EAST JOURNAL ON APPROXIMATIONS
Volume 16, Number 1 (2010), 93-107

A R C H I V E S:

ON SOME INTERPOLATION EXPANSIONS OF REGULAR–MONOTONE FUNCTIONS *

Yaroslav A. Tagamlitski

1. Introduction

Let the function f be defined and infinitely many times differentiable on the closed interval $[0,1]$. Bernstein calls such functions regular-monotone [1] , if

$$\epsilon_n f^{(n)}(x) \geq 0, \quad 0 \leq x \leq 1, \; n = 0, 1, \ldots \tag{1.1}$$

and

$$\epsilon_n = \pm 1, \tag{1.2}$$

where the constants $\{\epsilon_n\}$ do not depend on x but can vary with n.

Bernstein had investigated the properties of the regular-monotone functions in several papers [2, 3, 4] and had found interesting representations of these functions in the cases when

$$\epsilon_n = 1, \quad \epsilon_n = (-1)^n, \quad \epsilon_n = (-1)^{\left[\frac{n}{2}\right]}, \quad \epsilon_n = (-1)^{\left[\frac{n+1}{2}\right]}.$$

Our student Sendov, using methods from functional analysis, gave a nice generalization [5] of the results of Bernstein by finding the corresponding expansions in the case when $\{\epsilon_n\}$ is an arbitrary periodic sequence satisfying (1.2). In his paper [6], applying techniques from classical analysis, Sendov further generalizes his previous results by reducing even more the assumptions on the sequence $\{\epsilon_n\}$.

* The original document, written in Bulgarian, has been published in: Izvestia Math. Institute **3**, 2 (1959), 187–200. The present translation into English was prepared by Guergana Petrova.

Notice, however, that for an arbitrary choice of the constants $\{\epsilon_n\}$ for which (1.2) holds, there exist functions f satisfying (1.1) that are not idenicaly zero. For example, such functions are the Abel-Goncharov polynomials defined as follows:

$$P_0(x) = \epsilon_0, \quad P_n(x) = \epsilon_0 \int_{x_0}^{x} \int_{x_1}^{t_1} \dots \int_{x_{n-1}}^{t_{n-1}} \tau_0 \tau_1 \dots \tau_{n-1} \, dt_n \dots dt_2 dt_1, \qquad n = 1, 2, \dots,$$

where

$$x_k = \frac{1}{2}(1 - \epsilon_k \epsilon_{k+1}), \quad \tau_k = 1 - 2x_k, \quad k = 0, 1, \dots,$$

and therefore

$$\tau_k = \epsilon_k \epsilon_{k+1}, \quad k = 0, 1, \dots.$$

To show that the polynomials P_n indeed satisfy condition (1.1), it is enough to observe that if an integrant φ is non-negative and $0 \le x \le 1$, then*

$$\int_{x_k}^{x} \tau_k \varphi(t) \, dt \ge 0.$$

In this paper, we find a representation of regular-monotone functions for which the constants $\{\epsilon_n\}$ are arbitrary and do not satisfy any other condition but (1.2). Our argument is different from the approach of both Bernstein and Sendov.

2. Representation of regular-monotone functions

Let us consider the sequence $\{\epsilon_n\}$, where $\epsilon_n = \pm 1$. We denote by K the set of the corresponding regular-monotone functions, namely the set of all functions that are infinitely many times differentiable on $[0, 1]$ and satisfy (1.1). Let $f \in K$. It is a well known fact (which also can be shown directly) that

$$f(x) = \sum_{\nu=0}^{n} \epsilon_\nu f^{(\nu)}(x_v) P_\nu(x) + R_n(x), \tag{2.1}$$

where

$$R_n(x) = \epsilon_0 \int_{x_0}^{x} \int_{x_1}^{t_1} \dots \int_{x_n}^{t_n} \tau_0 \tau_1 \dots \tau_n \epsilon_{n+1} f^{(n+1)}(t) \, dt \dots dt_2 dt_1. \tag{2.2}$$

*We will often use this property in what follows.

Note that $P_n, R_n \in K$, $n = 0, 1, \ldots$, and in particular, $\epsilon_0 P_n(x) \geq 0$ and $\epsilon_0 R_n(x) \geq 0$, $n = 0, 1, \ldots$. This implies that for all positive integer values of n

$$\epsilon_0 f(x) \geq \sum_{\nu=0}^{n} \epsilon_\nu f^{(\nu)}(x_v) \epsilon_0 P_\nu(x),$$

and therefore the series with non-negative terms

$$\sum_{\nu=0}^{\infty} \epsilon_\nu f^{(\nu)}(x_v) \epsilon_0 P_\nu(x),$$

converges. Thus, the limit $\varphi(x) = \lim_{n\to\infty} R_n(x)$ exists and we have

$$f(x) = \sum_{\nu=0}^{\infty} \epsilon_\nu f^{(\nu)}(x_v) P_\nu(x) + \varphi(x). \tag{2.3}$$

One can easily check that the function φ satisfies the condition

$$\varphi^{(k)}(x_k) = 0, \quad k = 0, 1, \ldots, \tag{2.4}$$

and $\varphi \in K$. Indeed, from (2.1), we have that for $k \leq n$

$$R_n^{(k)}(x) = -\sum_{\nu=k}^{n} \epsilon_\nu f^{(\nu)}(x_v) P_\nu^{(k)}(x) + f^{(k)}(x).$$

Note that the series

$$\sum_{\nu=k}^{\infty} \epsilon_\nu f^{(\nu)}(x_v) \epsilon_k P_\nu^{(k)}(x)$$

is uniformly convergent because it is bounded from above by the convergent series

$$\sum_{\nu=k}^{\infty} \epsilon_\nu f^{(\nu)}(x_v) \epsilon_k P_\nu^{(k)}(1 - x_k),$$

whose terms do not depend on x. Thus, each of the sequences

$$R_k^{(k)}(x), R_{k+1}^{(k)}(x), \ldots, \quad k = 0, 1, \ldots,$$

converges uniformly on $[0, 1]$, and therefore the function φ is differentiable infinitely many times and

$$\varphi^{(k)}(x) = \lim_{n\to\infty} R_n^{(k)}(x).$$

This proves that $\varphi \in K$ and φ satisfies condition (2.4).

We will prove the main result of this paper by considering several cases. First, we will investigate the case when for infinitely many values of ν, $x_\nu = 0$. For this purpose, we consider the function $\varphi(ux)$, where $0 \leq u < 1$. Obviously, $\varphi(ux) \in K$ for fixed u. Formula (2.1) gives

$$\varphi(ux) = \sum_{\nu=0}^{n} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x) + S_n(x),$$

where

$$S_n(x) = \epsilon_0 u^{n+1} \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_n}^{t_n} \tau_0 \tau_1 \ldots \tau_n \epsilon_{n+1} \varphi^{(n+1)}(ut)\, dt \ldots dt_2 dt_1.$$

If we denote by $S(x) = \lim_{n\to\infty} S_n(x)$, we have

$$\varphi(ux) = \sum_{\nu=0}^{\infty} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x) + S(x).$$

Let $x_{n+1} = 0$. Then $\epsilon_{n+1}\epsilon_{n+2} = 1$ and

$$\epsilon_{n+1}\varphi^{(n+1)}(t) - \epsilon_{n+1}\varphi^{(n+1)}(ut) = \epsilon_{n+2} t(1-u)\varphi^{(n+2)}(\xi) \geq 0, \quad ut < \xi < t,$$

which gives

$$\epsilon_{n+1}\varphi^{(n+1)}(ut) \leq \epsilon_{n+1}\varphi^{(n+1)}(t).$$

It follows that

$$\begin{aligned} \epsilon_0 S_n(x) &\leq u^{n+1} \int_{x_0}^{x} \int_{x_1}^{t_1} \cdots \int_{x_n}^{t_n} \tau_0 \tau_1 \ldots \tau_n \epsilon_{n+1} \varphi^{(n+1)}(t)\, dt \ldots dt_2 dt_1 \\ &= u^{n+1} \epsilon_0 \varphi(x), \end{aligned}$$

and therefore $\epsilon_0 S_n(x) \leq u^{n+1}\epsilon_0\varphi(x)$. Knowing that $\epsilon_0 S_n(x) \geq 0$ and letting n tend to infinity, we obtain that $S(x) = 0$. In summary, if we have that for infinitely many positive integer values ν we have $x_\nu = 0$, then

$$\varphi(ux) = \sum_{\nu=0}^{\infty} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x), \quad 0 \leq x \leq 1, \quad 0 \leq u < 1. \tag{2.5}$$

If for all big enough integer values of n we have $x_n = 0$, then the case is trivial since for these values of n we have $\varphi^{(n)}(ux_n) = \varphi^{(n)}(0) = \varphi^{(n)}(x_n) = 0$, and therefore the right hand-side of (2.5) is a finite sum. If we let $u \to 1$, we obain

$$\varphi(x) = \sum_{\nu=0}^{\infty} \epsilon_\nu \varphi^{(\nu)}(x_\nu) P_\nu(x) = 0,$$

and then the expansion (2.3) becomes

$$f(x) = \sum_{\nu=0}^{\infty} \epsilon_\nu f^{(\nu)}(x_\nu) P_\nu(x). \tag{2.6}$$

Thus, we have shown that if for all big enough integer values of n, $x_n = 0$, the regular-monotone function f has a series expansion (2.6) using the Abel-Goncharov polynomials.

The case when for infinitely many values of ν, $x_\nu = 1$ is more complicated. To study this case, we will need to determine the sign of

$$\frac{d}{dx}\frac{P_n(x)}{P_m(x)}, \quad n > m \geq 0, \quad 0 < x < 1. \tag{2.7}$$

For this purpose, we chose a non-negative integer $k \leq m$, $m > 0$, and consider

$$\begin{aligned}
\frac{d}{dx}\frac{P_n^{(k)}(x)}{P_m^{(k)}(x)} &= \frac{P_m^{(k+1)}(x)}{P_m^{(k)}(x)}\left[\frac{P_n^{(k+1)}(x)}{P_m^{(k+1)}(x)} - \frac{P_n^{(k)}(x)}{P_m^{(k)}(x)}\right] \\
&= \frac{P_m^{(k+1)}(x)}{P_m^{(k)}(x)}\left[\frac{P_n^{(k+1)}(x)}{P_m^{(k+1)}(x)} - \frac{P_n^{(k)}(x) - P_n^{(k)}(x_k)}{P_m^{(k)}(x) - P_m^{(k)}(x_k)}\right] \\
&= \frac{P_m^{(k+1)}(x)}{P_m^{(k)}(x)}\left[\frac{P_n^{(k+1)}(x)}{P_m^{(k+1)}(x)} - \frac{P_n^{(k+1)}(\xi)}{P_m^{(k+1)}(\xi)}\right],
\end{aligned}$$

where ξ is a point between x and x_k. This gives that

$$\frac{d}{dx}\frac{P_n^{(k)}(x)}{P_m^{(k)}(x)} = \tau_k(x-\xi)\frac{\tau_k P_m^{(k+1)}(x)}{P_m^{(k)}(x)}\psi'(\eta), \quad \text{where } \psi(x) := \frac{P_n^{(k+1)}(x)}{P_m^{(k+1)}(x)}, \tag{2.8}$$

and η is a number between x and ξ. Since

$$\tau_k(x-\xi) > 0, \quad \frac{\tau_k P_m^{(k+1)}(x)}{P_m^{(k)}(x)} > 0,$$

we have that

$$\frac{d}{dx}\frac{P_n^{(k)}(x)}{P_m^{(k)}(x)} \quad \text{and } \psi'(\eta)$$

have the same sign and therefore the question of determining the sign of (2.8) is equivalent to the question of determining the sign of

$$\frac{d}{dx}\frac{P_n^{(k+1)}(x)}{P_m^{(k+1)}(x)}.$$

Note that

$$\frac{d}{dx}\frac{P_n^{(m)}(x)}{P_m^{(m)}(x)} = \epsilon_m P_n^{(m+1)}(x)$$

and its sign is the same as the sign of τ_m even when $m = 0$. Therefore, we have determined that the sign of

$$\frac{d}{dx}\frac{P_n^{(k)}(x)}{P_m^{(k)}(x)} \tag{2.9}$$

for $n > m \geq k \geq 0$ coincides with the sign of τ_m for all non-negative integer values of k that do not exceed m. In particular, since (2.7) is obtained from (2.9) when $k = 0$, it has the same sign as τ_m for $n > m \geq 0$ and $0 < x < 1$.

Now, we are ready to consider the case when $x_\nu = 1$ for infinitely many values of ν. Let $n_1 < n_2 < n_3 < \ldots$ be the sequence of all non-negative integer numbers for which $x_{n_k} = 1$. In this case the fraction

$$\frac{P_{n_{k+1}}(x)}{P_{n_k}(x)}$$

is monotone decreasing for $0 \leq x \leq 1$, because the sign of the derivative

$$\frac{d}{dx}\frac{P_{n_{k+1}}(x)}{P_{n_k}(x)}$$

coincides with the sign of $\tau_{n_k} = -1$. Since $\epsilon_0 P_n(1 - x_0) > 0$, we can consider the fraction

$$\frac{P_{n_{k+1}}(1-x_0)}{P_{n_k}(1-x_0)}.$$

The monotonicity gives that

$$\frac{P_{n_{k+1}}(x)}{P_{n_k}(x)} \geq \frac{P_{n_{k+1}}(1-x_0)}{P_{n_k}(1-x_0)}, \tag{2.10}$$

when $x_0 = 0$ and

$$\frac{P_{n_{k+1}}(x)}{P_{n_k}(x)} \leq \frac{P_{n_{k+1}}(1-x_0)}{P_{n_k}(1-x_0)}, \tag{2.11}$$

when $x_0 = 1$, namely in both cases we have

$$\tau_0 \left[\frac{P_{n_{k+1}}(x)}{P_{n_{k+1}}(1-x_0)} - \frac{P_{n_k}(x)}{P_{n_k}(1-x_0)}\right] \geq 0. \tag{2.12}$$

Therefore the sequence of non-negative and bounded by 1 functions

$$\frac{P_{n_1}(x)}{P_{n_1}(1-x_0)}, \frac{P_{n_2}(x)}{P_{n_2}(1-x_0)}, \ldots \tag{2.13}$$

is monotone (monotone increasing or decreasing depending on whether $x_0 = 0$ or $x_0 = 1$) and thus it converges. Let us denote its limit by $\epsilon_0 P(x)$.

Let us fix $x \in [0,1]$, chose $\epsilon > 0$ and m such that

$$\left| \frac{P_{n_k}(x)}{P_{n_k}(1-x_0)} - \epsilon_0 P(x) \right| < \epsilon \tag{2.14}$$

for $k > m$. In this case when $\nu > n_m$ we have

$$\begin{aligned} |\epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x) \; &- \; \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(1-x_0) \epsilon_0 P(x)| \\ &\leq \; \epsilon \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) \epsilon_0 P_\nu(1-x_0). \end{aligned} \tag{2.15}$$

Indeed, for $x_\nu = 1$ this inequality follows from (2.14) and for $x_\nu = 0$ we have $\varphi^{(\nu)}(ux_\nu) = 0$.

Let us consider

$$\begin{aligned} &\varphi(ux) - \epsilon_0 P(x) \varphi(1-x_0) \\ &\quad = \sum_{\nu=0}^{\infty} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x) - \epsilon_0 P(x) \sum_{\nu=0}^{\infty} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(1-x_0) \\ &\quad = \sum_{\nu=0}^{n_m} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x) - \epsilon_0 P(x) \sum_{\nu=0}^{n_m} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(1-x_0) \\ &\qquad + \sum_{\nu=n_m+1}^{\infty} \left[\epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(x) - \epsilon_0 P(x) \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) P_\nu(1-x_0) \right]. \end{aligned}$$

This equality and (2.15) give

$$\begin{aligned} &|\varphi(ux) - \epsilon_0 P(x) \varphi(1-x_0)| \\ &\quad \leq \sum_{\nu=0}^{n_m} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) |P_\nu(x)| + \epsilon_0 P(x) \sum_{\nu=0}^{n_m} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) |P_\nu(1-x_0)| \\ &\qquad + \epsilon \sum_{\nu=n_m+1}^{\infty} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) \epsilon_0 P_\nu(1-x_0) \\ &\quad \leq \sum_{\nu=0}^{n_m} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) |P_\nu(x)| \\ &\qquad + \epsilon_0 P(x) \sum_{\nu=0}^{n_m} \epsilon_\nu u^\nu \varphi^{(\nu)}(ux_\nu) |P_\nu(1-x_0)| + \epsilon \epsilon_0 \varphi(ux), \end{aligned}$$

and if we let $u \to 1$, we obtain

$$|\varphi(x) - \epsilon_0 P(x) \varphi(1-x_0)| \leq \epsilon \epsilon_0 \varphi(x),$$

and therefore Z

$$\varphi(x) = \epsilon_0 \varphi(1-x_0)P(x).$$

If $\varphi(1-x_0) \neq 0$, then the function P is infinitely many times differentiable for $0 \leq x \leq 1$ and belongs to K. If $\varphi(1-x_0) = 0$, then $\varphi(x) = 0$. This way (2.3) becomes

$$f(x) = \sum_{\nu=o}^{\infty} \epsilon_\nu f^{\nu)}(x_\nu) P_\nu(x) + AP(x), \tag{2.16}$$

where A is non negative constant, $P \in K$ does not depend on f and satisfies the conditions

$$P^{(k)}(x_k) = 0, \quad k = 0, 1, \ldots. \tag{2.17}$$

The case when we have only finite number of values of ν for which $x_\nu = 0$ is simple since for all big enough values for ν we have $x_\nu = 1$. In this case, we investigate the function $\varphi(1-u+ux)$, where $0 \leq u < 1$. Similarly to the way we showed (2.5), we can prove that

$$\varphi(1-u+ux) = \sum_{\nu=0}^{\infty} \epsilon_\nu u^\nu \varphi^{(\nu)}(1-u+ux_\nu) P_\nu(x).$$

In this case the infinite series reduces to a finite sum and therefore when we let $u \to 1$, we obtain $\varphi(x) = 0$, namely

$$f(x) = \sum_{\nu=0}^{\infty} \epsilon_\nu f^{(\nu)}(x_\nu) P_\nu(x). \tag{2.18}$$

This shows that (2.16) holds in this case as well and $AP(x) \equiv 0$.

We have shown that if for all big enough integer values of ν either only $x_\nu = 0$ or only $x_\nu = 1$, then (2.16) becomes (2.18) and therefore if we have

$$f^{(k)}(x_k) = 0, \quad k = 0, 1, 2, \ldots$$

then $f(x) \equiv 0$. We will show that in all other cases there exists a function $P \in K$, satisfying

$$P^{(k)}(x_k) = 0, \quad k = 0, 1, 2, \ldots$$

that is not identically zero. We consider the sequence

$$\epsilon_0 \frac{P_{n_1}(x)}{P_{n_1}(1-x_0)}, \epsilon_0 \frac{P_{n_2}(x)}{P_{n_2}(1-x_0)}, \ldots,$$

where $n_1 < n_2 < \ldots$ is the sequence of these integer positive values of ν for which $x_\nu = 1$, and we will denote by $P(x)$ its limit. Let us denote

$$g_i(x) := \epsilon_0 \frac{P_{n_i}(x)}{P_{n_i}(1-x_0)} \in K.$$

Let ξ_1, ξ_2, ξ_3 be three different values between 0 and 1. We have

$$\frac{g_i(\xi_1)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}+\frac{g_i(\xi_2)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}+\frac{g_i(\xi_3)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}=\frac{g_i''(\xi)}{2!},$$

where $\xi\in(0,1)$. It follows that

$$\epsilon_2\left[\frac{g_i(\xi_1)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}+\frac{g_i(\xi_2)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}+\frac{g_i(\xi_3)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}\right]\geq 0$$

which after taking a limit results in

$$\epsilon_2\left[\frac{P(\xi_1)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}+\frac{P(\xi_2)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}+\frac{P(\xi_3)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}\right]\geq 0$$

Therefore the function $\epsilon_2 P$ is convex and therefore P is continuous in all interior points of $(0,1)$.

Next, we will investigate the points $x=0$ and $x=1$. Let m be positive integer for which $x_m=0$, namely $\tau_m=1$. By assumption, there are such numbers (even infinitely many). We consider the ratio

$$\frac{P_{n_i}(x)}{P_m(x)}$$

when $n_i>m$. This ratio is monotone increasing, as noted above, since $\tau_m=1$. Thus, if $x_0=0$, then

$$\frac{P_{n_i}(x)}{P_m(x)}\leq\frac{P_{n_i}(1)}{P_m(1)}$$

and if $x_0=1$, then

$$\frac{P_{n_i}(x)}{P_m(x)}\geq\frac{P_{n_i}(0)}{P_m(0)},$$

so in both cases

$$\tau_0\left[\frac{P_m(x)}{P_m(1-x_0)}-\frac{P_{n_i}(x)}{P_{n_i}(1-x_0)}\right]\geq 0.$$

We let $i\to\infty$ and obtain

$$\tau_0\frac{P_m(x)}{P_m(1-x_0)}\geq\epsilon_1 P(x). \tag{2.19}$$

It follows from (2.12) that the sequence

$$\tau_0\frac{P_{n_1}(x)}{P_{n_1}(1-x_0)},\tau_0\frac{P_{n_2}(x)}{P_{n_2}(1-x_0)},\ldots \tag{2.20}$$

is monotone increasing and threfore for every positive integer i we have

$$\epsilon_1 P(x) \geq \tau_0 \frac{P_{n_i}(x)}{P_{n_i}(1-x_0)}. \tag{2.21}$$

Now we are ready to show that the function P is continuous at $x = 0$ and $x = 1$. Let ξ be either 0 or 1. Let us chose $\epsilon > 0$. We can select $\delta > 0$ in such a way that for $|x - \xi| < \delta$ we have

$$\tau_0 \frac{P_m(x)}{P_m(1-x_0)} - \tau_0 \frac{P_m(\xi)}{P_m(1-x_0)} < \epsilon, \tag{2.22}$$

$$\tau_0 \frac{P_{n_i}(\xi)}{P_{n_i}(1-x_0)} - \tau_0 \frac{P_{n_i}(x)}{P_{n_i}(1-x_0)} < \epsilon, \quad i \text{ is fixed} \tag{2.23}$$

We also have

$$\tau_0 \frac{P_m(\xi)}{P_{n_i}(1-x_0)} = \tau_0 \frac{P_{n_i}(\xi)}{P_{n_i}(1-x_0)} = \epsilon_1 P(\xi),$$

and therefore using (2.19) and (2.21), we derive from (2.22) and (2.23)

$$\epsilon_1 P(x) - \epsilon_1 P(\xi) < \epsilon, \quad \epsilon_1 P(\xi) - \epsilon_1 P(x) < \epsilon$$

which shows that P is continuous at ξ.

We proved that the monotone sequence (2.20) of continuous functions converges on $[0, 1]$ to a continuous function. By Dini's theorem, this sequence is uniformly convergent.

Similar arguments applied to the sequence with general term

$$\frac{P'_{n_i}(x)}{P'_{n_i}(1-x_1)} \tag{2.24}$$

show that this sequence is uniformly convergent on $[0, 1]$ as well. Indeed, the polynomials

$$P'_1(x), P'_2(x), \ldots$$

are Abel-Goncharov polynomials corresponding to the sequence

$$\epsilon_1, \epsilon_2, \ldots$$

$x_{n_i} = 1$. We also know that there is at least one zero (by assumption there are infinitely many zeroes) among the numbers

$$x_1, x_2, \ldots$$

Note that the sequence with general term

$$\frac{P_{n_i}(x)}{P'_{n_i}(1-x_1)}$$

is convergent for $x = x_0$ and therefore is convergent for every $x \in [0,1]$, because (2.24) is uniformly convergent. In particular, it is convergent for $x = 1 - x_0$. We will show that its limit is different from zero. This is indeed the case since otherwise for every x from the closed interval we have

$$\lim_{i\to\infty} \frac{P_{n_i}(x)}{P'_{n_i}(1-x_1)} = 0,$$

because

$$|P_{n_i}(x)| \leq |P_{n_i}(1-x_0)|.$$

It follows, taking into account the uniform convergence of the sequence (2.24) that

$$\lim_{i\to\infty} \frac{P'_{n_i}(x)}{P'_{n_i}(1-x_1)} = 0,$$

which is not true for $x = 1 - x_1$. Thus, we proved that the limit of the sequence with general term

$$\frac{P_{n_i}(1-x_0)}{P'_{n_i}(1-x_1)}$$

is different from zero and therefore the sequence with general term

$$\frac{P'_{n_i}(1-x_1)}{P_{n_i}(1-x_0)}$$

is convergent. Thus, the sequence with general term

$$\frac{P'_{n_i}(x)}{P_{n_i}(1-x_0)} = \frac{P'_{n_i}(1-x_1)}{P_{n_i}(1-x_0)} \frac{P'_{n_i}(x)}{P'_{n_i}(1-x_1)}$$

is uniformly convergent on $[0,1]$, and therefore P is differentiable in $[0,1]$ and

$$P(x) = \lim_{i\to\infty} \epsilon_0 \frac{P'_{n_i}(x)}{P_{n_i}(1-x_0)}$$

We can use similar arguments to show that the sequence with general term

$$\frac{P^{(k)}_{n_i}(x)}{P_{n_i}(1-x_0)}$$

is uniformly convergent on $[0,1]$ and therefore

$$P^{(k)}(x) = \lim_{i\to\infty} \epsilon_0 \frac{P^{(k)}_{n_i}(x)}{P_{n_i}(1-x_0)}.$$

This is enough to claim that $P \in K$ and

$$P^{(k)}(x_k) = 0, \quad k = 0, 1, 2, \ldots$$

Notice that $P(1-x_0)=1$ and therefore P is not identically zero.

Now, it is easy to prove that the sequence

$$\epsilon_0 \frac{P_1(x)}{P_1(1-x_0)}, \epsilon_0 \frac{P_2(x)}{P_2(1-x_0)}, \dots \tag{2.25}$$

converges uniformly to P on $[0,1]$. Indeed, to show that it is enough to consider the sequence

$$m_1 < m_2 < \dots,$$

of these positive integers for which $x_{m_i}=0$. We consider the sequence

$$\epsilon_0 \frac{P_{m_1}(x)}{P_{m_1}(1-x_0)}, \epsilon_0 \frac{P_{m_2}(x)}{P_{m_2}(1-x_0)}, \dots \tag{2.26}$$

for which it is easy to see (similarly to the case of the sequence (2.20)) that it is convergent. Let us denote by Q its limit. As we did when considering P, one can show that the sequence (2.26) is uniformly convergent, $Q \in K$ and

$$Q^{(k)}(x_k)=0, \quad k=0,1,2,\dots$$

This implies that we can expand Q in series using (2.16). This gives

$$Q(x)=AP(x)$$

Notice that $Q(1-x_0)=1$ and $P(1-x_0)=1$ and therefore $A=1$, namely

$$Q(x)=P(x)$$

This way we show that the sequence (2.25) converges to P uniformly on $[0,1]$. The same way one can show that the sequence of the derivatives of order k

$$\epsilon_0 \frac{P_1^{(k)}(x)}{P_1(1-x_0)}, \epsilon_0 \frac{P_2^{(k)}(x)}{P_2(1-x_0)}, \dots$$

is also uniformly convergent on $[0,1]$.

Finally, let us mention that using differentiation term by term (which is easily justified) it can be seen that every function that has expansion of the form

$$f(x)=\sum_{\nu=0}^{\infty} a_\nu P_\nu(x) + AP(x)$$

for $x \in [0,1]$ with $a_\nu \geq 0$ and $A \geq 0$ always belongs to K. This way we have proved the following theorem.

Theorem 1. *Every function $f \in K$ has expansion of the form*

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu P_\nu(x) + AP(x), \tag{2.27}$$

where a_ν and A are non negative constants and the function $P \in K$ does not depend on f and satisfies the conditions

$$P^{(k)}(x_k) = 0, \quad k = 0, 1, 2, \ldots .$$

The coefficients a_ν are determined by

$$a_\nu = \epsilon_\nu P^{(\nu)}(x_\nu).$$

If for all big enough integer values of ν we have either only $x_\nu = 0$ or only $x_\nu = 1$, then $P \equiv 0$. In all other cases P is not identically zero and we have that on $[0,1]$, P is the uniform limit

$$P(x) = \lim_{n\to\infty} \epsilon_0 \frac{P_n(x)}{P_n(1-x_0)}.$$

Conversely, if a function f has on $[0,1]$ an expansion of the form (2.27) with $a_\nu \geq 0$ and $A \geq 0$, then it belongs to K.

In conclusion, we will consider several examples.

1. If for all positive integer values of n we have $\epsilon_n = 1$, then $x_n = 0$. In this case the expansion (2.27) has the form

$$f(x) = \sum_{\nu=0}^{\infty} f^{(\nu)}(0) \frac{x^\nu}{\nu!},$$

which is the Bernstein result in [2].

2. If for all positive integer values of n we have $\epsilon_n = (-1)^n$, then $x_n = 1$. In this case the expansion (2.27) has the form

$$f(x) = \sum_{\nu=0}^{\infty} (-1)^\nu f^{(\nu)}(1) \frac{(1-x)^\nu}{\nu!},$$

This result is not different from the above mentioned one and it derived from it if we substitute x by $1-x$.

3. Let for all positive integer values of n we have $\epsilon_n = (-1)^{\frac{n}{2}}$. In this case the sequence of x's

$$x_0 = 0,\ x_1 = 1,\ x_2 = 0,\ x_3 = 1, \ldots,$$

is periodic with period 2. We consider the function

$$\varphi(x) = \sin \frac{\pi}{2} x.$$

Obviously $\varphi \in K$,

$$\varphi^{(k)}(x_k) = 0, \quad k = 0, 1, 2, \ldots,$$

the expansion (2.27) gives

$$\sin \frac{\pi}{2} x = AP(x).$$

Apparently $A \neq 0$ and we can determine P. In this case (2.27) has the form

$$f(x) = \sum_{\nu=0}^{\infty} \epsilon_\nu f^{(\nu)}(x_\nu) P_\nu(x) + C \sin \frac{\pi}{2} x.$$

This is the expansion for the cyclic monotone functions, derived by Bernstein in [3].

4. Let the sequence

$$\epsilon_0, \epsilon_2, \ldots,$$

is periodic with period q, i.e. for all positive integer values of n we have $\epsilon_n = \epsilon_{n+q}$. In this case the function $P^{(q)} \in K$ and therefore admits expansion in the form (2.27). On the other side,

$$P^{(q+k)}(x_k) = 0, \quad k = 0, 1, \ldots,$$

and therefore this expansion has the form

$$P^{(q)}(x) = AP(x),$$

which is a simple linear differential equation with constant coefficients. This way we obtain the result of Sendov from [5], where he generalizes Bernstein's results considered in the first three examples.

References

[1] S. Bernstein, Lecons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle, Paris, 1926.

[2] S. Bernstein, Sur la définition et les propriétés des fonctions analytiques d'une variable réelle, *Math. Annalen* **75** (1914), 449–468.

[3] S. Bernstein, On some properties of the cyclic monotone functions, *Izvestia AN USSR* **14** (1950), 381–404 (in Russian).

[4] S. Bernstein, On some properties of the regular monotone functions, *Collected papers* Vol. 1, 350–360 (in Russian).

[5] Bl. Sendov, On a class of regular monotone functions, *Doklady AN USSR* **110** (1956), 27–30 (in Russian).

[6] Bl. Sendov, On some properties of the regular monotone functions, *Notices of the BAS Math Institute* **3**, 2 (1958) (in Bulgarian).