Morse theory for Lagrange multipliers
and adiabatic limits

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Outline

1. Morse homology
2. Lagrange multiplier
3. Adiabatic limits
I. Morse homology
Let $M$ be a smooth manifold, $f \in C^\infty(M)$. $f$ is a Morse function if $df$ is a transverse section of $T^*M$. 
Let $M$ be a smooth manifold, $f \in C^\infty(M)$. $f$ is a **Morse function** if $df$ is a transverse section of $T^*M$.

By **Morse lemma**, near $p \in \text{Crit}f$, there is a coordinate chart $(x_1, \ldots, x_n)$ such that

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$
Let $M$ be a smooth manifold, $f \in C^\infty(M)$. $f$ is a Morse function if $df$ is a transverse section of $T^*M$.

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$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$  

Morse index: $\text{ind}(p, f) = k$, the number of negative eigenvalues of the Hessian.
For any metric $g$ on $M$, the metric dual of $df$ is the gradient $\nabla f$ of $f$. The negative gradient flow of $f$ is the ODE

$$x : \mathbb{R} \to M, \quad x'(t) + \nabla f(x(t)) = 0.$$
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The integral of the flow defines a 1-parameter diffeomorphisms group $(\phi_t)_{t \in \mathbb{R}}$ of $M$, i.e.

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**unstable/stable manifolds** of $p \in \text{Crit} f$:

$$W^u(p) = \left\{ x \in M \mid \lim_{t \to -\infty} \phi_t(x) = p \right\}, \ \dim W^u(p) = \text{ind}(p)$$

$$W^s(p) = \left\{ x \in M \mid \lim_{t \to +\infty} \phi_t(x) = p \right\}, \ \dim W^s(p) = n - \text{ind}(p).$$
The pair \((f, g)\) is **Morse-Smale** if

\[ \forall p_-, p_+ \in \text{Crit} f, \quad W^u(p_-) \pitchfork W^s(p_+). \]
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In this case, the **moduli space** of solutions to the ODE which are asymptotic to \(p_\pm\) at \(\pm\infty\)

\[
\tilde{\mathcal{M}}(p_-, p_+) = W^u(p_-) \cap W^s(p_+)
\]

\[
= \left\{ x : \mathbb{R} \to M \mid x'(t) + \nabla f(x(t)) = 0, \lim_{t \to \pm\infty} x(t) = p_\pm \right\}
\]

is a smooth manifold. It has dimension \(\text{ind}(p_-) - \text{ind}(p_+)\).
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\[ \mathcal{M}(p_-, p_+) := \widetilde{\mathcal{M}}(p_-, p_+)/\mathbb{R}. \]
(We need **Palais-Smale condition** in noncompact case):

\[
\text{ind}(p_-) - \text{ind}(p_+) = 1 \implies |M(p_-, p_+)| < \infty.
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(We need **Palais-Smale condition** in noncompact case):

\[ \text{ind}(p_-) - \text{ind}(p_+) = 1 \implies \# \mathcal{M}(p_-, p_+) < \infty. \]

**Morse-Smale-Witten complex:**

\[ CM_k(f; \mathbb{Z}_2) = \bigoplus_{p \in \text{Crit}_k f} \mathbb{Z}_2 \langle p \rangle, \quad \delta_{f,g} : CM_k \to CM_{k-1}. \]

\[ \delta_{f,g}(p) = \sum_{q \in \text{Crit}_{k-1} f} \# \mathcal{M}(p, q) \cdot q. \]
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Slogan: “Boundary operator defined by the (oriented) counting of isolated trajectories.”
A nontrivial fact is that \((\delta_f, g)^2 = 0\). Then \((CM_*(f; \mathbb{Z}_2), \delta_{f, g})\) is a chain complex, and we define the Morse homology associated to the pair \((f, g)\) by

\[
HM_*(f, g; \mathbb{Z}_2) := H\left(CM_*(f; \mathbb{Z}_2), \delta_{f, g}\right).
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HM_*(f, g; \mathbb{Z}_2) := H(CM_*(f; \mathbb{Z}_2), \delta_f, g).
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**Theorem**

A generic pair \((f, g)\) is Morse-Smale. The Morse homology is (canonically) independent of the choice of \(f, g\). For compact \(M\), \(HM_*(f, g; \mathbb{Z}_2)\) is isomorphic to the homology of \(M\).
II. Morse theory for Lagrange multipliers
$X$: compact manifold
\[ f, \mu \in C^\infty(X) \]
\[ 0: \text{regular value of } \mu \]
\[ \overline{X} := \mu^{-1}(0). \]
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\[ \text{Lagrange multiplier:} \]
\[ F: X \times \mathbb{R} \to \mathbb{R} \]
\[ (x, \eta) \mapsto f(x) + \eta \mu(x) \]
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Lagrange multiplier:

$F : X \times \mathbb{R} \to \mathbb{R}$
$(x, \eta) \mapsto f(x) + \eta \mu(x)$

$\text{Crit} F = \{(x, \eta) \mid df(x) + \eta d\mu(x) = 0, \ \mu(x) = 0\}$

$\simeq \text{Crit} (f|_{\overline{X}}) =: \{p_1, p_2, \ldots, p_k\}$. 
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\[
\simeq \text{Crit} (f|_{\overline{X}}) =: \{p_1, p_2, \ldots, p_k\}.  
\]

In general, we can consider \( \mu = (\mu_1, \ldots, \mu_k): X \to \mathbb{R}^k \) and
\[
F(x, \eta_1, \ldots, \eta_k) = f(x) + \sum \eta_i \mu_i(x).  
\]
If $p \in \text{Crit} f|_X$, and locally

$$\mu = x_n, \quad f|_X = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2.$$
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\]

Then

\[
\nabla^2 (f + \eta x_n) = \nabla^2 F = \begin{bmatrix}
A_{n-1} & * & 0 \\
* & * & 1 \\
0 & 1 & 0
\end{bmatrix} \sim \begin{bmatrix}
A_{n-1} & 0 & 0 \\
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\]
If \( p \in \text{Crit} f|_{\overline{X}}, \) and locally

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\]

So the two extra direction, \( x_n \) and \( \eta \) give additional one positive and one negative eigenvalues of the Hessian, and

\[
\text{ind} \ (p, F) = \text{ind} \ (p, \text{Crit} f|_{\overline{X}}) + 1.
\]
Choose a metric $g_X$ on $X$, and the standard metric $e$ on $\mathbb{R}$, we define a family of metrics on $X \times \mathbb{R}$ by

$$g_\lambda = g_X \oplus \lambda^{-2} e, \quad \lambda > 0.$$
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The negative gradient flow equation is

$$\begin{align*}
    x'(t) + \nabla f(x(t)) + \eta(t) \mu(x(t)) &= 0, \\
    \eta'(t) + \lambda^2 \mu(x(t)) &= 0,
\end{align*}$$

Denote by $M_{\lambda}(p^-, p^+)$ the moduli space of solutions connecting $p^-$ and $p^+$ modulo time translation. Elements are called $\lambda$-trajectories.

We try to define the Morse homology for $(F, g_\lambda)$, denoted by $HM_{\lambda}^*(F; \mathbb{Z}_2)$, by counting isolated $\lambda$-trajectories.
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Denote by $\mathcal{M}_\lambda (p_-, p_+)$ the moduli space of solutions connecting $p_-$ and $p_+$ modulo time translation. Elements are called $\lambda$-trajectories.
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Second, for generic $(f, \mu, g_\lambda)$, there is an isolated set $\Lambda^{\text{sing}} \subset \mathbb{R}^+$ such that for $\lambda \in \Lambda^{\text{reg}} := \mathbb{R}^+ \setminus \Lambda^{\text{sing}}$, $(F, g_\lambda)$ is Morse-Smale. So $HM^\lambda_*(F; \mathbb{Z}_2)$ is defined.
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Moreover, for $\lambda_1, \lambda_2 \in \Lambda^{reg}$, there is a canonical isomorphism (induced from a chain map)

$$\Phi : HM_{\lambda_1}^*(F; \mathbb{Z}_2) \simeq HM_{\lambda_2}^*(F; \mathbb{Z}_2).$$
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The dynamics are varying with $\lambda$, so we are counting different objects for different $\lambda$. It yields different chain complexes $CM^\lambda_\ast(F)$ which have isomorphic homology.
A natural question to ask is

What are the “limits” of $CM^\lambda_*(F)$ as $\lambda \to 0$ and $\lambda \to \infty$?
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What are the “limits” of $CM^\lambda_*(F)$ as $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$?

More approachable questions:

1. Given a sequence $\lambda_i$ with $\lim \lambda_i = 0$ or $\infty$, and a sequence $\gamma_i \in M_\lambda(p_-, p_+)$. Is there a good limit (up to choosing a subsequence) of $\gamma_i$?
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2. If we can describe limiting trajectories, do they always arise as limits of $\lambda$-trajectories?
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2. If we can describe limiting trajectories, do they always arise as limits of $\lambda$-trajectories?

3. Does the counting of the limiting trajectories defines a chain complex which has the same homology?
In general, if \( x'(t) + \nabla h(x(t)) = 0 \), then the energy of \( x \) is:

\[
\int_{\mathbb{R}} |x'(t)|^2 = h(x(-\infty)) - h(x(+\infty)).
\]

In the case of \((F, g_\lambda)\), for any solution \((x, \eta) : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}^n\),

\[
|\nabla f(x) + \eta \nabla \mu(x)|^2_{L^2} + \lambda^2 |\mu(x)|^2_{L^2} = F(p^-) - F(p^+).
\]

\( \lambda \rightarrow \infty \)

\[ \Rightarrow \text{trajectories converge into } \mathbb{R} = \mu^{-1}(0). \]

Indeed, the \( \mathbb{R} \)-component of the trajectory converges to (broken) trajectories of \((f, g) := (f|_{\mathbb{R}}, g|_{\mathbb{R}})\). So the limiting dynamical system is well-understood.
\[ \lambda \to \infty \]

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$\lambda \to \infty$ trajectories converge into $\overline{X} = \mu^{-1}(0)$.

Indeed, the $X$-component of the trajectory converges to (broken) trajectories of $(\overline{f}, \overline{g}) := (f|_{\overline{X}}, g|_{\overline{X}})$. So the limiting dynamical system is well-understood.
**Theorem**

If $\lambda_i \to \infty$, and $\gamma_i : \mathbb{R} \to X \times \mathbb{R}$ is a sequence of $\lambda_i$-trajectories connecting $p_-$ and $p_+$, then there is a subsequence of $\gamma_i$ which converges to a broken trajectory of $-\nabla F$ connecting $p_-$ and $p_+$. 

---

**Corollary (Folklore)**

For $\lambda \in \Lambda_{\text{reg}}$, there is an isomorphism $\text{HM}_{\lambda}^* (F; \mathbb{Z}_2)[1] \cong \text{HM}_* (f; \mathbb{Z}_2) \cong H_* (X; \mathbb{Z}_2)$.
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Conversely, if $\text{ind} p_- - \text{ind} p_+ = 1$, then there exists $\lambda_0 >> 0$ such that for any trajectory $\overline{y}$ of $-\nabla f$ connecting $p_-$ and $p_+$, for any $\lambda > \lambda_0$, there exists a unique $\lambda$-trajectory (up to time translation) connecting $p_-$ and $p_+$ which is “close” to $\overline{y}$. 

**Corollary (Folklore)**

For $\lambda \in \Lambda_{\text{reg}}$, there is an isomorphism $\text{HM}_\lambda^\ast(F; \mathbb{Z}_2) \cong \text{HM}^\ast(f; \mathbb{Z}_2) \cong \text{H}^\ast(X; \mathbb{Z}_2)$. 

Scheckter-G.X. Lagrange Multiplier
Theorem

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Conversely, if $\text{ind} p_- - \text{ind} p_+ = 1$, then there exists $\lambda_0 >> 0$ such that for any trajectory $\bar{y}$ of $-\nabla f$ connecting $p_-$ and $p_+$, for any $\lambda > \lambda_0$, there exists a unique $\lambda$-trajectory (up to time translation) connecting $p_-$ and $p_+$ which is “close” to $\bar{y}$.

Corollary (Folklore)

For $\lambda \in \Lambda^{\text{reg}}$, there is an isomorphism

$$HM^\lambda_*(F; \mathbb{Z}_2)[1] \simeq HM_*(\bar{f}; \mathbb{Z}_2) \simeq H_*(\bar{X}; \mathbb{Z}_2)$$
Now we consider the other limit of $CM^\lambda_*(F)$ as $\lambda \to 0$. The equation is

$$\begin{cases} 
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\end{cases}$$

This is a special case of the fast-slow ODE:

$$\begin{cases}
x'(t) &= A(x(t), y(t)),
y'(t) &= \epsilon B(x(t), y(t)).
\end{cases}$$

So in normal scale, the variable $\eta$ is “freezed”. Then the variable $x$ travels along the flow of $-\nabla(f + \eta \mu)$. 

\[\text{Schecter-G.X.}\]

\[\text{Lagrange Multiplier}\]
Set $\lambda = 0$. It gives the equation for the fast flow is
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On the other hand, in large scale (long time), $\eta$ can still change at places where $x$ also changes slowly, i.e., the slow manifold

$$C_X := \{(x, \eta) \mid \nabla f(x) + \eta \nabla \mu(x) = 0\}.$$
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So for generic data, $C_X$ is a smooth curve in $X \times \mathbb{R}$. Moreover

$$(x, \eta) \in C_X \implies x \in \text{Crit}(f + \eta \mu), \quad \text{Crit}F = C_X \cap (\overline{X} \times \mathbb{R}).$$
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So for generic data, $C_X$ is a smooth curve in $X \times \mathbb{R}$. Moreover

$$(x, \eta) \in C_X \implies x \in \text{Crit}(f + \eta \mu), \ \text{Crit}F = C_X \cap (X \times \mathbb{R}).$$  

Moreover, in generic case $F|_{C_X}$ is a Morse function, whose negative gradient flow is called the slow flow. Crit$F$ is part of the equillibria of the slow flow.
A fast-slow trajectory connecting $p_-$ and $p_+$ is a finite concatenations

$$\gamma = (\ldots, \gamma^F_k, \gamma^S_k, \gamma^F_{k+1}, \gamma^S_{k+1}, \ldots)$$

of trajectories of the fast or slow flow. $\mathcal{M}_{FS}(p_-, p_+)$ is their moduli space.
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**Theorem (Schecter-G.X.)**

Given $p_\pm \in \text{Crit}F$, a sequence $\lambda_i \to 0$ and a sequence of $\lambda_i$-trajectories $\gamma_i \in \mathcal{M}_{\lambda_i}(p_-, p_+)$, there is a subsequence which converges to a fast-slow trajectory connecting $p_-$ and $p_+$. 
A fast-slow trajectory connecting $p_-$ and $p_+$ is a finite concatenations

$$\gamma = (\ldots, \gamma^F_k, \gamma^S_k, \gamma^F_{k+1}, \gamma^S_{k+1}, \ldots)$$

of trajectories of the fast or slow flow. $\mathcal{M}_{FS}(p_-, p_+)$ is their moduli space.

**Theorem (Schecter-G.X.)**

Given $p_\pm \in \text{Crit} F$, a sequence $\lambda_i \to 0$ and a sequence of $\lambda_i$-trajectories $\gamma_i \in \mathcal{M}_{\lambda_i}(p_-, p_+)$, there is a subsequence which converges to a fast-slow trajectory connecting $p_-$ and $p_+$.

To show that all fast-slow trajectories arise as limits of $\lambda$-trajectories, we have to do the “gluing” part. In the study of fast-slow ODE, this story is called geometric singular perturbation theory.
The “gluing part” of the main theorem is

**Theorem (Schecter–G.X., 2012)**

Suppose \((f, \mu, g_X)\) is generic. Then for any if \(\text{ind}p_+ - \text{ind}p_- = 1\), \(\mathcal{M}_{FS}(p_-, p_+)\) consists of isolated objects. Moreover, there exists a \(\lambda_0 > 0\) such that for every \(\lambda \in (0, \lambda_0)\) and for every \(\gamma \in \mathcal{M}_{FS}(p_-, p_+)\), there exists a unique \(\lambda\)-trajectory \(\gamma\lambda \in \mathcal{M}_\lambda(p_-, p_+)\) which is close to \(\gamma\).
The “gluing part” of the main theorem is

**Theorem (Schecter–G.X., 2012)**

Suppose $(f, \mu, g_X)$ is generic. Then for any if $\text{ind}p_- - \text{ind}p_+ = 1$, $\mathcal{M}_{FS}(p_-, p_+)$ consists of isolated objects. Moreover, there exists a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ and for every $\gamma \in \mathcal{M}_{FS}(p_-, p_+)$, there exists a unique $\lambda$-trajectory $\gamma_\lambda \in \mathcal{M}_\lambda(p_-, p_+)$ which is close to $\gamma$.

**Corollary**

The counting of isolated fast-slow trajectories defines a chain complex $CM^{FS}(f, \mu; \mathbb{Z}_2)$, whose homology is isomorphic to $HM^\lambda_*(F; \mathbb{Z}_2)$. So there is an isomorphism

$$HM^{FS}_*(f, \mu; \mathbb{Z}_2) \cong HM^\lambda_{\text{big}}_*(F; \mathbb{Z}_2) \cong HM^\lambda_{\text{small}}_*(F; \mathbb{Z}_2) \cong H_*(\overline{X}; \mathbb{Z}_2)[-1].$$
To see what we are actually counting in the fast-slow complex, we make a digression into another geometric application of geometric singular perturbation theory.
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We see \( \text{Crit} f_\epsilon = \text{Crit} h \). Then we consider the flow of \(-\nabla f_\epsilon\). In a tubular neighborhood of \( S \), the equation is

\[
\begin{cases}
    v'(t) = -\nabla f(v(t), s(t)), \\
    s'(t) = -\epsilon \nabla h(v(t), s(t)).
\end{cases}
\]
The fast flow is just the flow of $-\nabla f$. The slow manifold is $S$. The slow flow is the flow of $-\nabla h$ inside $S$. 
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More importantly, the fast flow is normally hyperbolic. This means that the linearization of the term $-\nabla f(v, s)$ in the normal direction of $S$ is nondegenerate.
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The fast-slow trajectories in this case are the “\textit{cascades}”, i.e., a concatenation of flow lines of $-\nabla f$ between two points in two different components of $S$, and flow lines of $-\nabla h$ inside $S$. 
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**Theorem (Banyaga-Hurtubise, 2013)**

If $\text{ind}(p_-) - \text{ind}(p_+) = 1$, then for any $\epsilon$ small enough, there is a one-to-one correspondence between “cascade trajectories” and trajectories of $-\nabla f_\epsilon$ connecting $p_-$ and $p_+$. 
Morse homology
Lagrange multipliers
Adiabatic limits

$\lambda \to \infty$
$\lambda \to 0$
In our fast-slow complex, there are only finitely many “fast tunnels” which can be a component of a fast-slow trajectory connecting $p_-$ and $p_+$ if they have adjacent indices.

They are the **handle-slides, cusp trajectories, and some index 1 trajectories**.
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There may be handle-slides, which are fast trajectories connecting $(x_-, \eta), (x_+, \eta) \in C_X$ with $\text{ind}(x_-, f+\eta \mu) = \text{ind}(x_+, f+\eta \mu)$. 
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There also may be birth-deaths, i.e., for some $(x, \eta) \in C_X$ such that $x$ is a degenerate critical point of $f + \eta \mu$. A cusp trajectory is a type of fast trajectory connecting a birth-death with a non-birth-death point on $C_X$. 
If $p_-$ is a local minimum of the slow flow, then the fast-slow trajectory must start with an index 1 fast trajectory from $p_-$; if $p_+$ is a local maximum of the slow flow, then the fast-slow trajectory must end with an index 1 fast trajectory to $p_+$. 
If $p_-$ is a local minimum of the slow flow, then the fast-slow trajectory must start with an index 1 fast trajectory from $p_-$. If $p_+$ is a local maximum of the slow flow, then the fast-slow trajectory must end with an index 1 fast trajectory to $p_+$. We have then a map of transportation, consists of those “fast tunnels” and the “slow tracks” gives only finitely many possible routes to transport from $p_-$ to $p_+$. We are actually counting those routes.
The proof relies on Schecter’s general exchange lemma, which basically gives the local normal form of the system near $C_X$, even around the birth-deaths.
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A model case: $f = -\frac{1}{3}x^3$, $\mu = x + 1$, $C_X = \{\eta = x^2\}$. The origin is a birth-death.

\[
z_1 = x, \quad z_2 = \eta, \quad \epsilon = \lambda^2.
\]
Thank you!