Exercises 1.

1. Compute the given arithmetic expressions. Give the answer in the form $a+bi$ for $a, b \in \mathbb{R}$ and in the polar form $|z|(p+qi)$ where $|p+qi|=1$.

   a) $(-i)^{24}$, b) $(3-i)(4+3i)$

2. Find all solutions in $\mathbb{C}$ of the given equations:

   a) $z^3 = 8$, b) $z^4 = -16$

3. Let $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$. Use the trigonometric identities

$\sin(a+b) = \dots$

$\cos(a+b) = \dots$

to derive

$$z_1 z_2 = |z_1||z_2|\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right].$$

(4) Derive Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ formally from following series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!},$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!},$$

$x \in \mathbb{C}$.

(4.) Determine which of the following binary operations are associative:

(a) the operation $*$ on $\mathbb{Z}$ defined by

$$a * b = a - b$$

(b) the operation $*$ on $\mathbb{R}$ defined by

$$a * b = a + b + ab$$

(c) the operation $*$ on $\mathbb{Q}$ defined by

$$a * b = \frac{a+b}{5}$$

(d) the operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(a, b) * (c, d) = (ad + bc, bd)$$

Decide which of the binary operations in these examples are commutative.

5. Let G be a group.

(a) Let $x \in G$. Prove that $x^2 = 1$ if and only if the order $|x|$ is either 1 or 2.

(b) Prove that if $|x| = n$, $x \in G$ then

$x^{-1} = x^{n-1}$

(c) Let $x, y \in G$. Prove that $xy = yx$ if and only if $y^{-1}xy = x$; if and only if $x^{-1}y^{-1}xy = 1$.

(d) Prove that $(x_1 x_2 \cdots x_n)^{-1} = x_n^{-1} x_{n-1}^{-1} \cdots x_1^{-1}$ for all $x_1, \dots, x_n \in G$.

6. Prove that for $x, y \in G$

$|x| = |y^{-1}xy|$,

$|xy| = |yx|$.

7. Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.

(8) There is an isomorphism of U_7 with $\mathbb{Z}_7$ in which $\xi = e^{i(2\pi/7)}$ corresponds to 4. Find the element in $\mathbb{Z}_7$ to which ξ^m must correspond for $m = 0, 2, 3, 4, 5,$ and 6.

(9) Why can there be no isomorphism of U_6 with $\mathbb{Z}_6$ in which $\xi = e^{i(\pi/3)}$ corresponds to 4?

(10) Find all solutions x of the given equation.

a) $x +_{2\pi} \frac{3\pi}{2} = \frac{3\pi}{4}$ in $\mathbb{R}_{2\pi}$

b) $x +_4 x +_4 x +_4 x = 0$ in $\mathbb{Z}_4$.