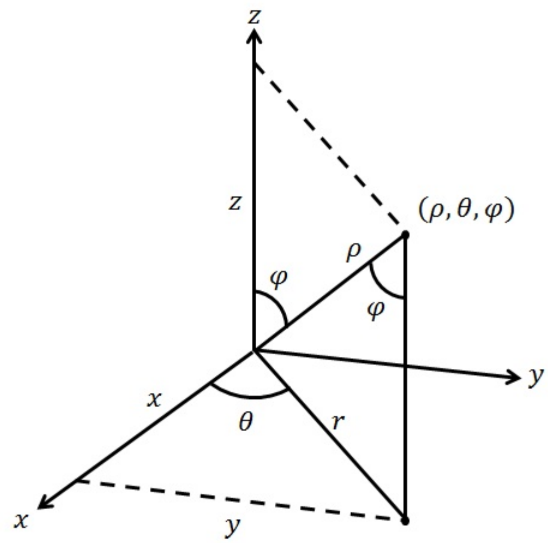

Figure 3: Relationship between rectangular and spherical coordinates.

Example: Convert the point $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$ from spherical to rectangular coordinates.

Since $\rho = 4$, $\theta = \pi/4$, and $\phi = \pi/6$,

$$x = \rho \sin\phi \cos\theta = 4 \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{4}\right) = 4\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2},$$
$$y = r \sin\phi \sin\theta = 4 \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{4}\right) = 4\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2},$$
$$z = \rho \cos\phi = 4 \cos\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}.$$

Thus, the point is $(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$ in rectangular coordinates.

Example: Convert the point $(1, -1, -\sqrt{2})$ from rectangular to spherical coordinates.

Since $x = 1$, $y = -1$, and $z = -\sqrt{2}$,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2,$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-1) = \frac{7\pi}{4},$$
$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}.$$

Thus, the point is $\left(2, \frac{7\pi}{4}, \frac{3\pi}{4}\right)$ in spherical coordinates.

Example: Find an equation in spherical coordinates for the surface $3x^2 - x + 3y^2 + 3z^2 = 0$.

Since $\rho^2 = x^2 + y^2 + z^2$ and $x = \rho \sin\phi \cos\theta$, it follows that

$$\begin{aligned} 3x^2 + 3y^2 + 3z^2 &= x \\ 3\rho^2 &= \rho \sin\phi \cos\theta \\ \rho &= \frac{1}{3} \sin\phi \cos\theta. \end{aligned}$$

Example: A solid lies above the cone $5z = \sqrt{x^2 + y^2}$ and outside the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of spherical coordinates.

Since $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho\cos\phi$, it follows that

$$\begin{aligned} x^2 + y^2 + z^2 &\geq z \\ \rho^2 &\geq \rho\cos\phi \\ \rho &\geq \cos\phi. \end{aligned}$$

The first equation gives

$$\begin{aligned} 5z &\geq \sqrt{x^2 + y^2} \\ 25z^2 &\geq x^2 + y^2 \\ 25\rho^2\cos^2\phi &\geq \rho^2\sin^2\phi \\ 25\cos^2\phi &\geq 1 - \cos^2\phi \\ 26\cos^2\phi &\geq 1 \\ \cos\phi &\geq \frac{1}{\sqrt{26}}. \end{aligned}$$