HW 2

① Determine which of the following binary operations are associative:

a) the operation on $\mathbb{N} = \{1, 2, \dots\}$ defined by
$$x * y = x^y;$$

b) the operation on $\mathbb{Z}$ defined by
$$x * y = x^2 + y^2;$$

c) the operation on $\mathbb{R}$ defined by
$$x * y = \sin x \sin y.$$

② Solve Problem 24 from Exercises 2 of the Fraleigh's book. Justify your answers or give references to the corresponding statements in the book.

③ Let G be a group. For each $x \in G$ and $a, b \in \mathbb{Z}^+$

a) prove that $x^{a+b} = x^a x^b$,

b) prove that $\left(x^{a}\right)^{-1} = x^{-a}$;

c) establish part a) for arbitrary integers a and b in $\mathbb{Z}$ (positive, negative or zero)

(4) if x is an element of finite order n in G, prove that the elements $1, x, x^2, \ldots, x^{n-1}$ are all distinct. Deduce that $|x| \leq |G|$.

(5) if x is an element of infinite order in G, prove that the elements x^n, $n \in \mathbb{Z}$ are all distinct.

(6) Determine whether the given subset of the complex numbers is a subgroup of the group $(\mathbb{C}, +)$:

a) $3\mathbb{R} = \{3x \mid x \in \mathbb{R}\}$

b) $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$

c) The set $\{e^n \mid n \in \mathbb{Z}\}$ where $e = 2.7\ldots$ is Euler number.

7. Determine whether the given set of invertible $n \times n$ matrices with real entries is a subgroup of

$$GL_n(\mathbb{R}) = \{A \in Mat_n(\mathbb{R}) \mid \det A \neq 0\}:$$

a) the $n \times n$ matrices with determinant -1;

b) the upper-triangular $n \times n$ matrices with 1's on the diagonal.

8. Solve Problem 25 from Exercises 4 of the Fraleigh's book. Justify your answers or give references to the corresponding statements in the book.

(9) Prove that the "one-sided definition" presented at the top of page 43 of the Fraleigh's Book is equivalent to the definition 4.1.