HW 6.

(1). Answer each of the questions from Problem 32 in Exercises 11 of the Fraleigh's book. Justify your answers or give a reference to the corresponding statement in the book.

(2) The same as (1) concerning Problem 36 in Exercises 11.

(3) Let G be an abelian group. Show that the elements of finite order in G form a subgroup. This subgroup is called the torsion subgroup of G.

(4) Find the order of the torsion subgroup of

$$\mathbb{Z} \times \mathbb{Z}_{125} \times \mathbb{Z} \times \mathbb{Z}_{82}; \quad \text{of} \quad \mathbb{Z}^3 \times \mathbb{Z}_{72} \times \mathbb{Z} \times \mathbb{Z}_8.$$

(5) Find the torsion subgroup of:

a) the multiplicative group $\mathbb{R}^*$ of nonzero real numbers;

b) the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers.

⑥ Read the definition of torsion coefficients given in Problem 44 of Exercises 11 of the book and find the torsion coefficients of

a) $\mathbb{Z}_8 \times \mathbb{Z}_{27} \times \mathbb{Z}_{25}$

b) $\mathbb{Z}_{72} \times \mathbb{Z}_{56} \times \mathbb{Z}_{125} \times \mathbb{Z}_{132}$.

Describe an algorithm to find the torsion coefficients of a direct product of cyclic groups.

⑦ Let H and K be subgroups of a group G satisfying the three properties:

a) Every element of G is of the form hk for some $h \in H$ and $k \in K$,

b) $hk = kh$ for all $h \in H$ and $k \in K$,

c) $H \cap K = \{e\}$

Show that for each $g \in G$, the expression $g = hk$ for $h \in H$ and $k \in K$ is unique. Then let each g be renamed (h, k). Show that, under this renaming, G becomes isomorphic to $H \times K$.

(8) Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

(9) Determine whether the given map φ is a homomorphism:

1. $\varphi: (\mathbb{R},+) \to (\mathbb{Z},+)$ given by $\varphi(x) =$ the greatest integer $\leq x$.

2. $\varphi: (\mathbb{R}^*,\cdot) \to (\mathbb{R}^*,\cdot)$ given by $\varphi(x) = |x|$.

3. $\varphi: (\mathbb{Z},+) \to (\mathbb{R},+)$ given by $\varphi(n) = n$.

4. $\varphi: \mathbb{Z}_{25} \to \mathbb{Z}_3$ be given by $\varphi(x) =$ the remainder of x when divided by 3.

5. $\varphi: \mathbb{Z}_6 \to \mathbb{Z}_3$ be given by $\varphi(x) =$ the reminder of x when divided by 3.

(10) Compute the indicated quantities for the given homomorphism φ.

a) $\mathrm{Ker}(\varphi)$ for $\varphi: S_4 \to \mathbb{Z}_2$ in Example 13.3 of the book. $\varphi(\sigma)$ and $\varphi(\tau)$ where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

b) $\mathrm{Ker}(\varphi)$ and $\varphi(33)$ for $\varphi: \mathbb{Z} \to \mathbb{Z}_5$ such that $\varphi(1) = 3$.

c) $\mathrm{Ker}(\varphi)$ and $\varphi(113)$ for $\varphi: \mathbb{Z} \to \mathbb{Z}_{10}$ such that $\varphi(1) = 6$.

d) $\mathrm{Ker}(\varphi)$ and $\varphi(15)$ for $\varphi: \mathbb{Z} \to S_8$ such that $\varphi(1) = (1,3,2)(4,7,5)(6,8)$.