Extended Gaussian Type Cubatures for the Ball

Hao Nguyen*, Guergana Petrova†

Abstract

We construct cubatures that approximate the integral of a function $u$ over the unit ball by the linear combination of surface integrals over the unit sphere of normal derivatives of $u$ and surface integrals of $u$ and $\Delta^2 u$ over $m$ spheres, centered at the origin. We derive explicitly the weights and the nodes of these cubatures, and prove that they are the only ones that are exact for all $(2m+2)$-harmonic functions. There are no other cubatures of these types that integrate exactly polyharmonic functions of higher order.

AMS subject classification: 65D32, 65D30, 41A55.

Key Words: Extended cubature formulae, polyharmonic functions, polyharmonic degree of precision.

1 Introduction

Recently, there has been a substantial effort to extend some of the classical results on quadrature formulas to higher dimensions. New cubature formulas for balls, simplices, spheres and parallelepipeds, based on integrals over low dimensional manifolds, have been suggested, see [3, 4, 8, 9, 12, 14, 15], and the references therein.

In this paper, we construct high dimensional analogues to the result from [10], where Turan’s problem [17] of finding a quadrature formula of the form

$$\int_{-1}^{1} f(x) \, dx \approx \sum_{k=1}^{m} (a_k f(x_k) + b_k f''(x_k)),$$  \hfill (1.1)

that is exact for all univariate polynomials of degree at least $2m$ has been solved. It

---

*WorldQuant LLC, Hanoi, Vietnam Email: haospt@gmail.com
†Corresponding author, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA. Email: gpetrova@math.tamu.edu
was shown in [10] that the quadrature formula

\[
\int_{-1}^{1} f(x) \, dx \approx \frac{2}{m(m+3)} \sum_{k=1}^{m} \left( \frac{f(x_k)}{P_{m+1}^2(x_k)} + \frac{(1-x_k^2)f''(x_k)}{(m+1)(m+2)P_{m+1}^2(x_k)} \right),
\]

where \( P_{m+1} \) is the Legendre polynomial and \( \{x_i\}_{i=1}^{m} \) are the zeroes of \( P_{m+1}^n \), has algebraic degree of precision \( 2m+1 \). This result has been extended in [11], where the authors present a cubature for approximating the integral of a function \( u \) over the unit ball \( B := \{ x \in \mathbb{R}^n : \|x\| := (\sum_{i=1}^{n} x_i^2)^{1/2} \} < 1 \) in \( \mathbb{R}^n \) using, instead of point evaluations, integrals over spheres \( S(r_k) := \{ x \in \mathbb{R}^n : \|x\| = r_k \} \), namely

\[
\int_{B} u(x) \, dx \approx \sum_{k=1}^{m} \left( A_k \int_{S(r_k)} u(\xi) \, d\sigma(\xi) + B_k \int_{S(r_k)} \Delta u(\xi) \, d\sigma(\xi) \right). \tag{1.2}
\]

They showed that formula (1.2) is exact for all polyharmonic functions of order \( 2m+1 \), and explicitly computed the weights \( \{A_k\} \), \( \{B_k\} \) and the nodes \( \{r_k\} \).

Here, we present Gauss-Lobatto analogues to cubature (1.2), where we use information along the boundary \( S(1) \) of the integration domain \( B \) and integrals over spheres \( \{S(\tau_j)\} \) of the function and its second order Laplacain \( \Delta^2 \). To further describe our results, we need some notation. Let us denote by \( B(r) \) the euclidean ball in \( \mathbb{R}^n \) with radius \( r \). Recall that a function \( u \), defined on \( B \), is called a polyharmonic function of order \( p \) (or \( p \)-harmonic function), see [1, 13], if \( u \in C^{2p-1}(\overline{B}) \cap C^{2p}(B) \) and it satisfies the equation

\[
\Delta^p u(x) = 0, \quad x \in B, \quad \text{where} \quad \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \Delta^p := \Delta(\Delta^{p-1}). \tag{1.3}
\]

In particular, when \( p = 1 \) \( (p = 2) \), \( u \) is called harmonic (biharmonic). The set of all \( p \)-harmonic functions on \( B \) is denoted by \( H^p(B) \). We also denote by \( \frac{\partial u}{\partial \nu} \) the normal derivative of \( u \), where \( \nu \) is the outward unit normal to the sphere \( S(1) \).

In this paper, we construct cubature of type I, that is cubature of the form

\[
\int_{B} u(x) \, dx \approx A \int_{S(1)} u(\xi) \, d\sigma(\xi) + \sum_{j=1}^{m} \left( B_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi) + C_j \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi) \right) \tag{1.4}
\]

and cubature of type II,

\[
\int_{B} u(x) \, dx \approx F \int_{S(1)} \frac{\partial u}{\partial \nu}(\xi) \, d\sigma(\xi) + \sum_{j=1}^{m} \left( G_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi) + H_j \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi) \right) \tag{1.5}
\]

that are exact for polyharmonic functions of order as high as possible. We call such formulas Gauss-Lobatto-Turan cubatures. We view them as multidimensional
analogues of the classical Lobatto quadratures since they use information along the boundary $S(1)$ of the integration domain $B$, such as $\int_{S(1)} \frac{\partial u}{\partial \nu}(\xi) \, d\sigma(\xi)$ or $\int_{S(1)} u(\mathbf{x}) \, d\sigma(\xi)$.

They are also multidimensional generalizations of the Turan’s problem (1.1), because they involve higher order derivatives of the integrand, that is $\left\{ \int_{S(\tau_j)} \Delta^2 u(\mathbf{x}) \, d\sigma(\xi) \right\}_{j}$.

Since all polynomials in $n$ variables of degree at most $2p - 1$ are $p$-harmonic functions, namely

$$\pi_{2p-1}(\mathbb{R}^n) \subset H^p(B),$$

finding cubature formulas of type I or type II that are exact for $H^p(B)$ for $p$ as large as possible is a natural generalization of the notion of Gaussian quadratures in the one dimensional case. The largest natural number $\ell$ for which (1.4) (or (1.5)) is exact for all $u \in H^\ell(B)$ is called a Polyharmonic Degree of Precision (PDP) of (1.4) (or (1.5)). Formulas for numerical integration with the best possible PDP are called Gaussian cubatures.

In the process of deriving cubatures (1.4) and (1.5), we obtain also a formula of the form

$$\int_{B} u(\mathbf{x}) \, d\mathbf{x} \approx E_0 \int_{S(1)} u(\xi) \, d\sigma(\xi) + E_1 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi) + \sum_{j=1}^{m} D_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi),$$

that has PDP(1.7) = $2m + 2$. We call it the Gauss-Lobatto cubature for the ball because it is a natural generalization of a one dimensional Gauss-Lobatto quadrature, and there are no other cubatures of this form that integrate exactly polyharmonic functions of higher order. We also explicitly construct a cubature of the type

$$\int_{B} u(\mathbf{x}) \, d\mathbf{x} \approx P_0 \int_{S(1)} u(\xi) \, d\sigma(\xi) + P_1 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi) + \sum_{j=1}^{m} Q_j \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi),$$

exact for all $u \in H^{2m+2}(B)$, which we call the Lobatto-Turan cubature for the ball. We show that this is the unique cubature of type (1.8) with PDP(1.8) = $2m + 2$ and no other cubature of this type can be exact for polyharmonic functions of higher order.

The paper is organized as follows. In §2, we provide several results, needed for the construction of the above cubatures. The computation of the weights and the nodes of a classical quadrature, used in the construction of the new formulas is performed in §3. We obtain a new Gauss-Lobatto cubature for the ball in §4, and a new Lobatto-Turan cubature for the ball in §5. The Gauss-Lobatto-Turan cubatures of type I and type II are described in §6.
2 Preliminaries

In this section, we discuss some results, needed later in the paper. First, we state a well known fact about the different representations of \( p \)-harmonic functions, see [7], Lemma 2.

**Lemma 2.1** Let \( \phi_0, \ldots, \phi_{p-1} \), be a basis in the space of univariate algebraic polynomials of degree \( p - 1 \). For each \( u \in H^p(B) \), there exist unique functions \( b_0, \ldots, b_{p-1} \), each harmonic in \( B \), such that
\[
u(x) = \sum_{j=0}^{p-1} \phi_j(|x|^2)b_j(x), \quad x \in B.
\] (2.1)

The particular choice of basis \( \phi_j(t) = t^j, j = 0, \ldots, p - 1 \), recovers the classical Almansi’s expansion, see [1], Proposition 1.3, p. 4. The next result we describe is Lemma 2 from [6].

**Lemma 2.2** For every harmonic function \( b \), we have
\[
\int_{S(r)} b(x) \, dx = \gamma_n r^{n-1} b(0), \quad \gamma_n := \frac{n\pi^{n/2}}{\Gamma(n/2 + 1)},
\] (2.2)

where \( \gamma_n \) is the area of the unit sphere \( S(1) \) in \( \mathbb{R}^n \). Moreover
\[
\int_{S(1)} \frac{\partial^i b}{\partial \nu^i}(\xi) \, d\sigma(\xi) = 0, \quad \text{for } i = 1, 2, \ldots.
\] (2.3)

3 An auxiliary quadrature

In this section, we compute the weights and nodes of the quadrature formula
\[
\int_{-1}^{1} (1 + x)^{\frac{n}{2} - 1} f(x) \, dx \approx e_0 f(1) + e_1 f'(1) + \sum_{j=1}^{m} d_j f(x_j),
\] (3.1)

which we use to construct the cubatures of **type I** and **type II**. The existence and uniqueness of such formula with highest algebraic degree of precision equal to \( 2m + 1 \) is a classical result. Its nodes \( x_1, \ldots, x_m \) are the zeroes of the Jacobi polynomials \( P_m^{(2,\frac{n}{2}-1)} \). These polynomials are orthogonal on \((-1, 1)\) with respect to the weight function \((1 - x)^2(1 + x)^{n/2-1}\). In the notation used here, they are normalized such that \( P_m^{(2,\frac{n}{2}-1)}(1) = \frac{1}{2}(m + 2)(m + 1)\).
If we use the Pochhammer symbol, defined by

\((a)_0 = 1, \quad (a)_j = a(a+1)\ldots (a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}, \quad j = 1, 2, \ldots,\)

with \(\Gamma\) being the Gamma function, we recall that the generalized hypergeometric series is given by

\[ pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j z^j}{(b_1)_j \cdots (b_q)_j j!}. \]

Note that Jacobi polynomials \(P^{(2, \frac{n}{2}-1)}_m(x)\) are hypergeometric polynomials, namely that

\[ P^{(2, \frac{n}{2}-1)}_m(x) = \frac{1}{2} (m+1)(m+2) \frac{1}{2} F_1(-m, m+n/2+2; 3; (1-x)/2) \quad (3.2) \]

For simplicity, we denote by

\[ R(x) := \frac{1}{2} F_1(-m, m+n/2+2; 3; (1-x)/2). \]

Clearly, \(R(1) = 1\). Further, we will need the following technical lemma which explicitly computes the integral below.

**Lemma 3.1**

\[ I(p) := \int_{-1}^{1} (1+x)^{\frac{n}{2}-1}(1-x)^p R(x) \, dx = \frac{2^{n/2+p} p! (n/2)_m (2-p)_m}{(n/2)_{p+1} (3)_m (n/2 + p + 1)_m} \]

**Proof:** We use the change of variables \(1+x = 2t\) to derive

\[ I(p) = 2^{n/2+p} \int_0^1 t^{n/2-1} (1-t)^p F_1(-m, m+n/2+2; 3; 1-t) \, dt \]

\[ = 2^{n/2+p} \sum_{j=0}^m \frac{(-m)_j (m+n/2+2)_j}{(3)_j j!} \Gamma(n/2) \Gamma(n/2+j+p+1) \frac{1}{(j+1)!} \]

\[ = \frac{2^{n/2+p} p!}{(n/2)_{p+1}} \sum_{j=0}^m \frac{(-m)_j (m+n/2+2)_j (p+1)_j}{(3)_j (n/2 + p + 1)_j j!} \]

\[ = \frac{2^{n/2+p} p!}{(n/2)_{p+1}} \cdot 3 \, F_2(-m, m+n/2 + 2, p+1; 3, n/2 + p + 1; 1). \]
Application of the Saalschütz’s formula, see [2, p. 9, Eq. (1)],

\[ _3F_2(-m, a, b; c, 1 + a + b - c - m; 1) = \frac{(c - a)_m(c - b)_m}{(c)_m(c - a - b)_m}, \]  

(3.3)

for \( a = m + n/2 + 2, b = p + 1, c = 3 \), gives

\[ I(p) = \frac{2^{n/2+p}p!(n/2)_m(2-p)_m}{(n/2)p+1(n/2+p+1)_m}, \]

and the proof is completed.

The next lemma provides explicitly the nodes and weights of quadrature (3.1).

**Lemma 3.2** The quadrature formula (3.1) with weights

\[ e_0 = 2^{n/2+2} \frac{(8m^2 + 4mn + 16m + 3n + 6)}{3(m + 1)(m + 2)(n + 2m)(n + 2m + 2)}, \]

\[ e_1 = -2^{n/2+4} \frac{1}{(m + 1)(m + 2)(n + 2m)(n + 2m + 2)}, \]

\[ d_j = 2^{n/2+2} \frac{(4m + n + 6)^2(m + 2)}{(m + 1)(2m + n + 4)^2(2m + n)(2m + n + 2)}, \]

\[ (1 + x_j) \frac{(1 + x_j)}{(1 - x_j)} P^{(2,n/2-1)}_{m+1}(x_j)^2, \]

and nodes \( \{x_j\}_{j=1}^m \) the zeroes of the Jacobi polynomials \( P^{(2,n/2-1)}_m \) is the only quadrature of this form that has ADP(3.1) = \( 2m + 1 \).

**Proof:** Since the existence and uniqueness of this quadrature is a classical result, we shall simply compute its weights. First, we compute \( e_0 \) and \( e_1 \). We apply (3.1) to \((1 - x)R(x) \in \pi_{m+1}(\mathbb{R}) \). Using Lemma 3.1, we have

\[ e_1 = -I(1) = -\frac{2n/2+1m!}{(3)_m(n/2 + m)_2} = -\frac{2n/2+4}{(m + 1)(m + 2)(n + 2m)(n + 2m + 2)}. \]

To calculate the weight \( e_0 \), we apply (3.1) to the polynomial \( R \) and derive that

\[ e_0 R(1) + e_1 R'(1) = I(0). \]

Since

\[ R'(1) = \frac{2}{(m + 1)(m + 2)} \frac{d}{dx} P^{(2,n/2-1)}_m(1) = \frac{2m + n + 4}{2(m + 1)(m + 2)} P^{(3,n/2)}_{m-1}(1) = \frac{m(2m + n + 4)}{12}, \]

and from Lemma 3.1, we have

\[ I(0) = \frac{2^{n/2+2}}{(m + 2)(n + 2m)}, \]
we obtain
\[ e_0 = I(0) - e_1 R'(1) = \frac{2^5+2(8m^2 + 4mn + 16m + 3n + 6)}{3(m + 1)(m + 2)(n + 2m)(n + 2m + 2)}. \]

Next, to calculate the coefficients \( \{d_j\}_{j=1}^m \), we apply (3.1) to \((1 - x)^2 S(x)\), where \( S \) is any polynomial from \( \pi_{2m-1}(IR) \). It follows that
\[
\int_{-1}^{1} (1 + x)^{n/2-1}(1 - x)^2 S(x) \, dx = \sum_{j=1}^{m} d_j (1 - x_j)^2 S(x_j),
\]
and therefore this is the unique Gaussian quadrature on \((-1, 1)\) with the weight \((1 + x)^{n/2-1}(1 - x)^2\). The nodes of this formula are the zeroes of \( P_{m+1}^{(2,n/2-1)}(x) \) and the coefficients \( \lambda_j \) are given by, see [16, p. 352, Eq.(15.3.1)],
\[
\lambda_j = 2^{n/2+2} \frac{4(m + 1)(m + 2)}{(2m + n)(2m + n + 2)} \frac{1}{(1 - x_j^2)} \left\{ \frac{d}{dx} P_{m}^{(2,n/2-1)}(x_j) \right\}^{-2} (1 - x_j^2)^2,
\]
where in the last equality we have used the fact, see [16, (4.5.7)],
\[
\frac{d}{dx} P_{m}^{(2,n/2-1)}(x_j) = -\frac{2(m + 1)(2m + n + 4)}{4m + n + 6} \frac{P_{m+1}^{(2,n/2-1)}(x_j)}{(1 - x_j^2)^2}.
\]
Therefore, we have
\[
d_j = \lambda_j (1 - x_j)^{-2} = 2^{n/2+2} \frac{(4m + n + 6)^2(m + 2)}{(m + 1)(2m + n + 4)^2(2m + n)(2m + n + 2)} \cdot \frac{1 + x_j}{(1 - x_j)} \left[ P_{m+1}^{(2,n/2-1)}(x_j) \right]^2,
\]
and the proof is completed.

Next, we provide several tables with the weights and nodes, see Tables 1-7, of the quadrature
\[
\int_{-1}^{1} (1 + x)^{n/2} f(x) \, dx \approx e_0 f(1) + e_1 f'(1) + \sum_{j=1}^{m} d_j f(x_j), \tag{3.4}
\]
which is (3.1) for \( n = 3 \).

4 Gauss-Lobatto cubature for the ball

In this section, we investigate cubatures of type (1.7) that have maximal possible Polyharmonic Degree of Precision. Clearly, the PDP(1.7) \( \leq 2m + 2 \), since it is not
Table 1: Formula (3.4), $m=1$, $e_0 = 0.915872$, $e_1 = -0.215499$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>0.969746</td>
</tr>
<tr>
<td>$x_j$</td>
<td>-0.333333</td>
</tr>
</tbody>
</table>

Table 2: Formula (3.4), $m=2$, $e_0 = 0.513806$, $e_1 = -0.059861$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>0.396975</td>
<td>0.974837</td>
</tr>
<tr>
<td>$x_j$</td>
<td>-0.645661</td>
<td>0.184123</td>
</tr>
</tbody>
</table>

Table 3: Formula (3.4), $m=3$, $e_0 = 0.325698$, $e_1 = -0.022856$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>0.198750</td>
<td>0.593721</td>
<td>0.767450</td>
</tr>
<tr>
<td>$x_j$</td>
<td>-0.780044</td>
<td>-0.212793</td>
<td>0.463425</td>
</tr>
</tbody>
</table>

Table 4: Formula (3.4), $m=4$, $e_0 = 0.224164$, $e_1 = -0.010549$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>0.070277</td>
<td>0.372180</td>
<td>0.588545</td>
<td>0.587653</td>
</tr>
<tr>
<td>$x_j$</td>
<td>-0.850186</td>
<td>-0.444336</td>
<td>0.099507</td>
<td>0.623586</td>
</tr>
</tbody>
</table>

Table 5: Formula (3.4), $m=5$, $e_0 = 0.163466$, $e_1 = -0.005526$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>0.046593</td>
<td>0.167766</td>
<td>0.314448</td>
<td>0.424946</td>
<td>0.447498</td>
</tr>
<tr>
<td>$x_j$</td>
<td>-0.917675</td>
<td>-0.684043</td>
<td>-0.336962</td>
<td>0.067332</td>
<td>0.463376</td>
</tr>
</tbody>
</table>

Table 6: Formula (3.4), $m=6$, $e_0 = 0.124387$, $e_1 = -0.003169$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>0.032450</td>
<td>0.119572</td>
<td>0.236310</td>
<td>0.337241</td>
<td>0.394313</td>
<td>0.380360</td>
</tr>
<tr>
<td>$x_j$</td>
<td>-0.935446</td>
<td>-0.750014</td>
<td>-0.467351</td>
<td>-0.123498</td>
<td>0.237711</td>
<td>0.570260</td>
</tr>
</tbody>
</table>

Table 7: Formula (3.4), $m=7$, $e_0 = 0.097784$, $e_1 = -0.001946$. 

exact for the polynomial

\[(1 - |x|^2)^2 (|x|^2 - \tau_1^2)^2 \cdots (|x|^2 - \tau_m^2)^2 \in \pi_{4m+4}(IR^n).\]

The existence and uniqueness of formulas of type (1.7) is closely related to the existence and uniqueness of quadratures of type (3.1), as the following lemma, similar to Lemma 3 in [7], claims.

**Lemma 4.1** Let \(\mu\) be a weight function on \((0, 1)\) and \(0 < \tau_1 < \ldots < \tau_m < 1\). Then cubature

\[
\int_B \mu(|x|)u(x) \, dx \approx \tilde{E}_0 \int_{S(1)} u(\xi) d\sigma(\xi) + \tilde{E}_1 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi) + \sum_{j=1}^N \tilde{D}_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi),
\]

is exact for \(H^p(B)\) if and only if the quadrature

\[
\int_0^1 \mu(t)t^{n-1}f(t^2) \, dt \approx \tilde{E}_0 f(1) + 2\tilde{E}_1 f'(1) + \sum_{j=1}^N \tilde{D}_j \tau_j^{n-1}f(\tau_j^2), \tag{4.1}
\]

is exact for all polynomials in \(\pi_{p-1}(IR)\).

**Proof:** For the \(p\)-harmonic function \(u\), we use Lemma 2.1 with \(\phi_j(t) = t^j\), and obtain

\[
u(u) = \sum_{j=0}^{p-1} |x|^{2j}b_j(|x|). \tag{4.2}
\]

We associate with \(u\) the polynomial \(P\), where

\[
P(t) := \sum_{j=0}^{p-1} t^j b_j(0) \in \pi_{p-1}(IR).
\]

We integrate (4.2) over the sphere \(S(t)\) and obtain

\[
\int_{S(t)} u(\xi) \, d\sigma(\xi) = \sum_{j=0}^{p-1} t^{2j} \int_{S(t)} b_j(\xi) \, d\sigma(\xi) = \gamma_n t^{n-1} \sum_{j=0}^{p-1} t^{2j} b_j(0) = \gamma_n t^{n-1} P(t^2), \tag{4.3}
\]

where we have used Lemma 2.2 in the next to the last equality. We also compute the normal derivative of \(u\),

\[
\frac{\partial}{\partial \nu} u(|x|) = \sum_{j=0}^{p-1} |x|^{2j} \frac{\partial}{\partial \nu} b_j(|x|) + \sum_{j=1}^{p-1} 2j|x|^{2j-1} b_j(|x|),
\]
and again by integrating over $S(t)$ and using Lemma 2.2, we get

$$
\int_{S(t)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi) = \sum_{j=0}^{p-1} t^{2j} \int_{S(t)} \frac{\partial}{\partial \nu} b_j(|\xi|) \, d\sigma(\xi) + \sum_{j=1}^{p-1} 2j t^{2j-1} \int_{S(t)} b_j(|\xi|) \, d\sigma(\xi)
$$

$$
= \gamma_n t^{n-1} \sum_{j=1}^{p-1} 2j t^{2j-1} b_j(0) = \gamma_n t^{n-1} \frac{d}{dt} [P(t^2)] = 2\gamma_n t^n P(t^2). \quad (4.4)
$$

Change of variables and (4.3) give

$$
\int_B \mu(|x|) u(x) \, dx = \int_0^1 \int_{S(t)} \mu(|\xi|) u(\xi) \, d\sigma(\xi) \, dt = \gamma_n \int_0^1 \mu(t) t^{n-1} P(t^2) \, dt. \quad (4.5)
$$

On the other hand, applying (4.3) and (4.4), we derive

$$
\tilde{E}_0 \int_{S(1)} u(\xi) d\sigma(\xi) + \tilde{E}_1 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi) + \sum_{j=1}^N \tilde{D}_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi)
$$

$$
= \gamma_n \tilde{E}_0 P(1) + 2\gamma_n \tilde{E}_1 P'(1) + \gamma_n \sum_{j=1}^N \tilde{D}_j \tau_j^{n-1} P(\tau_j^2). \quad (4.6)
$$

Comparing (4.5) and (4.6) completes the proof of the lemma.

Application of Lemma 4.1 and Lemma 3.2 lead to the proof of the existence and uniqueness of the Gauss-Lobatto formula for the ball, stated in the next theorem.

**Theorem 4.2** There is a unique Gauss-Lobatto cubature for the ball of type (1.7) with $PDE(1.7) = 2m + 2$. Its nodes are given by $\tau_j = \sqrt{(1 + x_j)/2}$, where $\{x_j\}_{j=1}^m$ are the zeros of the Jacobi polynomial $P_m^{(2, \frac{n}{2}-1)}$, and its weights are

$$
E_0 = 2^{-\frac{n}{2}-1} e_0, \quad E_1 = 2^{-\frac{n}{2}-3} e_1, \quad D_j = 2^{-\frac{3}{2}} (x_j + 1)^{-\frac{n}{2} + \frac{3}{2}} d_j, \quad j = 1, \ldots, m,
$$

where $e_0, e_1$ and $\{d_j\}_{j=1}^m$ are the coefficients from Lemma 3.2.

**Proof:** It follows from Lemma 4.1 with $\mu \equiv 1$, $N = m$, and $p = 2m + 2$, that formula (1.7) is the only cubature with $PDP = 2m + 2$ if and only if

$$
\int_0^1 t^{n-1} f(t^2) \, dt \approx E_0 f(1) + 2E_1 f'(1) + \sum_{j=1}^m D_j \tau_j^{n-1} f(\tau_j^2)
$$

is the only quadrature, exact for all polynomials from $\pi_{2m+1}(I)$. After change of variables $1 + x = 2t^2$ and the new notation,

$$
g(x) := f \left( \frac{1 + x}{2} \right)
$$
the above quadrature quadrature can be rewritten as
\[
\int_{-1}^{1} (1 + x)^{\frac{n-1}{2}} g(x) \, dx \approx 2^{\frac{n+1}{2}} E_0 g(1) + 2^{\frac{n+3}{2}} E_1 g'(1) + 2^{\frac{n+1}{2}} \sum_{j=1}^{m} D_j \tau_j^{n-1} g(2\tau_j^2 - 1).
\]
We apply Theorem 3.2 and derive that
\[
E_0 = 2^{-\frac{n}{2}} - 1 e_0, \quad E_1 = 2^{-\frac{n-3}{2}} e_1, \quad D_j = 2^{-\frac{n-1}{2}} \tau_j^{1-n} d_j, \quad \tau_j = \sqrt{(1 + x_j)/2}.
\]

**Remark 4.3** The weights from Theorem 4.2 can be computed explicitly and are
\[
E_0 = \frac{2(8m^2 + 4mn + 16m + 3n + 6)}{3(m+1)(m+2)(n+2m)(n+2m+2)},
\]
\[
E_1 = -\frac{2}{(m+1)(m+2)(n+2m)(n+2m+2)},
\]
and for \( j = 1, \ldots, m, \)
\[
D_j = 2^{\frac{n+1}{2}} \frac{4(4m + n + 6)^2(m+2)}{(m+1)(2m + n + 4)^2(2m + n)(2m + n+2)} \frac{(1 + x_j)^{-\frac{n}{2}+\frac{3}{2}}}{(1 - x_j)^2 \left[P_{m+1}^{(2n/2-1)}(x_j)\right]^2},
\]
where \( \{x_j\}_{j=1}^{m} \) are the zeroes of the Jacobi polynomials \( P_{m}^{(2n/2-1)} \).

Next, we provide the coefficients and nodes of cubature (1.7), see Tables 8-14 in the two-dimensional case \((n = 2)\).

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_j )</td>
<td>0.592593</td>
</tr>
<tr>
<td>( \tau_j )</td>
<td>0.500000</td>
</tr>
</tbody>
</table>

Table 8: Formula (1.7), \( m = 1 \), \( E_0 = 0.203704 \), \( E_1 = -0.013889 \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_j )</td>
<td>0.431427</td>
<td>0.328753</td>
</tr>
<tr>
<td>( \tau_j )</td>
<td>0.350021</td>
<td>0.737666</td>
</tr>
</tbody>
</table>

Table 9: Formula (1.7), \( m = 2 \), \( E_0 = 0.106481 \), \( E_1 = -0.003472 \).
<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_j$</td>
<td>0.270174</td>
<td>0.291116</td>
<td>0.204948</td>
<td></td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.204613</td>
<td>0.139218</td>
<td>0.839644</td>
<td></td>
</tr>
</tbody>
</table>

Table 10: Formula (1.7), $m=3$, $E_0 = 0.065000$, $E_1 = -0.001250$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_j$</td>
<td>0.278077</td>
<td>0.253301</td>
<td>0.204613</td>
<td>0.139218</td>
<td></td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.253301</td>
<td>0.488468</td>
<td>0.719060</td>
<td>0.892105</td>
<td></td>
</tr>
</tbody>
</table>

Table 11: Formula (1.7), $m=4$, $E_0 = 0.043704$, $E_1 = -0.000556$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_j$</td>
<td>0.235883</td>
<td>0.221766</td>
<td>0.191887</td>
<td>0.150503</td>
<td>0.100503</td>
<td></td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.185954</td>
<td>0.416510</td>
<td>0.624409</td>
<td>0.796451</td>
<td>0.922526</td>
<td></td>
</tr>
</tbody>
</table>

Table 12: Formula (1.7), $m=5$, $E_0 = 0.031368$, $E_1 = -0.000283$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_j$</td>
<td>0.204752</td>
<td>0.196262</td>
<td>0.176714</td>
<td>0.149117</td>
<td>0.114937</td>
<td>0.075877</td>
<td></td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.160949</td>
<td>0.362717</td>
<td>0.549943</td>
<td>0.713468</td>
<td>0.845979</td>
<td>0.941704</td>
<td></td>
</tr>
</tbody>
</table>

Table 13: Formula (1.7), $m=6$, $E_0 = 0.023597$, $E_1 = -0.000159$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_j$</td>
<td>0.180852</td>
<td>0.175577</td>
<td>0.162143</td>
<td>0.142898</td>
<td>0.118710</td>
<td>0.090471</td>
<td>0.059276</td>
<td></td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.141890</td>
<td>0.321075</td>
<td>0.490463</td>
<td>0.643430</td>
<td>0.774611</td>
<td>0.879498</td>
<td>0.954559</td>
<td></td>
</tr>
</tbody>
</table>

Table 14: Formula (1.7), $m=7$, $E_0 = 0.018390$, $E_1 = -0.000096$.

5 Lobatto-Turan cubature for the ball

In this section, we discuss cubatures of the form (1.8) that are exact for all functions $u \in H^{2m+2}(B)$. We start with a simple lemma.

Lemma 5.1 If $u$ has continuous Laplacians of second order $\Delta^2 u$ in a neighborhood of $B$, then the following formula holds:

$$
\int_B u(x)dx = \frac{1}{n} \int_{S(1)} u(\xi)d\sigma(\xi) - \frac{1}{n(n+2)} \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) d\sigma(\xi)
$$

$$
+ \frac{1}{8n(n+2)} \int_B (1-|x|^2)^2 \Delta^2 u(x) dx.
$$

12
Proof: We apply the divergence theorem first to the vector field \( F \),
\[
F := (1 - |x|^2)^2 \left( \frac{\partial}{\partial x_1} (\Delta u), \ldots, \frac{\partial}{\partial x_n} (\Delta u) \right)
\]
and obtain
\[
\frac{1}{4} \int_B (1 - |x|^2)^2 \Delta^2 u(x) \, dx = \int_B (1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (\Delta u) \, dx,
\]
and then to the vector field \( G \),
\[
G := (1 - |x|^2) \Delta u(x_1, \ldots, x_n),
\]
and derive
\[
\int_B (1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (\Delta u) \, dx = 2 \int_B \Delta u(x) \, dx - (n + 2) \int_B (1 - |x|^2) \Delta u(x) \, dx.
\]
It follows from the last two formulas that
\[
\int_B (1 - |x|^2)^2 \Delta^2 u(x) \, dx = 8 \int_B \Delta u(x) \, dx - 4(n + 2) \int_B (1 - |x|^2) \Delta u(x) \, dx.
\]
Similarly, we derive that
\[
\int_B \Delta u(x) \, dx = \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi),
\]
and
\[
\int_B (1 - |x|^2) \Delta u(x) \, dx = 2 \int_{S(1)} u(\xi) \, d\sigma(\xi) - 2n \int_B u(x) \, dx.
\]
It follows from (5.2), (5.3) and (5.4) that
\[
\int_B (1 - |x|^2)^2 \Delta^2 u(x) \, dx = 8n(n + 2) \int_B u(x) \, dx - 8(n + 2) \int_{S(1)} u(\xi) \, d\sigma(\xi)
\]
\[
+ 8 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi),
\]
and the proof is completed.
Theorem 5.2 There is a unique cubature of type (1.8), exact for all \( u \in H^{2m+2}(B) \), with weights
\[
P_0 = \frac{1}{n}, \quad P_1 = -\frac{1}{n(n+2)},
\]
and for \( j = 1, \ldots, m \),
\[
Q_j = 2^{\frac{n+2}{2}} \frac{(1-x_j)^2 (1+x_j)^{-\frac{n+2}{2}+\frac{1}{2}}}{n(n+2)} d_j,
\]
with \( \{d_j\}_{j=1}^m \) being the weights from Lemma 3.2. Its nodes are \( \tau_j = \sqrt{(1+x_j)/2} \), where \( \{x_j\}_{j=1}^m \) are the zeros of the Jacobi polynomial \( P^{(2,n/2-1)}_m \). There is no other formula with higher polyharmonic degree of precision.

Proof: Let us first notice that, if \( u \in H^{2m+2}(B) \), then the function
\[
(1-|x|^2)^2 \Delta^2 u(x) \in H^{2m+2}(B),
\]
see [14], and we can use the Gauss-Lobatto cubature (1.7), to compute the integral \( \int_B (1-|x|^2)^2 \Delta^2 u(x) \, dx \). Since
\[
\int_{S(1)} (1-|x|^2)^2 \Delta^2 u(x) \, d\sigma(\xi) = 0, \quad \int_{S(1)} \frac{\partial}{\partial \nu} ((1-|x|^2)^2 \Delta^2 u(x)) \, d\sigma(\xi) = 0,
\]
we have
\[
\int_B (1-|x|^2)^2 \Delta^2 u(x) \, dx = \sum_{j=1}^m D_j (1-\tau_j^2)^2 \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi).
\]
The latter formula and Lemma 5.1 give the new cubature
\[
\int_B u(x) \, dx = \frac{1}{n} \int_{S(1)} u(\xi) d\sigma(\xi) - \frac{1}{n(n+2)} \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi) \, d\sigma(\xi) + \frac{1}{8n(n+2)} \sum_{j=1}^m D_j (1-\tau_j^2)^2 \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi),
\]
and the proof is completed.

Remark 5.3 The weights from Theorem 5.2 can be computed explicitly and they are for \( j = 1, \ldots, m \),
\[
Q_j = \frac{2^\frac{n+2}{2} (4m+n+6)^2 (m+2)}{n(n+2)(m+1)(2m+n+4)(2m+n)(2m+n+2)} \cdot \frac{(1-x_j)(1+x_j)^{-\frac{n+2}{2}+\frac{1}{2}}}{P^{(2,n/2-1)}_{m+1}(x_j)^2},
\]
where \( \{x_j\}_{j=1}^m \) are the zeroes of the Jacobi polynomial \( P^{(2,n/2-1)}_m \).
The coefficients and nodes of cubature (1.8) in the three-dimensional case \((n = 3)\) are presented in Tables 15-21.

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_j)</td>
<td>0.001905</td>
<td>0.000404</td>
</tr>
<tr>
<td>(\tau_j)</td>
<td>0.577350</td>
<td>0.769455</td>
</tr>
</tbody>
</table>

Table 15: Formula (1.8), \(m = 1\), \(P_0 = 0.333333\), \(P_1 = -0.066667\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_j)</td>
<td>0.002235</td>
<td>0.000817</td>
</tr>
<tr>
<td>(\tau_j)</td>
<td>0.420915</td>
<td>0.627378</td>
</tr>
</tbody>
</table>

Table 16: Formula (1.8), \(m = 2\), \(P_0 = 0.333333\), \(P_1 = -0.066667\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_j)</td>
<td>0.002109</td>
<td>0.000817</td>
<td>0.000111</td>
</tr>
<tr>
<td>(\tau_j)</td>
<td>0.331630</td>
<td>0.627378</td>
<td>0.855402</td>
</tr>
</tbody>
</table>

Table 17: Formula (1.8), \(m = 3\), \(P_0 = 0.333333\), \(P_1 = -0.066667\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_j)</td>
<td>0.001705</td>
<td>0.001029</td>
<td>0.000320</td>
<td>0.000038</td>
</tr>
<tr>
<td>(\tau_j)</td>
<td>0.273692</td>
<td>0.527098</td>
<td>0.741454</td>
<td>0.900996</td>
</tr>
</tbody>
</table>

Table 18: Formula (1.8), \(m = 4\), \(P_0 = 0.333333\), \(P_1 = -0.066667\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_j)</td>
<td>0.001533</td>
<td>0.001109</td>
<td>0.000502</td>
<td>0.000255</td>
<td>0.000065</td>
</tr>
<tr>
<td>(\tau_j)</td>
<td>0.202885</td>
<td>0.397465</td>
<td>0.575777</td>
<td>0.730525</td>
<td>0.855388</td>
</tr>
</tbody>
</table>

Table 19: Formula (1.8), \(m = 5\), \(P_0 = 0.333333\), \(P_1 = -0.066667\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_j)</td>
<td>0.001533</td>
<td>0.001109</td>
<td>0.000624</td>
<td>0.000255</td>
<td>0.000065</td>
<td>0.000007</td>
</tr>
<tr>
<td>(\tau_j)</td>
<td>0.202885</td>
<td>0.397465</td>
<td>0.575777</td>
<td>0.730525</td>
<td>0.855388</td>
<td>0.945326</td>
</tr>
</tbody>
</table>

Table 20: Formula (1.8), \(m = 6\), \(P_0 = 0.333333\), \(P_1 = -0.066667\).
Table 21: Formula (1.8), $m=7$, $P_0 = 0.333333$, $P_1 = -0.066667$.

6 Gauss-Lobatto-Turan cubatures for the ball

Now, we can combine cubatures (1.7) and (1.8), both having $PDP= 2m + 2$, and obtain the Gauss-Lobatto-Turan cubature of type I and type II.

**Theorem 6.1** There is a unique Gauss-Lobatto-Turan cubature of type I, (1.4), with weights

$$A = \frac{E_0 P_1 - P_0 E_1}{P_1 - E_1}, \quad B_j = \frac{P_1}{P_1 - E_1} D_j, \quad C_j = -\frac{E_1}{P_1 - E_1} Q_j, \quad j = 1, \ldots, m,$$

and of type II, (1.5), with weights

$$F = \frac{E_1 P_0 - P_1 E_0}{P_0 - E_0}, \quad G_j = \frac{P_0}{P_0 - E_0} D_j, \quad H_j = -\frac{E_0}{P_0 - E_0} Q_j, \quad j = 1, \ldots, m.$$

Here $E_0, E_1, \{D_j\}_{j=1}^m$, and $P_0, P_1$ and $\{Q_j\}_{j=1}^m$ are the weights from Theorem 4.2 and Theorem 5.2, respectively. The nodes of both cubatures are $\tau_j = \sqrt{(1 + x_j)/2}$, $j = 1, \ldots, m$, where $\{x_j\}_{j=1}^m$ are the zeros of the Jacobi polynomial $P_m^{(2, \frac{1}{2})-1}$. There are no formulas of these types with polyharmonic degree of precision higher than $2m + 2$.

**Proof:** We multiply (1.7) (which by Theorem 4.2 exists and is exact for all polyharmonic functions of order $2m + 2$) by $P_1$ and (1.8) by $-E_1$, add them together and obtain

$$\int_B u(x) \, dx = \frac{E_0 P_1 - P_0 E_1}{P_1 - E_1} \int_{S(1)} u(\xi) \, d\sigma(\xi) + \frac{P_1}{P_1 - E_1} \sum_{j=1}^m D_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi)$$

$$- \frac{E_1}{P_1 - E_1} \sum_{j=1}^m Q_j \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi).$$

Similarly, we derive

$$\int_B u(x) \, dx = \frac{E_1 P_0 - P_1 E_0}{P_0 - E_0} \int_{S(1)} u(\xi) \, d\sigma(\xi) + \frac{P_0}{P_0 - E_0} \sum_{j=1}^m D_j \int_{S(\tau_j)} u(\xi) \, d\sigma(\xi)$$

$$- \frac{E_0}{P_0 - E_0} \sum_{j=1}^m Q_j \int_{S(\tau_j)} \Delta^2 u(\xi) \, d\sigma(\xi),$$
which is exact for all polyharmonic functions of order $2m + 2$.

The coefficients and nodes of cubature (1.4) in the three-dimensional case ($n = 3$) are shown in Tables 22-28.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.600000</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000317</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.577350</td>
</tr>
</tbody>
</table>

Table 22: Formula (1.4), $m = 1$, $A = 0.133333$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.412463</td>
<td>0.303093</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000092</td>
<td>-0.000017</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.420915</td>
<td>0.769455</td>
</tr>
</tbody>
</table>

Table 23: Formula (1.4), $m=2$, $A = 0.080808$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.324382</td>
<td>0.270757</td>
<td>0.188263</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000032</td>
<td>-0.000002</td>
<td>-0.000001</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.331630</td>
<td>0.627378</td>
<td>0.855402</td>
</tr>
</tbody>
</table>

Table 24: Formula (1.4), $m=3$, $A = 0.053333$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.268732</td>
<td>0.238475</td>
<td>0.190583</td>
<td>0.128869</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000013</td>
<td>-0.000002</td>
<td>-0.000000</td>
<td>-0.000000</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.273692</td>
<td>0.527098</td>
<td>0.741454</td>
<td>0.928016</td>
</tr>
</tbody>
</table>

Table 25: Formula (1.4), $m=4$, $A = 0.037559$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.229640</td>
<td>0.210905</td>
<td>0.180811</td>
<td>0.141005</td>
<td>0.093821</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000006</td>
<td>-0.000004</td>
<td>-0.000002</td>
<td>-0.000000</td>
<td>-0.000000</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.233020</td>
<td>0.453481</td>
<td>0.649506</td>
<td>0.810546</td>
<td>0.928016</td>
</tr>
</tbody>
</table>

Table 26: Formula (1.4), $m=5$, $A = 0.027778$.  

17
Table 27: Formula (1.4), $m=6$, $A = 0.021333$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.200521</td>
<td>0.188123</td>
<td>0.168027</td>
<td>0.141059</td>
<td>0.108344</td>
<td>0.071360</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000003</td>
<td>-0.000002</td>
<td>-0.000001</td>
<td>-0.000001</td>
<td>-0.000000</td>
<td>-0.000000</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.202885</td>
<td>0.397465</td>
<td>0.575777</td>
<td>0.730525</td>
<td>0.855388</td>
<td>0.945326</td>
</tr>
</tbody>
</table>

Table 28: Formula (1.4), $m=7$, $A = 0.016878$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_j$</td>
<td>0.177954</td>
<td>0.169329</td>
<td>0.155262</td>
<td>0.136208</td>
<td>0.112781</td>
<td>0.085751</td>
<td>0.056095</td>
</tr>
<tr>
<td>$C_j$</td>
<td>-0.000002</td>
<td>-0.000001</td>
<td>-0.000001</td>
<td>-0.000000</td>
<td>-0.000000</td>
<td>-0.000000</td>
<td>-0.000000</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.179659</td>
<td>0.353543</td>
<td>0.516066</td>
<td>0.662005</td>
<td>0.786674</td>
<td>0.886076</td>
<td>0.957072</td>
</tr>
</tbody>
</table>

Table 29: Formula (1.5), $m=1$, $F = 0.044444$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_j$</td>
<td>1.000000</td>
</tr>
<tr>
<td>$H_j$</td>
<td>-0.001799</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.577350</td>
</tr>
</tbody>
</table>

Table 30: Formula (1.5), $m=2$, $F = 0.021333$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_j$</td>
<td>0.544451</td>
<td>0.400082</td>
</tr>
<tr>
<td>$H_j$</td>
<td>-0.000837</td>
<td>-0.000151</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.420915</td>
<td>0.769455</td>
</tr>
</tbody>
</table>

Table 31: Formula (1.5), $m=3$, $F = 0.012698$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_j$</td>
<td>0.302857</td>
<td>0.268758</td>
<td>0.214784</td>
</tr>
<tr>
<td>$H_j$</td>
<td>-0.000257</td>
<td>-0.000139</td>
<td>-0.000043</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.273692</td>
<td>0.527098</td>
<td>0.741454</td>
</tr>
</tbody>
</table>

Table 32: Formula (1.5), $m=4$, $F = 0.008466$. 

<table>
<thead>
<tr>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_j$</td>
<td>0.302857</td>
<td>0.268758</td>
<td>0.214784</td>
<td>0.145233</td>
</tr>
<tr>
<td>$H_j$</td>
<td>-0.000257</td>
<td>-0.000139</td>
<td>-0.000043</td>
<td>-0.000005</td>
</tr>
<tr>
<td>$\tau_j$</td>
<td>0.273692</td>
<td>0.527098</td>
<td>0.741454</td>
<td>0.900996</td>
</tr>
</tbody>
</table>
The coefficients and nodes of cubature (1.5) in the three-dimensional case ($n = 3$) are shown in Tables 29-35.

Finally, we demonstrate the PDP of formulas (1.4) and (1.5) in the two dimensional case ($n = 2$). We apply both formulas to the function $u(x, y) = (x^2 + y^2)^{2m+1} \in H^{2m+2}(B)$ and show that they are exact up to the machine error, see Table 36.

**Acknowledgment:** The authors would like to thank Professor Dimitar K. Dimitrov for the fruitful discussions on the topic and Matthew Hielsberg for the help in carrying the numerical simulations. This research was supported by Office of Naval Research Contract ONR N00014-11-1-0712 and the NSF Grant DMS-1222715.

**References**


