HW 8.

(1) Let G be a group, and let $g \in G$. Let $\varphi_g : G \to G$ be defined by $\varphi_g(x) = gxg^{-1}$ for $x \in G$. Show that φ_g is a homomorphism.

(2) Let $\varphi : G \to G'$ be a group homomorphism.

a) Show that if $|G|$ is finite, then $|\varphi[G]|$ is finite and is a divisor of $|G|$.

b) Show that if $|G'|$ is finite, then, $|\varphi[G]|$ is finite and is a divisor of $|G'|$.

(3) The sign of an even permutation is $+1$ and the sign of an odd permutation is -1.

a) Show that the map $\varphi : S_n \to \{-1, 1\}$ defined by

$$\varphi(\sigma) = \operatorname{sign}(\sigma)$$

is a homomorphism of S_n onto the multiplicative group $\{-1, 1\}$.

b) What is the kernel of φ?

(4) Answer each of the questions from Problem 23 in Exercises 14 of the Book. Justify your answers or give a reference to the corresponding statement in the Book.

(5) Show that A_n is a normal subgroup of S_n and compute S_n / A_n.

(6) A subgroup H is conjugate to a subgroup K of a group G if there is $g \in G$ such that $gHg^{-1} = K$. Show that conjugacy is an equivalence relation on the collection of subgroups of G.

(7) Show that if H and K are normal subgroups in G then $H \cap K$ also is a normal subgroup in G.

Classify the given group according to the fundamental theorem of finitely generated abelian groups.

a) $(\mathbb{Z}_3 \times \mathbb{Z}_5)/\langle(0,1)\rangle$.

b) $(\mathbb{Z}_3 \times \mathbb{Z}_9)/\langle(0,3)\rangle$

c) $(\mathbb{Z}_3 \times \mathbb{Z}_9)/\langle(1,3)\rangle$.

d) $(\mathbb{Z} \times \mathbb{Z})/\langle(1,0)\rangle$

e) $(\mathbb{Z} \times \mathbb{Z})/\langle(0,2)\rangle$.