On the isotropy constant of projections of polytopes

Julio Bernués
(joint work with D. Alonso-Gutiérrez, J. Bastero and P. Wolff)

University of Zaragoza

College Station, July 20th 2009
Let $K \subset \mathbb{R}^d$ be a symmetric convex body. Its isotropy constant $L_K$ is

$$dL_K^2 := \min \left\{ \frac{1}{|TK|^{1+\frac{2}{d}}} \int_{TK} |x|^2 dx : T \in GL(d) \right\}$$
Let $K \subset \mathbb{R}^d$ be a symmetric convex body. Its isotropy constant $L_K$ is

$$dL_k^2 := \min \left\{ \frac{1}{|TK|^{1 + \frac{2}{d}}} \int_{TK} |x|^2 dx : T \in GL(d) \right\}$$

- $L_K$ is a linear invariant.
Let \( K \subset \mathbb{R}^d \) be a symmetric convex body. Its isotropy constant \( L_K \) is

\[
dL^2_K := \min \left\{ \frac{1}{|TK|^{1 + \frac{2}{d}}} \int_{TK} |x|^2 dx : T \in GL(d) \right\}
\]

- \( L_K \) is a linear invariant.
- By Hölder’s inequality, \( L_K \geq L_{B_2^d} \simeq 0.24 \)
Let $K \subset \mathbb{R}^d$ be a symmetric convex body. Its isotropy constant $L_K$ is

$$dL^2_K := \min \left\{ \frac{1}{|TK|^{1+\frac{2}{d}}} \int_{TK} |x|^2 dx : T \in GL(d) \right\}$$

- $L_K$ is a linear invariant.
- By Hölder’s inequality, $L_K \geq L_{B^d_2} \simeq 0.24$

**Problem:** Estimate $L_K$ from above.

$L_K$ is known to be bounded by an absolute constant for many families of convex bodies.
Let $K \subset \mathbb{R}^d$ be a symmetric convex body. Its isotropy constant $L_K$ is

$$dL_K^2 := \min \left\{ \frac{1}{|TK|^{1+\frac{2}{d}}} \int_{TK} |x|^2 dx : T \in GL(d) \right\}$$

- $L_K$ is a linear invariant.
- By Hölder’s inequality, $L_K \geq L_{B_2^d} \simeq 0.24$

**Problem:** Estimate $L_K$ from above.

**Conjecture:** $L_K \leq L_{B_\infty^d} \simeq 0.29$

$L_K$ is known to be bounded by an absolute constant for many families of convex bodies.
For random convex bodies, (2008-2009)
Introduction

For random convex bodies, (2008-2009)

Theorem (Klartag-Kozma)

Let $G_0, \ldots, G_n$ be i.i.d. gaussian random vectors in $\mathbb{R}^d$, $n > d$. For $K = \text{conv}\{\pm G_0, \ldots, \pm G_n\}$ we have that with probability greater than $1 - Ce^{-cd}$

$$L_K \leq C.$$ 

(extra arguments needed for $n \leq cd$)

Theorem (Alonso-Gutierrez and Dafnis, Giannopoulos, Guedon)

If $n \geq cd$ and $\{P_i\}_{i=0}^n$ are independent random points

- on $S^{d-1}$
- on an isotropic unconditional convex body in $\mathbb{R}^d$

and $K = \text{conv}\{\pm P_1, \ldots, \pm P_n\}$ then with probability $\geq 1 - c_1 e^{-c_2 d}$

$$L_K \leq C.$$
We will focus on the isotropy constant of projections of convex bodies.
Introduction

We will focus on the isotropy constant of *projections* of convex bodies.

**Notation:** From now on, $E$ is a $d$ dimensional subspace of $\mathbb{R}^n$ and $P_E$ is the orthogonal projection onto $E$. 
Introduction

We will focus on the isotropy constant of projections of convex bodies.

Notation: From now on, $E$ is a $d$ dimensional subspace of $\mathbb{R}^n$ and $P_E$ is the orthogonal projection onto $E$.

For any $d$ and $1 < p \leq \infty$, (M. Junge, 1994, 1995; E. Milman, 2006)

1. $L_{P_E}(B_p^n) \leq c\sqrt{p'}$, \quad $\frac{1}{p} + \frac{1}{p'} = 1$.
2. $L_{P_E}(B_1^n) \leq c \log n$.

$P_E(B_1^n)$ is a $d$-dimensional convex polytope of at most $n$ vertices.
An observation. Random projections of $B_1^n$

- Recall $G_{n,d} := \{ E \subseteq \mathbb{R}^n : E \text{ subspace $d$-dimensional} \}$. There exists a unique probability measure, $\mu_{n,d}$ (Haar measure) on $G_{n,d}$ invariant under orthogonal transformations.
An observation. Random projections of $B_1^n$

- Recall $G_{n,d} := \{ E \subseteq \mathbb{R}^n : E \text{ subspace } d\text{-dimensional} \}$. There exists a unique probability measure, $\mu_{n,d}$ (Haar measure) on $G_{n,d}$ invariant under orthogonal transformations.

- If $G$ is an $n \times d$, $G : \mathbb{R}^d \to \mathbb{R}^n$, random matrix with entries $g_{ij}$ i.i.d. standard gaussian r.v.’s then by uniqueness for every borelian $A \subset G_{n,d}$,

$$\mu_{n,d}(A) = \mathbb{P}\{\text{Im } G \in A\}$$
An observation. Random projections of $B_1^n$

- If $G$ is an $n \times d$ matrix and $E = \text{Im } G \in G_{n,d}$ then (linear algebra)

$$G|_E^t P_E B_1^n = \text{conv}\{\pm G_1, \ldots, \pm G_n\}$$

where $G_i$ are the rows of $G$. 
An observation. Random projections of $B_1^n$

- If $G$ is an $n \times d$ matrix and $E = \text{Im} \ G \in G_{n,d}$ then (linear algebra)
  
  $$G_{|E}^t P_E B_1^n = \text{conv}\{\pm G_1, \ldots, \pm G_n\}$$

  where $G_i$ are the rows of $G$.

- Thus, Klartag-Kozma result reads
  
  $$\mu_{n,d}\{E \in G_{n,d} : L_{P_E B_1^n} \leq C\} > 1 - ce^{-c'd}$$
An observation. Random projections of $B_1^n$

- If $G$ is an $n \times d$ matrix and $E = \text{Im } G \in G_{n,d}$ then (linear algebra)
  \[
  G^t_{\mid E} P_E B_1^n = \text{conv}\{\pm G_1, \ldots, \pm G_n\}
  \]
  where $G_i$ are the rows of $G$.

- Thus, Klartag-Kozma result reads
  \[
  \mu_{n,d}\{E \in G_{n,d} : L_{P_E B_1^n} \leq C\} > 1 - ce^{-c'd}
  \]

- For $d$ as small as $d < c \log n$ Dvoretzky’s theorem says that $P_E B_1^n$ is typically like the euclidean ball $B_2^d$ and so,
  \[
  \mu_{n,d}\{E \in G_{n,d} : L_{P_E B_1^n} \leq C\} > 1 - ce^{-c' \log n}
  \]
An observation. Random projections of $B_1^n$

In conclusion, there exists $C, c, c' > 0$ such that for all $1 \leq d \leq n$,

$$\mu_{n,d}\{E \in G_{n,d} : L_{P_EB_1^n} \leq C\} > 1 - ce^{-c' \max\{d, \log n\}}$$
An observation. Random projections of $B_1^n$

In conclusion, there exists $C, c, c' > 0$ such that for all $1 \leq d \leq n$,

$$\mu_{n,d}\{E \in G_{n,d} : L_{P_E B_1^n} \leq C\} > 1 - ce^{-c' \max\{d, \log n\}}$$

On the other hand recall the following:

**Fact**

If $L_{P_E B_1^n} \leq C$ for every $d$-dimensional $E$ and every $1 \leq d \leq n$, then $L_K \leq C$ for every symmetric convex body $K \subset \mathbb{R}^d$ in any dimension $d$. 
An observation. Random projections of $B_1^n$

In conclusion, there exists $C, c, c' > 0$ such that for all $1 \leq d \leq n$,

$$\mu_{n,d}\{E \in G_{n,d} : L_{PB_1^n} \leq C\} > 1 - ce^{-c'\max\{d,\log n\}}$$

On the other hand recall the following:

**Fact**

If $L_{PB_1^n} \leq C$ for every $d$-dimensional $E$ and every $1 \leq d \leq n$, then $L_K \leq C$ for every symmetric convex body $K \subset \mathbb{R}^d$ in any dimension $d$.

In this sense, “most” symmetric convex bodies $K \subset \mathbb{R}^d$ have isotropy constant bounded.
Main result. Projections of $B_1^n$

**Theorem**

$$L_{PE}(B_1^n), L_{PE}(S_n) \leq C \sqrt{\frac{n}{d}}$$

where $S_n$ is the $n$-dimensional regular simplex.
Main result. Projections of $B_1^n$

**Theorem**

\[ L_{P_E}(B_1^n), L_{P_E}(S_n) \leq C \sqrt{\frac{n}{d}} \]

where $S_n$ is the $n$-dimensional regular simplex.

**Corollary**

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with $n$ vertices. Then

\[ L_K \leq C \sqrt{\frac{n}{d}} \]
Proof: Starting point

For any convex polytope $K \subset \mathbb{R}^n$,

$$dL_{P_E(K)}^2 \leq \frac{1}{|P_E(K)|^{2/d}} \frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \lambda_E(dx)$$
Proof: Starting point

For any convex polytope $K \subset \mathbb{R}^n$,

$$dL^2_{P_E(K)} \leq \frac{1}{|P_E(K)|^{2/d}} \frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \lambda_E(dx)$$

- Estimate from above $\frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \lambda_E(dx)$
- Estimate from below $|P_E(K)|$
Main Lemma
Let $\mathbf{K} \subset \mathbb{R}^n$ be a convex polytope and $\mathbf{E}$ a $d$-dimensional subspace of $\mathbb{R}^n$. There exists a subset $\tilde{\mathbf{F}}$ of $d$-dimensional faces of $\mathbf{K}$ such that
\[
\{ \text{PE}(\text{relint } F) \mid F \in \tilde{\mathbf{F}} \}
\]
is a family of pair-wise disjoint sets,
\[
\bigcup \{ \text{PE}(F) ; F \in \tilde{\mathbf{F}} \} = \text{PE}(\mathbf{K}),
\]
and for each $F \in \tilde{\mathbf{F}}$, $\text{PE}|_F : F \to \text{PE}(F)$ is an affine isomorphism.
Main Lemma

Let $K \subseteq \mathbb{R}^n$ be a convex polytope and $E$ a $d$ dimensional subspace of $\mathbb{R}^n$. There exists a subset $\mathcal{F}$ of $d$ dimensional faces of $K$ such that

- $\{P_E(\text{relint } F) \mid F \in \mathcal{F}\}$ is a family of pair-wise disjoint sets,
- $\bigcup\{P_E(F); F \in \mathcal{F}\} = P_E(K), \ a.e. \ \lambda_E$
- For each $F \in \mathcal{F}$, $P_E|F: F \to P_E(F)$ is an affine isomorphism.
Integrating on $P_E(K)$
Integrating on $P_E(K)$

**Corollary**

Let $K \subset \mathbb{R}^n$ be a convex polytope and $E$ a $d$ dimensional subspace of $\mathbb{R}^n$. There exists a subset $\tilde{F}$ of $d$ dimensional faces of $K$ such that for any integrable function $f : P_E(K) \to \mathbb{R}$,

$$\int_{P_E(K)} f(x) \lambda_E(dx) = \sum_{F \in \tilde{F}} \int_{P_E(F)} f(x) \lambda_E(dx) =$$

$$= \sum_{F \in \tilde{F}} \frac{|P_E(F)|}{|F|} \int_F f(P_Ey) \lambda_{affF}(dy)$$

In particular for $f \equiv 1$, $|P_E(K)| = \sum_{F \in \tilde{F}} |P_E(F)|$
Sketch of the proof

\[
\frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \, dx = \frac{1}{|P_E(K)|} \sum_{F \in \mathcal{F}} \int_{P_E(F)} |x|^2 \, dx =
\]
Sketch of the proof

\[ \frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \, dx = \frac{1}{|P_E(K)|} \sum_{F \in \tilde{F}} \int_{P_E(F)} |x|^2 \, dx = \]

\[ \frac{1}{|P_E(K)|} \sum_{F \in \tilde{F}} \frac{|P_E(F)|}{|F|} \int_{P_E(F)} |P_Ey|^2 \, dy \leq \sum_{F \in \tilde{F}} \frac{|P_E(F)|}{|P_E(K)| |F|} \frac{1}{|F|} \int_{P_E(F)} |y|^2 \, dy \]
Sketch of the proof

\[
\frac{1}{|P_E(K)|} \int_{P_E(K)} |x|^2 \, dx = \frac{1}{|P_E(K)|} \sum_{F \in \mathcal{F}} \int_{P_E(F)} |x|^2 \, dx = 
\]

\[
\frac{1}{|P_E(K)|} \sum_{F \in \mathcal{F}} \frac{|P_E(F)|}{|F|} \int_F |P_Ey|^2 \, dy \leq \sum_{F \in \mathcal{F}} \frac{|P_E(F)|}{|P_E(K)|} \frac{1}{|F|} \int_F |y|^2 \, dy
\]

\[
\leq \sup_{F \in \mathcal{F}} \frac{1}{|F|} \int_F |y|^2 \, dy \quad \text{(since} \quad 1 = \sum_{F \in \mathcal{F}} \frac{|P_E(F)|}{|P_E(K)|} \text{)}
\]
In the symmetric case, $K = B_1^n$, denoting $\Delta_d$ the $d$-dimensional regular simplex in $\langle e_1, \ldots, e_{d+1} \rangle$,

$$\frac{1}{|P_E(B_1^n)|} \int_{P_E(B_1^n)} |x|^2 \, dx \leq \sup_{F \in \mathcal{F}} \frac{1}{|F|} \int_F |y|^2 \, dy = \frac{1}{|\Delta_d|} \int_{\Delta_d} |x|^2 \, dx = \frac{2}{d + 2}$$
In the symmetric case, \( K = B_1^n \), denoting \( \Delta_d \) the \( d \)-dimensional regular simplex in \( \langle e_1, \ldots, e_{d+1} \rangle \),

\[
\frac{1}{|P_E(B_1^n)|} \int_{P_E(B_1^n)} |x|^2 \, dx \leq \sup_{F \in \mathcal{F}} \frac{1}{|F|} \int_F |y|^2 \, dy = \frac{1}{|\Delta_d|} \int_{\Delta_d} |x|^2 \, dx = \frac{2}{d + 2}
\]

On the other hand, \( n^{-1/2} B_2^n \subseteq B_1^n \implies n^{-1/2} P_E(B_2^n) \subseteq P_E(B_1^n) \). Therefore

\[
|P_E(B_1^n)|^{1/d} \geq \frac{c}{\sqrt{nd}}
\]
In the symmetric case, $K = B_1^n$, denoting $\Delta_d$ the $d$-dimensional regular simplex in $\langle e_1, \ldots, e_{d+1} \rangle$,

$$\frac{1}{|P_E(B_1^n)|} \int_{P_E(B_1^n)} |x|^2 \, dx \leq \sup_{F \in \mathcal{F}} \frac{1}{|F|} \int_F |y|^2 \, dy = \frac{1}{|\Delta_d|} \int_{\Delta_d} |x|^2 \, dx = \frac{2}{d+2}$$

On the other hand, $n^{-1/2}B_2^n \subseteq B_1^n \implies n^{-1/2}P_E(B_2^n) \subseteq P_E(B_1^n)$. Therefore

$$|P_E(B_1^n)|^{1/d} \geq \frac{c}{\sqrt{nd}}$$

Combining all estimates we get

$$L^2_{P_E(B_1^n)} \leq \frac{1}{d |P_E(B_1^n)|^{2/d}} \frac{1}{|P_E(B_1^n)|} \int_{P_E(B_1^n)} |x|^2 \, dx \leq \frac{1}{d} \frac{nd}{c^2} \frac{2}{d + 2} \leq c' \frac{n}{d}$$
Application

Our result,

Theorem

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with $n$ vertices. Then

$$L_K \leq C\sqrt{\frac{n}{d}}$$
Our result,

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with $n$ vertices. Then

$$L_K \leq C \sqrt{\frac{n}{d}}$$

solves the difficulties in the papers above.
Our result, 

**Theorem**

Let $K \subseteq \mathbb{R}^d$ be a convex polytope with $n$ vertices. Then 

$$L_K \leq C \sqrt{\frac{n}{d}}$$

solves the difficulties in the papers above:

If the number of vertices is proportional to the dimension, $n \leq cd$ then $L_K \leq C$ deterministically.
The isotropy constant of projections of $B^n_p$, $1 < p \leq 2$.

For hyperplanes,

**Theorem.**

There exists $C > 0$ such that for every $1 < p \leq 2$ and any hyperplane $H = \theta^\perp$,

$$L_{P_H}(B^n_p) \leq C$$
The isotropy constant of projections of $B^n_p$, $1 < p \leq 2$.

For hyperplanes,

**Theorem.**

There exists $C > 0$ such that for every $1 < p \leq 2$ and any hyperplane $H = \theta^\perp$, 

$$L_{P_H}(B^n_p) \leq C$$

**Proof of the Theorem.** (Ideas from Barthe-Naor). From the definition,

$$(n-1)L^2_{P_H}(B^n_p) \leq \frac{1}{|P_H(B^n_p)|^{\frac{2}{n-1}}} \cdot \frac{1}{|P_H(B^n_p)|} \int_{P_H(B^n_p)} |x|^2 \, dx$$
The isotropy constant of hyperplane projections of $B^n_p, 1 < p \leq 2$.

The first step is Cauchy's formula: For good $f$ we have

\[
\int_{P_H(B^n_p)} f(x) dx = \frac{1}{2} |\partial B^n_p| \int_{\partial B^n_p} f(P_H(y)) |\langle N(y), \theta \rangle| d\sigma^n_p(y)
\]

where $N(y)$ is the unit normal vector to $\partial B^n_p$ and $\sigma^n_p$ the normalized area measure. We apply it to $f = 1$ and $f(y) = |y|^2$.
The isotropy constant of hyperplane projections of $B_p^n$, $1 < p \leq 2$.

The first step is **Cauchy’s formula**: For good $f$ we have

$$
\int_{P_H(B_p^n)} f(x) dx = \frac{1}{2} |\partial B_p^n| \int_{\partial B_p^n} f(P_H(y)) |\langle N(y), \theta \rangle| d\sigma_p^n(y)
$$

where $N(y)$ is the unit normal vector to $\partial B_p^n$ and $\sigma_p^n$ the normalized area measure. We apply it to $f = 1$ and $f(y) = |y|^2$.

**Question.** In order to extend the proof to lower dimensional projections, $E$, we need a formula for

$$
\int_{P_E(B_p^n)} f(x) dx
$$
Fact 1 Naor, Romik 2003. The relation between two measures on $\partial B^n_p$: $\sigma^n_p$ (the normalized area measure) and $\mu^n_p$ (the cone probability measure) is given by

$$\frac{d\sigma^n_p}{d\mu^n_p}(x) = \frac{n|B^n_p|}{|\partial B^n_p|} |\nabla (\| \cdot \|_p)(x)| \quad a.e. \quad x \in \partial B^n_p$$
The proof

Fact 1 Naor, Romik 2003. The relation between two measures on $\partial B^n_p$: $\sigma^n_p$ (the normalized area measure) and $\mu^n_p$ (the cone probability measure) is given by

$$\frac{d\sigma^n_p}{d\mu^n_p}(x) = \frac{n|B^n_p|}{|\partial B^n_p|} |\nabla(\|\cdot\|_p)(x)| \quad a.e. \quad x \in \partial B^n_p$$

Fact 2 Schechtmann, Zinn 1990. There is a representation of the cone measure $\mu^n_p$. Let $g_1, \ldots, g_n$ be i.i.d. copies of a r.v. with density $c_p e^{-|t|^p}, \ t \in \mathbb{R}$. Define $S = (\sum_{i=1}^n |g_i|^p)^{1/p}$. Then the probability determined by $(g_1/S, \ldots, g_n/S) \in \partial B^n_p$ is $\mu^n_p$. Moreover, $(g_1/S, \ldots, g_n/S)$ is independent of $S$. 
By Cauchy's formula and good $f$,

$$
\int_{\mathcal{P}_H(B^n_p)} f(x) dx = \frac{1}{2} |\partial B^n_p| \int_{\partial B^n_p} f(P_H(y)) |\langle N(y), \theta \rangle| d\sigma^n_p(y)
$$

($N(y)$ is the unit normal vector to $\partial B^n_p$)

$$
= \frac{n}{2} |B^n_p| \int_{\partial B^n_p} f(P_H(y)) |\langle \nabla (|| \cdot ||_p)(y), \theta \rangle| d\mu^n_p(y)
$$

$$
= \frac{n}{2} |B^n_p| \int_{\partial B^n_p} f(P_H(y)) \left| \sum_{i=1}^n |y_i|^{p-1} \text{sgn}(y_i) \theta_i \right| d\mu^n_p(y)
$$

Use it with $f = 1$ (for the volume) and $f(y) = |y|^2$ (and also $|P_H(y)| \leq |y|$).

Now use the representation of $d\mu^n_p$, $y_i \rightarrow \frac{g_i}{S}$. 
\[
\frac{1}{|P_{H}(B_{p}^{n})|} \int_{P_{H}(B_{p}^{n})} |x|^{2} \, dx \leq \frac{\mathbb{E} \sum_{i=1}^{n} \frac{|g_{i}|^{2}}{S^{2}} \left| \sum_{i=1}^{n} \frac{|g_{i}|^{p-1}}{S_{p-1}} \text{sgn}(g_{i}) \theta_{i} \right|}{\mathbb{E} \left| \sum_{i=1}^{n} \frac{|g_{i}|^{p-1}}{S_{p-1}} \text{sgn}(g_{i}) \theta_{i} \right|}
\]

= (by independence)

\[
\frac{\mathbb{E} S^{p-1}}{\mathbb{E} S^{p+1}} \sum_{i=1}^{n} \frac{\mathbb{E}|g_{i}|^{2} \left| \sum_{i=1}^{n} |g_{i}|^{p-1} \text{sgn}(g_{i}) \theta_{i} \right|}{\mathbb{E} \left| \sum_{i=1}^{n} |g_{i}|^{p-1} \text{sgn}(g_{i}) \theta_{i} \right|}
\]

Since \( S = (\sum_{i=1}^{n} |g_{i}|^{p})^{1/p} \),

\[
\frac{\mathbb{E} S^{p-1}}{\mathbb{E} S^{p+1}} \leq \frac{c}{n^{2/p}}
\]
\[ \mathbb{E} \left| \sum_{i=1}^{n} |g_i|^{p-1} \text{sgn}(g_i) \theta_i \right| = \mathbb{E}_{\varepsilon} \mathbb{E}_g \left| \sum_{i=1}^{n} |\varepsilon_i g_i|^{p-1} \varepsilon_i \text{sgn}(g_i) \theta_i \right| \]
\[ \geq \text{(by Khinchine's inequality)} \]
\[ \geq C \mathbb{E}_g \left( \sum_{i=1}^{n} |g_i|^{2p-2} \theta_i^2 \right)^{1/2} \]
\[ \geq \text{(by Jensen's inequality)} \]
\[ \geq C \mathbb{E} \sum_{i=1}^{n} |g_i|^{p-1} \theta_i^2 = C \mathbb{E} |g|^{p-1} \geq C > 0 \]

With the same argument,

\[ \mathbb{E}|g_i|^2 \left| \sum_{i=1}^{n} |g_i|^{p-1} \text{sgn}(g_i) \theta_i \right| \leq C \]
The isotropy constant of hyperplane projections of $B_{p}^{n}, 1 < p \leq 2$.

Putting all estimates together, $\frac{1}{|P_{H}(B_{p}^{n})|} \int_{P_{H}(B_{p}^{n})} |x|^{2} dx \leq C \frac{n}{n^{2/p}}$

By the same argument, $|P_{H}(B_{p}^{n})| \frac{2}{n-1} \geq \frac{C}{n^{2/p}}$ and so,

$$(n - 1)L_{P_{H}(B_{p}^{n})}^{2} \leq \frac{1}{|P_{H}(B_{p}^{n})|^{2/n-1}} \cdot \frac{1}{|P_{H}(B_{p}^{n})|} \int_{P_{H}(B_{p}^{n})} |x|^{2} dx \leq Cn$$
The isotropy constant of hyperplane projections of $B^n_p$, $1 < p \leq 2$.

Putting all estimates together, \[
\frac{1}{|P_H(B^n_p)|} \int_{P_H(B^n_p)} |x|^2 dx \leq C \frac{n}{n^{2/p}}
\]

By the same argument, \[|P_H(B^n_p)|^{2/n-1} \geq \frac{C}{n^{2/p}}\] and so,

\[
(n-1)L^2_{P_H(B^n_p)} \leq \frac{1}{|P_H(B^n_p)|^{2/n-1}} \cdot \frac{1}{|P_H(B^n_p)|} \int_{P_H(B^n_p)} |x|^2 dx \leq Cn
\]

**Question.** In order to extend the proof to lower dimensional projections, $E$, we need a formula for

\[
\int_{P_E(B^n_p)} f(x) dx
\]