Random Matrices with Independent Columns

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College Station, July 2009

Based on joint work with
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Outline

1 Introduction
   - Basic definitions
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2 Motivations
   - Sampling convex bodies
   - Properties of random polytopes
   - Smallest singular value

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3 Operator Norm
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4. Kannan-Lovasz-Simonovits Question
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6. Smallest Singular Value
The basic model

Definition

Let $\Gamma$ be an $n \times N$ matrix with columns $X_1, \ldots, X_N$, where $X_i$’s are independent random vectors with values in $\mathbb{R}^n$.

Questions

What is the operator norm of $\Gamma$: $\ell^2_{\mathbb{N}} \rightarrow \ell^2_{\mathbb{N}}$?

When is $\Gamma^T$ close to a multiple of isometry?

How does $\Gamma$ act on sparse vectors?

What is the smallest singular value of $\Gamma$?
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Assumptions on $X_i$

- $X_i$'s are isotropic, i.e.
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- $X_i$ are $\psi_\alpha$ random vectors ($\alpha \in [1, 2]$), i.e. for some $C$ and all $y \in \mathbb{R}^n$,
  $$\| \langle X_i, y \rangle \|_{\psi_\alpha} \leq C |y|,$$

  where
  $$\| Y \|_{\psi_\alpha} = \inf\{ a > 0 : \mathbb{E} \exp((X/a)^\alpha) \leq 2\}$$
Consequences
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- $\mathbb{E}|X_i|^2 = \sum_{j=1}^{n} \mathbb{E}(X_i, e_j)^2 = n$
- For any $y \in S^{n-1}$ and $t \geq 0$,
  \[
  \mathbb{P}(|\langle X_i, y \rangle| \geq t) \leq 2 \exp\left(-\frac{t}{C^\alpha}\right).
  \]
Consequences

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Fact

For every random vector $X$ not supported on any $n-1$ dimensional hyperplane, there exists an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $TX$ is isotropic.
Consequences

- $E|X_i|^2 = \sum_{j=1}^{n} E\langle X_i, e_j \rangle^2 = n$
- For any $y \in S^{n-1}$ and $t \geq 0$,

  $$P(|\langle X_i, y \rangle| \geq t) \leq 2 \exp\left(-\left(\frac{t}{C}\right)^\alpha\right).$$

Fact

*For every random vector $X$ not supported on any $n-1$ dimensional hyperplane, there exists an affine map $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $TX$ is isotropic.*

If for a set $K \subseteq \mathbb{R}^n$ the random vector distributed uniformly on $K$ is isotropic, we say that $K$ is isotropic.
Examples

- $G = (g_1, \ldots, g_n)$, where $g_i$ are i.i.d. $\mathcal{N}(0, 1)$, is $\psi_2$ and isotropic,
- $R = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i$ are i.i.d., $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$, is $\psi_2$ and isotropic,
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- a vector drawn from the uniform distribution on $\sqrt{n}S^{n-1}$ is $\psi_2$ and isotropic,
- a random vector distributed uniformly in an isotropic convex body is $\psi_1$ (C. Borell)
- more generally an isotropic random vector with log-concave density $f$ is $\psi_1$
Motivations: sampling convex bodies

Problem

Let $K \subseteq \mathbb{R}^n$ be a convex body, s.t. $B_2^n \subseteq K \subseteq R B_2^n$. Assume we have access to an oracle (a black box), which given $x \in \mathbb{R}^n$ tells us whether $x \in K$.

How to generate random points uniformly distributed in $K$?

How to compute the volume of $K$?

This can be done by using Markov chains. Their speed of convergence depends on the position of the convex body.

Preprocessing: First put $K$ in the isotropic position (again by randomized algorithms).
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- This can be done by using Markov chains.
- Their speed of convergence depends on the position of the convex body.
- Preprocessing: First put $K$ in the isotropic position (again by randomized algorithms).
Centering the body is not comp. difficult – takes $O(n)$ steps.

The question boils down to:

How to approximate the covariance matrix of $X$ - uniformly distributed on $K$ by the empirical covariance matrix

$$\frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i.$$ 

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Given an isotropic convex body in $\mathbb{R}^n$, how large $N$ should we take so that

$$\left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - Id \right\|_{\ell_2 \to \ell_2} \leq \varepsilon$$

with high probability?
Interpretation in terms of $\Gamma$. 

We have

$$\left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - \text{Id} \right\|_{\ell_2 \rightarrow \ell_2} = \sup_{y \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, y \rangle^2 - 1 \right|$$

$$= \sup_{y \in S^{n-1}} \left| \frac{1}{N} |\Gamma^T y|^2 - 1 \right|$$
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Let $\Gamma$ be a matrix with independent columns $X_1, \ldots, X_N$ drawn from an isotropic convex body (log-concave measure) in $\mathbb{R}^n$. 

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So the (geometric) question is

Let $\Gamma$ be a matrix with independent columns $X_1, \ldots, X_N$ drawn from an isotropic convex body (log-concave measure) in $\mathbb{R}^n$.

How large should $N$ be so that $N^{-1/2} \Gamma^T : \mathbb{R}^n \to \mathbb{R}^N$ was an almost isometry?
Kannan, Lovasz, Simonovits (1995) – $N = \mathcal{O}(n^2)$
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- Aubrun (2006) – unconditional bodies: $N = O(n)$
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Remark

If $\frac{1}{\sqrt{N}} \Gamma^T$ is an almost isometry then obviously $\|\Gamma\| \leq C\sqrt{N}$, so the KLS question and the question about $\|\Gamma\|$ are related.
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If \( \frac{1}{\sqrt{N}} \Gamma^T \) is an almost isometry then obviously \( \| \Gamma \| \leq C \sqrt{N} \), so the KLS question and the question about \( \| \Gamma \| \) are related.

It turns out that to answer KLS it is enough to have good bounds on

\[
A_m := \sup_{z \in S^{N-1}, |\text{supp } z| \leq m} |\Gamma z|
\]

Theorem (Litvak, Pajor, Tomczak-Jaegermann, R.A.)

If \( N \leq \exp(\sqrt{n}) \) and the vectors \( X_i \) are log-concave then for \( t > 1 \), with probability at least \( 1 - \exp(-ct \sqrt{n}) \),

\[
A_m \leq C t (\sqrt{n} + \sqrt{m \log(2N/m)})
\]

In particular, with high probability \( \| \Gamma \| \leq C (\sqrt{n} + \sqrt{N}) \).
Remark

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A_m := \sup_{\substack{z \in S^{N-1} \setminus \text{supp } z \leq m}} |\Gamma z|
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**Theorem (Litvak, Pajor, Tomczak-Jaegermann, R.A.)**

*If \( N \leq \exp(c \sqrt{n}) \) and the vectors \( X_i \) are log-concave then for \( t > 1 \), with probability at least \( 1 - \exp(-ct \sqrt{n}) \),

\[
\forall m \leq N \ A_m \leq Ct \left( \sqrt{n} + \sqrt{m} \log \left( \frac{2N}{m} \right) \right).
\]

*In particular, with high probability \( \| \Gamma \| \leq C(\sqrt{n} + \sqrt{N}) \).*
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If we knew the support of \( x \), to determine \( x \) it would be enough to take \( m \) measurements along basis vectors.
Imagine we have a vector $x \in \mathbb{R}^N$ ($N$ large), which is supported on a small number of coordinates (say $|\text{supp } x| = m \ll N$).

If we knew the support of $x$, to determine $x$ it would be enough to take $m$ measurements along basis vectors.

What if we don’t know the support?
Imagine we have a vector \( x \in \mathbb{R}^N \) (\( N \) large), which is supported on a small number of coordinates (say \( |\text{supp } x| = m \ll N \)).

If we knew the support of \( x \), to determine \( x \) it would be enough to take \( m \) measurements along basis vectors.

What if we don’t know the support?

**Answer** (Donoho, Candes, Tao, Romberg) Take measurements in random directions \( Y_1, \ldots, Y_n \) and set

\[
\hat{x} = \arg\min \{ \|y\|_1 : \langle Y_i, y \rangle = \langle Y_i, x \rangle \}
\]
Compressed sensing and neighbourly polytopes

**Definition**

A polytope $K \subseteq \mathbb{R}^n$ is called $m$-neighbourly if any set of vertices of $K$ of cardinality at most $m + 1$ is the vertex set of a face.

Theorem (Donoho)

Let $\Gamma$ be an $n \times N$ matrix with columns $X_1, \ldots, X_N$. The following conditions are equivalent

(i) For any $x \in \mathbb{R}^N$ with $|\text{supp } x| \leq m$, $x$ is the unique solution of the minimization problem

\[
\min \|t\|_1, \Gamma t = \Gamma x.
\]

(ii) The polytope $K(\Gamma) = \text{conv}(\pm X_1, \ldots, \pm X_N)$ has $2N$ vertices and is $m$-symmetric-neighbourly.
A (centrally symmetric) polytope $K \subseteq \mathbb{R}^n$ is called $m$-(symmetric)-neighbourly if any set of vertices of $K$ of cardinality at most $m + 1$ (containing no opposite pairs) is the vertex set of a face.
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A (centrally symmetric) polytope $K \subseteq \mathbb{R}^n$ is called \textit{m-(symmetric)-neighbourly} if any set of vertices of $K$ of cardinality at most $m + 1$ (containing no opposite pairs) is the vertex set of a face.

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Definition (Restricted Isometry Property (Candès, Tao))

For an $n \times N$ matrix $\Gamma$ define the **isometry constant** $\delta_m = \delta_m(\Gamma)$ as the smallest number such that

$$(1 - \delta_m) |x|^2 \leq |\Gamma x|^2 \leq (1 + \delta_m) |x|^2$$

for all $m$-sparse vectors $x \in \mathbb{R}^N$. 

Theorem (Candès)

If $\delta_2 m(\Gamma) < \sqrt{2 - 1}$ then for every $m$-sparse $x \in \mathbb{R}^n$, $x$ is the unique solution to

$$\min \|t\|_1, \quad \Gamma t = \Gamma x.$$ 

In consequence, the polytope $K(\Gamma)$ (resp. $K(\Gamma) = \text{conv}(X_1, \ldots, X_N)$) is $m$-symmetric-neighbourly (resp. $m$-neighbourly).
Compressed sensing and neighbourly polytopes

Definition (Restricted Isometry Property (Candès, Tao))

For an \( n \times N \) matrix \( \Gamma \) define the **isometry constant** \( \delta_m = \delta_m(\Gamma) \) as the smallest number such that

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\]

for all \( m \)-sparse vectors \( x \in \mathbb{R}^N \).

Theorem (Candès)

If \( \delta_{2m}(\Gamma) < \sqrt{2} - 1 \) then for every \( m \)-sparse \( x \in \mathbb{R}^n \), \( x \) is the unique solution to

\[
\min \| t \|_1, \quad \Gamma t = \Gamma x.
\]

In consequence, the polytope \( K(\Gamma) \) (resp. \( K_+(\Gamma) = \text{conv}(X_1, \ldots, X_N) \)) is \( m \)-symmetric-neighbourly (resp. \( m \)-neighbourly)
History

The following matrices satisfy RIP

- Gaussian matrices (Candes, Tao), $m \sim n / \log(2N/n)$
- Matrices with rows selected randomly from the Fourier matrix (Candes & Tao, Rudelson & Vershynin), $m \sim n / \log^4(2N/n)$
- Matrices with independent subgaussian isotropic rows (Mendelson, Pajor, Tomczak-Jaegermann), $m \sim n / \log(2N/n)$
- Matrices with independent log-concave isotropic columns (LPTA), $m \sim n / \log^2(2N/n)$
Neighbourly polytopes

**Theorem (LPTA)**

Let \( \theta \in (0, 1) \) and assume that \( N \leq \exp(c\theta^C n^c) \) and 
\[
C m \log^2 \left( \frac{2N}{\theta m} \right) \leq \theta^2 n.
\]
Then, with probability at least \( 1 - \exp(-c\theta^C n^c) \)
\[
\delta_m \left( \frac{1}{\sqrt{n}} \Gamma \right) \leq \theta.
\]

**Corollary (LPTA)**

Let \( X_1, \ldots, X_N \) be random vectors drawn from an isotropic convex body in \( \mathbb{R}^n \). Then, for \( N \leq \exp(cn^c) \), with probability at least \( 1 - \exp(-cn^c) \), the polytope \( K(\Gamma) \) (resp. \( K_+(\Gamma) \)) is \( m \)-symmetric-neighbourly (resp. \( m \)-neighbourly) with 
\[
m = \left\lfloor c \frac{n}{\log^2(CN/n)} \right\rfloor.
\]
Smallest singular value

**Definition**

For an $n \times n$ matrix $\Gamma$ let $s_1(\Gamma) \geq s_2(\Gamma) \geq \ldots \geq s_n(\Gamma)$ be the singular values of $\Gamma$, i.e. eigenvalues of $\sqrt{\Gamma \Gamma^T}$. In particular

$$s_1(\Gamma) = \|A\|, \quad s_n(\Gamma) = \inf_{x \in S^{n-1}} |\Gamma x| = \frac{1}{\|A^{-1}\|}$$

**Theorem (Edelman, Szarek)**

Let $\Gamma$ be an $n \times n$ random matrix with independent $\mathcal{N}(0, 1)$ entries. Let $s_n$ denote the smallest singular values of $\Gamma$. Then, for every $\varepsilon > 0$,

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2} \leq C\varepsilon,$$

where $C$ is a universal constant.
Theorem (Rudelson, Vershynin)

Let $\Gamma$ be a random matrix with independent entries $X_{ij}$, satisfying $\mathbb{E} X_{ij} = 0$, $\mathbb{E} X_{ij}^2 = 1$, $\|X_{ij}\|_{\psi_2} \leq B$. Then for any $\varepsilon \in (0, 1)$,

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + c^n,$$

where $C > 0$, $c \in (0, 1)$ depend only on $B$.

Theorem (Guedon, Litvak, Pajor, Tomczak-Jaegermann, R.A.)

Let $\Gamma$ be an $n \times n$ random matrix with independent isotropic log-concave rows. Then, for any $\varepsilon \in (0, 1)$,

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + C \exp(-cn^c)$$

and

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon^{n/(n+2)} \log^C(2/\varepsilon).$$
**Corollary**

For any $\delta \in (0, 1)$ there exists $C_\delta$ such that for any $n$ and $\varepsilon \in (0, 1)$,

$$
\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C_\delta \varepsilon^{1-\delta}.
$$

**Definition**

For an $n \times n$ matrix $\Gamma$ define the **condition number** $\kappa(\Gamma)$ as

$$
\kappa(\Gamma) = \|\Gamma\| \cdot \|\Gamma^{-1}\| = \frac{s_1(\Gamma)}{s_n(\Gamma)}.
$$

**Corollary**

If $\Gamma$ has independent isotropic log-concave columns, then for any $\delta > 0$, $t > 0$,

$$
\mathbb{P}(\kappa(\Gamma) \geq nt) \leq \frac{C_\delta}{t^{1-\delta}}.
$$
Recall: \( \Gamma \) - an \( n \times N \) matrix with independent columns \( X_1, \ldots, X_N \).

\[
A_m := \sup_{z \in S^{N-1}} |\Gamma z|, \quad |\text{supp } z| \leq m
\]

We are going to prove

**Theorem (Litvak, Pajor, Tomczak-Jaegermann, R.A.)**

If \( N \leq \exp(c\sqrt{n}) \) and the vectors \( X_i \) are log-concave then for \( t > 1 \), with probability at least \( 1 - \exp( -ct \sqrt{n} ) \),

\[
\forall m \leq N \quad A_m \leq C t \left( \sqrt{n} + \sqrt{m \log \left( \frac{2N}{m} \right)} \right).
\]

In particular, with high probability \( \| \Gamma \| \leq C (\sqrt{n} + \sqrt{N}) \).
An easy decoupling lemma

**Lemma**

Let $x_1, \ldots, x_N \in \mathbb{R}^n$. There exists a set $E \subset \{1, \ldots, N\}$, such that

$$\sum_{i \neq j} \langle x_i, x_j \rangle \leq 4 \sum_{i \in E} \sum_{j \in E^c} \langle x_i, x_j \rangle.$$ 

**Proof.**

$$2^{N-2} \sum_{i \neq j} \langle x_i, x_j \rangle = \sum_{E \subset \{1, \ldots, N\}} \sum_{i \in E} \sum_{j \in E^c} \langle x_i, x_j \rangle \leq 2^N \max_{E \subset \{1, \ldots, N\}} \sum_{i \in E} \sum_{j \in E^c} \langle x_i, x_j \rangle$$
Lemma

Let $m \leq N$, $\varepsilon, \alpha \in (0, 1]$ and $L \geq 2m \log \frac{12eN}{m\varepsilon}$. Then

$$
\mathbb{P} \left( \sup_{F \subset \{1, \ldots, N\}} \sup_{E \subset F} \sup_{z \in \mathcal{N}(F, \alpha, \varepsilon)} \left| \sum_{i \in E} \left\langle z_i X_i, \sum_{j \in F \setminus E} z_j X_j \right\rangle \right| > C_{\alpha} L A_m \right) \leq e^{-L/2}
$$

Lemma

Let $1 \leq k$, $m \leq N$, $\varepsilon, \alpha \in (0, 1]$, $\beta > 0$, and $L > 0$. Let $B(m, \beta)$ denote the set of vectors $x \in \beta B_2^N$ with $|\text{supp} x| \leq m$ and let $B$ be a subset of $B(m, \beta)$ of cardinality $M$. Then

$$
\mathbb{P} \left( \sup_{F \subset \{1, \ldots, N\}} \sup_{x \in B} \sup_{z \in \mathcal{N}(F, \alpha, \varepsilon)} \left| \sum_{i \in F} \left\langle z_i X_i, \sum_{j \notin F} x_j X_j \right\rangle \right| > C_{\alpha \beta} L A_m \right) 
\leq M \left( \frac{6eN}{k\varepsilon} \right)^k e^{-L}. 
$$
Proof of the first lemma

- Fix $F, E \subseteq F, z \in \mathcal{N}(F, \alpha, \varepsilon)$.
- Set $y = \sum_{j \in F \setminus E} z_j X_j$. 

The statement follows from the union bound.
Proof of the first lemma

- Fix $F, E \subset F, z \in \mathcal{N}(F, \alpha, \varepsilon)$.
- Set $y = \sum_{j \in F \setminus E} z_j X_j$.
- $|y| \leq A_m, \|z\|_\infty \leq \alpha$, hence

\[
\sum_{i \in E} \left| \langle z_i X_i, \sum_{j \in F \setminus E} z_j X_j \rangle \right| \leq \alpha A_m \sum_{i \in E} \left| \langle X_i, y/|y| \rangle \right|.
\]
Proof of the first lemma

- Fix $F, E \subset F, z \in \mathcal{N}(F, \alpha, \varepsilon)$.
- Set $y = \sum_{j \in F \setminus E} z_j X_j$.
- $|y| \leq A_m, \|z\|_\infty \leq \alpha$, hence

$$
\sum_{i \in E} \left| \langle z_i X_i, \sum_{j \in F \setminus E} z_j X_j \rangle \right| \leq \alpha A_m \sum_{i \in E} \left| \langle X_i, y / |y| \rangle \right|.
$$

- $y, (X_i)_{i \in E}$ independent, so Chebyshev’s inequality & $\psi_1$-property yield

$$
P\left( \sum_{i \in E} \left| \langle X_i, y / |y| \rangle \right| \geq CL \right) \leq e^{-L \mathbb{E}} \exp \left( \sum_{i \in E} \frac{|\langle X_i, y / |y| \rangle|}{C} \right)
\leq 2^{|E|} e^{-L} \leq 2^m e^{-L}.
$$

- The statement follows from the union bound.
Theorem (LPTA)

Let $X_1, \ldots, X_N$ be i.i.d. isotropic log-concave vectors in $\mathbb{R}^n$. For every $\varepsilon \in (0, 1)$ and $t \geq 1$ there exists $C(\varepsilon, t)$ s.t. if $N \geq C(\varepsilon, t)n$, then with probability at least $1 - \exp(-ct\sqrt{n})$, 

$$
\left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - \text{Id} \right\| = \sup_{y \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, y \rangle^2 - 1 \right| \leq \varepsilon.
$$

Moreover one can take $C(\varepsilon, t) = Ct^4 \varepsilon^{-2} \log(2t^2\varepsilon^{-2})$. 

Remark: In the proof we can assume that $N \leq \exp(\sqrt{n})$. 

Radosław Adamczak (MIM UW)
Theorem (LPTA)

Let $X_1, \ldots, X_N$ be i.i.d. isotropic log-concave vectors in $\mathbb{R}^n$. For every $\varepsilon \in (0, 1)$ and $t \geq 1$ there exists $C(\varepsilon, t)$ s.t. if $N \geq C(\varepsilon, t)n$, then with probability at least $1 - \exp(-ct\sqrt{n})$,

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Moreover one can take $C(\varepsilon, t) = Ct^4 \varepsilon^{-2} \log(2t^2 \varepsilon^{-2})$.

Remark: In the proof we can assume that $N \leq \exp(\sqrt{n})$. 
Strategy of the proof

1. Divide the stochastic process into the 'bounded' and 'unbounded' part

2. **Bounded part:**
   - reduce to an $\varepsilon$-net
   - use Bernstein’s inequality for individual vectors

3. **Unbounded part:** use estimates on $A_m$ to
   - show that the unbounded part has 'small support'
   - get estimates on the unbounded part
Sketch of the proof

It is enough to consider \( y \in \mathcal{N}, \mathcal{N} \) - a 1/4-net in \( S^{n-1} \) of card. \( 5^n \).

We decompose

\[
\langle X_i, y \rangle^2 = \langle X_i, y \rangle^2 \land R^2 + (\langle X_i, y \rangle^2 - R^2)1_{\{|\langle X_i, y \rangle| > R\}}
\]

We have

\[
\sup_{y \in \mathcal{N}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, y \rangle^2 - 1 \right| = \sup_{y \in \mathcal{N}} \left| \frac{1}{N} \sum_{i=1}^{N} (\langle X_i, y \rangle^2 - \mathbb{E}(\langle X_i, y \rangle^2) ) \right|
\]

\[
\leq \sup_{y \in \mathcal{N}} \left| \frac{1}{N} \sum_{i=1}^{N} (\langle X_i, y \rangle^2 \land R^2 - \mathbb{E}(\langle X_i, y \rangle^2 \land R^2)) \right|
\]

\[
+ \sup_{y \in \mathcal{N}} \frac{1}{N} \sum_{i=1}^{N} \langle X_i, y \rangle^2 1_{\{|\langle X_i, y \rangle| > R\}} + \sup_{y \in \mathcal{N}} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\langle X_i, y \rangle^2 1_{\{|\langle X_i, y \rangle| > R\}}
\]

\[= I + II + III.\]
\[
\mathbb{E}\langle X_1, y \rangle^2 1_{\{\|X_1, y\| > R\}} \leq (\mathbb{E}\langle X_1, y \rangle^4)^{1/2} \mathbb{P}(\|X_i, y\| > R)^{1/2}
\]
\[
\leq C \exp(-R/C).
\]
Theorem (Bernstein’s inequality)

Let $Y_1, \ldots, Y_N$ be i.i.d. centered r.v. with $\mathbb{E}Y_i^2 = \sigma^2$ and $\|Y_i\|_\infty \leq a$. For any $t \geq 0$,

$$\mathbb{P}
\left(\left|\frac{1}{N} \sum_{i=1}^{N} Y_i\right| \geq t\right) \leq 2 \exp \left(-cN \min \left(\frac{t^2}{\sigma^2}, \frac{t}{a}\right)\right).$$

\[\mathbb{E}\langle X_1, y \rangle^2 \mathbb{1}_{\{\langle X_1, y \rangle > R\}} \leq (\mathbb{E}\langle X_1, y \rangle^4)^{1/2} \mathbb{P}(\langle X_i, y \rangle > R)^{1/2} \leq C \exp(-R/C).\]
\[ \mathbb{E}\langle X_1, y \rangle^2 1_{\{\|X_1, y\| > R\}} \leq \left( \mathbb{E}\langle X_1, y \rangle^4 \right)^{1/2} \mathbb{P}(\|X_i, y\| > R)^{1/2} \leq C \exp(-R/C). \]

**Theorem (Bernstein’s inequality)**

Let \( Y_1, \ldots, Y_N \) be i.i.d. centered r.v. with \( \mathbb{E} Y_i^2 = \sigma^2 \) and \( \|Y_i\|_\infty \leq a \). For any \( t \geq 0 \),

\[ \mathbb{P}\left( \left\| \frac{1}{N} \sum_{i=1}^N Y_i \right\| \geq t \right) \leq 2 \exp\left( -cN \min\left( \frac{t^2}{\sigma^2}, \frac{t}{a} \right) \right). \]

This gives

\[ \mathbb{P}(I \geq \varepsilon) \leq 5^n \exp(-cN \min(\varepsilon^2, \varepsilon/R^2)). \]
The unbounded part

With pr. at least $1 - \exp(-t\sqrt{n})$ we have for all $m \leq N$,

$$A_m = \sup_{z \in S^{N-1}} \left| \sum_{i=1}^{N} z_i X_i \right| \leq Ct \left( \sqrt{n} + \sqrt{m} \log \left( \frac{2N}{n} \right) \right).$$
The unbounded part

With pr. at least $1 - \exp(-t\sqrt{n})$ we have for any $E \subseteq \{1, \ldots, N\}$,

$$\sup_{y \in S^{n-1}} \sum_{i \in E} \langle X_i, y \rangle^2 \leq Ct^2 \left( n + |E| \log \left( \frac{2N}{n} \right)^2 \right).$$
The unbounded part

With pr. at least $1 - \exp(-t\sqrt{n})$ we have for any $E \subseteq \{1, \ldots, N\}$,

$$\sup_{y \in S^{n-1}} \sum_{i \in E} \langle X_i, y \rangle^2 \leq Ct^2 \left( n + |E| \log \left( \frac{2N}{n} \right)^2 \right).$$

Set $E = E(y) = \{ i : |\langle X_i, y \rangle| \geq R \}$. Then

$$|E|R^2 \leq \sum_{i \in E} \langle X_i, y \rangle^2 \leq Ct^2 \left( n + |E| \log \left( \frac{2N}{n} \right)^2 \right).$$
The unbounded part

With pr. at least $1 - \exp(-t\sqrt{n})$ we have for any $E \subseteq \{1, \ldots, N\}$,

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For $R^2 \geq 2Ct^2 \log \left(\frac{2N}{n}\right)^2$ we get $|E| \leq Ct^2 nR^{-2}$. 
The unbounded part

With pr. at least $1 - \exp(-t\sqrt{n})$ we have for any $E \subseteq \{1, \ldots, N\}$,

$$\sup_{y \in S^{n-1}} \sum_{i \in E} \langle X_i, y \rangle^2 \leq Ct^2 \left( n + |E| \log \left( \frac{2N}{n} \right)^2 \right).$$

Set $E = E(y) = \{i: |\langle X_i, y \rangle| \geq R\}$. Then

$$|E|R^2 \leq \sum_{i \in E} \langle X_i, y \rangle^2 \leq Ct^2 \left( n + |E| \log \left( \frac{2N}{n} \right)^2 \right).$$

For $R^2 \geq 2Ct^2 \log \left( \frac{2N}{n} \right)^2$ we get $|E| \leq Ct^2 nR^{-2}$. Thus

$$\sup_{y \in S^{n-1}} \sum_{i \in E} \langle X_i, y \rangle^2 \leq 2Ct^2 n.$$
For $R \geq Ct \log(2N/n)$ we have

- With pr. $1 - 5^n \exp(-cN \min(\varepsilon^2, \varepsilon/R^2))$, $I \leq \varepsilon$, 
- With pr. $1 - \exp(-ct\sqrt{n})$, $II \leq Ct^2 n/N$,
- With pr. $1 - \exp(-R/C)$, $III \leq C \exp(-R/C)$.

Set $R = Ct \log(2N/n)$ and $N \geq C(\varepsilon, t)n$ to get $I + II + III \leq 3\varepsilon$. "w.h.p."
For $R \geq Ct \log(2N/n)$ we have

- With pr. $1 - 5^n \exp(-cN \min(\varepsilon^2, \varepsilon/R^2))$,
  \[ I \leq \varepsilon, \]

- With pr. $1 - \exp(-ct \sqrt{n})$
  \[ II \leq Ct^2 n/N, \]
Summarizing

For $R \geq Ct \log(2N/n)$ we have

- With pr. $1 - 5^n \exp(-cN \min(\varepsilon^2, \varepsilon/R^2))$, 
  $$I \leq \varepsilon,$$

- With pr. $1 - \exp(-ct\sqrt{n})$
  $$II \leq Ct^2 n/N,$$

- $III \leq C \exp(-R/C).$
Summarizing

For $R \geq Ct \log(2N/n)$ we have

- With pr. $1 - 5^n \exp(-cN \min(\varepsilon^2, \varepsilon/R^2))$,
  
  $$I \leq \varepsilon,$$

- With pr. $1 - \exp(-ct\sqrt{n})$
  
  $$II \leq Ct^2 n/N,$$

- $III \leq C \exp(-R/C)$.

Set $R = Ct \log(2N/n)$ and $N \geq C(\varepsilon, t)n$ to get

$$I + II + III \leq 3\varepsilon \text{ w.h.p.}$$
Theorem (LPTA)

Let $X_1, \ldots, X_N$ be i.i.d. isotropic log-concave vectors in $\mathbb{R}^n$. For every $p \geq 2$ and every $\varepsilon \in (0, 1)$ and $t \geq 1$ there exists $C(p, \varepsilon, t)$ s.t. if $N \geq C(p, \varepsilon, t)n^{p/2}$, then with probability at least $1 - \exp(-c_p t \sqrt{n})$,

\[
\sup_{y \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} |\langle X_i, y \rangle|^p - \mathbb{E}|\langle X_i, y \rangle|^p \right| \leq \varepsilon.
\]

Moreover one can take $C(\varepsilon, t) = C_p t^{2p} \varepsilon^{-2} \log^{2p-2}(2t^2 \varepsilon^{-2})$.

Previous contributions

- Giannopoulos, Milman (2000) – $N = \mathcal{O}((n \log n)^{p/2})$
- Guedon, Rudelson (2007) – $N = \mathcal{O}(n^{p/2} \log n)$.

Remark: For $p < 2$ it is enough to take $N = \mathcal{O}(n)$. 
Neighbourly polytopes

**Theorem (LPTA)**

Let \( \theta \in (0, 1) \) and assume that \( N \leq \exp(c\theta^Cn^c) \) and \( Cm \log^2 \left( \frac{2N}{\theta m} \right) \leq \theta^2 n \). Then, with pr. at least \( 1 - \exp(-c\theta^Cn^c) \),

\[
\sup_{z \in S^{N-1}} \frac{1}{n} \left| \left| \Gamma x \right|^2 - 1 \right| = \left| \frac{1}{n} \sum_{i=1}^{N} z_i X_i \right|^2 - 1 \leq \theta.
\]

**Corollary (LPTA)**

Let \( X_1, \ldots, X_N \) be random vectors drawn from an isotropic convex body in \( \mathbb{R}^n \). Then, for \( N \leq \exp(cn^c) \), with probability at least \( 1 - \exp(-cn^c) \), the polytope \( K(\Gamma) \) (resp. \( K_+(\Gamma) \)) is \( m \)-symmetric-neighbourly (resp. \( m \)-neighbourly) with

\[
m = \left\lfloor c \frac{n}{\log^2(CN/n)} \right\rfloor.
\]
Sketch of the proof

Denote

\[ C_m = \max_{i \leq N} |X_i|, \]

\[ B_m^2 = \sup_{z \in \mathbb{S}^{N-1}} \left| \frac{1}{n} \Gamma z \right|^2 - \sum_{i=1}^{N} z_i^2 |X_i|^2 \]

\[ = \sup_{z \in \mathbb{S}^{N-1}} \sum_{i \neq j} \langle z_i X_i, z_j X_j \rangle = \sup_{z \in \mathbb{S}^{N-1}} D_z. \]
Sketch of the proof

Denote

\[ C_m = \max_{i \leq N} |X_i|, \]

\[ B_m^2 = \sup_{z \in S^{N-1}} \sup_{|\text{supp } z| \leq m} \left| \frac{1}{n} |\Gamma z|^2 - \sum_{i=1}^{N} z_i^2 |X_i|^2 \right| \]

\[ = \sup_{z \in S^{N-1}} \sum_{i \neq j} \langle z_i X_i, z_j X_j \rangle = \sup_{z \in S^{N-1}} \langle D_z \rangle. \]

We have

\[ D_z - D_x = \langle \Gamma z, \Gamma (z - x) \rangle + \langle \Gamma (z - x), \Gamma x \rangle + \sum_{i=1}^{n} (x_i - z_i)(x_i + z_i) |X_i|^2. \]

Hence if \( |x - z| \leq \theta \) and they have the same support,

\[ D_z \leq D_x + 2\theta (A_m^2 + C_m^2) \leq D_x + 2\theta (B_m^2 + 2C_m^2). \]
\[ D_z \leq D_x + 2\theta(A_m^2 + C_m^2) \leq D_x + 2\theta(B_m^2 + 2C_m^2). \]
\[ D_z \leq D_x + 2\theta(A_m^2 + C_m^2) \leq D_x + 2\theta(B_m^2 + 2C_m^2). \]

We construct a set \( \mathcal{M}(\theta) \) such that

- \( \sup_{x \in \mathcal{M}} D_x \leq A_m\theta \sqrt{n} \) w.h.p.
\[ D_z \leq D_x + 2\theta(A_m^2 + C_m^2) \leq D_x + 2\theta(B_m^2 + 2C_m^2). \]

We construct a set \( \mathcal{M}(\theta) \) such that

- \( \sup_{x \in \mathcal{M}} D_x \leq A_m\theta \sqrt{n} \) w.h.p.
- for every \( z \in S^{n-1} \) with \( |\text{supp } z| \leq m \), there is \( x \in \mathcal{M}(\theta) \), \( \text{supp } z = \text{supp } x \), \( |x - z| < \theta \).
\[ D_z \leq D_x + 2\theta(A_m^2 + C_m^2) \leq D_x + 2\theta(B_m^2 + 2C_m^2). \]

We construct a set \( \mathcal{M}(\theta) \) such that

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We get

\[
B_m^2 = \sup_{\substack{z \in S^{n-1} \\ |\text{supp } z| \leq m}} D_z \leq \theta \sqrt{n} A_m + 2\theta(B_m^2 + C_m^2)
\]

\[
\leq \theta \sqrt{n} \sqrt{B_m^2 + C_m^2 + 2\theta(B_m^2 + C_m^2)}
\]

Now we solve for \( B_m \) and use the fact that \( C_m^2 \leq Cn \) w.h.p.
We get

\[ B_m^2 = \sup_{z \in S^{N-1} \atop |\text{supp } z| \leq m} \left| \frac{1}{n} |\Gamma z|^2 - \sum_{i=1}^{N} z_i^2 |X_i|^2 \right| \leq \theta n. \]
We get

\[
B_m^2 = \sup_{z \in S^{N-1}, |\text{supp } z| \leq m} \left| \frac{1}{n} |\Gamma z|^2 - \sum_{i=1}^{N} z_i^2 |X_i|^2 \right| \leq \theta n.
\]

How does it imply the theorem?
We get

\[ B_m^2 = \sup_{z \in S^{N-1}} \left| \frac{1}{n} |\Gamma z|^2 - \sum_{i=1}^{N} z_i^2 |X_i|^2 \right| \leq \theta n. \]

How does it imply the theorem?

\[ \left| \sum_{i=1}^{N} z_i^2 |X_i|^2 - n \right| \leq \sum_{i=1}^{n} z_i^2 \left| X_i \right|^2 - n \right| \leq \max_{i \leq N} \left| X_i \right|^2 - n. \]
We get

\[ B_m^2 = \sup_{z \in S^{N-1} \cap \text{supp } z \leq m} \left| \frac{1}{n} \| \Gamma z \|^2 - \sum_{i=1}^{N} z_i^2 |X_i|^2 \right| \leq \theta n. \]

How does it imply the theorem?

\[ \left| \sum_{i=1}^{N} z_i^2 |X_i|^2 - n \right| \leq \sum_{i=1}^{n} z_i^2 \left| |X_i|^2 - n \right| \leq \max_{i \leq N} \left| |X_i|^2 - n \right|. \]

**Theorem (Klartag)**

\[ \mathbb{P} \left( \left| |X_i| - n \right| \geq \theta n \right) \leq \exp(-c\theta^c n^c). \]

The statement follows by the union bound.
The smallest singular value of a square matrix

**Theorem (GLPTA)**

Let $\Gamma$ be an $n \times n$ random matrix with independent isotropic log-concave rows. Then, for any $\varepsilon \in (0, 1)$,

$$P(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + C \exp(-cn^c)$$

and

$$P(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon^{n/(n+2)} \log^C(2/\varepsilon).$$

**Corollary**

For any $\delta \in (0, 1)$ there exists $C_{\delta}$ such that for any $n$ and $\varepsilon \in (0, 1)$,

$$P(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C_{\delta}\varepsilon^{1-\delta}.$$
Compressible vectors

Let us define:

\[ \text{Sparse} = \text{Sparse}(\delta) = \{ x \in \mathbb{R}^n : \| \text{supp} \, x \| \leq \delta n \} \]
\[ \text{Comp} = \text{Comp}(\delta, \rho) = \{ x \in S^{n-1} : \text{dist}(x, \text{Sparse}(\delta)) \leq \rho \} \]
\[ \text{Incomp} = \text{Incomp}(\delta, \rho) = S^{n-1} \setminus \text{Comp}(\delta, \rho) \]

Lemma (Rudelson-Vershynin)

Let \( H_k = \text{span}((X_i)_{i \neq k}) \). For every \( \rho, \delta \in (0, 1) \) and \( \varepsilon > 0 \),

\[
\mathbb{P}( \inf_{x \in \text{Incomp}(\delta, \rho)} |\Gamma x| < \varepsilon \rho n^{-1/2} ) \leq \frac{1}{\delta n} \sum_{k=1}^{n} \mathbb{P}(\text{dist}(X_k, H_k) < \varepsilon ).
\]
The isotropy constant

**Definition**

For an isotropic log-concave measure $\mu$ on $\mathbb{R}^n$ define the **isotropy constant** of $\mu$ as

$$L_\mu = f(0)^{1/n},$$

where $f$ is the density of $\mu$. 

A famous open problem:

Is $L_\mu \leq C$?

**Fact** (not difficult, enough for us)

$L_\mu \leq C \sqrt{n}$. 

**Theorem** (Klartag)

$L_\mu \leq Cn^{1/4}$.
The isotropy constant

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where $f$ is the density of $\mu$.

**A famous open problem:** Is $L_\mu \leq C$?

**Fact (not difficult, enough for us)**

$$L_\mu \leq C\sqrt{n}.$$

**Theorem (Klartag)**

$$L_\mu \leq Cn^{1/4}.$$
Incompressible vectors

We have to estimate

\[ \mathbb{P}(\text{dist} (X_i, H_i) \leq \varepsilon), \]

where \( H_i \) is a random hyperplane independent of \( X_i \). Let \( y \) be a unit normal to \( H_i \), then

\[ \mathbb{P}(\text{dist} (X_i, H_i) \leq \varepsilon) = \mathbb{E}_{H_i} \mathbb{P}_{X_i}(|\langle X_i, y \rangle| \leq \varepsilon) \leq C\varepsilon, \]

as

\[ \mathbb{P}_{X_i}(|\langle X_i, y \rangle| \leq \varepsilon) = \int_{-\varepsilon}^{\varepsilon} f_{\langle X_i, y \rangle}(t)dt \leq C\varepsilon. \]
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Thus
\[ \mathbb{P}(\inf_{x \in \text{Incomp}(\delta, \rho)} |\Gamma x| < \varepsilon \rho n^{-1/2}) \leq C\varepsilon. \]
Lemma (Compressible to sparse reduction)

For any $\rho, \delta \in (0, 1)$ and $M \geq 1$, if

$$\inf_{x \in \text{Comp}(\delta, \rho/(2M))} |\Gamma x| \leq \rho \sqrt{n}$$

and $\|\Gamma\| \leq M \sqrt{n}$, then

$$\inf_{y \in \text{Sparse}(\delta), |y| = 1} |\Gamma y| \leq 4\rho \sqrt{n}.$$
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We already know that with pr. at least $1 - \exp(-cn^c)$ for all $m$-sparse vectors $z$ ($m = \delta n$, $\delta$ small enough),

$$\big||\Gamma z|^2 - n\big| \leq \frac{1}{4} n$$

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This implies that

$$|\Gamma z| \geq \sqrt{n}/2.$$
We have proved that

\[ P(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + \exp(-cn^c) \]
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How to prove

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Main ingredients
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Main ingredients

- We can assume that \( \varepsilon < \exp(-cn^c) \).
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- We can assume that \( \varepsilon < \exp(-cn^c) \).
- For a log-concave vector \( X \), \( \mathbb{P}(|X| \leq \rho \sqrt{n}) \leq C^n L_{\mu}^n \rho^n \leq C^n n^{n/2} \rho^n \).
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- If \( z \in S^{n-1} \) then \( \sum_{i=1}^n z_i X_i \) is log-concave (Prekopa-Leindler) and isotropic (easy).
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- The set \( \text{Sparse}(\rho) \) admits significantly smaller nets than \( S^{n-1}. \)
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Main ingredients

- We can assume that $\varepsilon < \exp(-cn^c)$.
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- If $z \in S^{n-1}$ then $\sum_{i=1}^n z_i X_i$ is log-concave (Prekopa-Leindler) and isotropic (easy).
- The set $\text{Sparse}(\rho)$ admits significantly smaller nets than $S^{n-1}$.
- One has to choose all the parameters (tedious but doable).
Thank you