Hamiltonian Floer homology in gauged $\sigma$-model

Guangbo Xu

Department of Mathematics, UC Irvine

Caltech Geometry&Toplogy Seminar, January 17, 2014
Outline

1. Arnold Conjecture and Hamiltonian Floer homology
2. Transversality
3. Vortex Floer homology for Hamiltonian $G$-manifolds
1. Arnold Conjecture and Hamiltonian Floer homology
Let $X$ be a compact $2n$-dimensional manifold. A symplectic form is a closed 2-form $\omega \in \Omega^2(X)$ such that $\omega^n$ is a volume form.
Let $X$ be a compact $2n$-dimensional manifold. A symplectic form is a closed 2-form $\omega \in \Omega^2(X)$ such that $\omega^n$ is a volume form.

For any smooth function $H$ on a symplectic manifold $(X, \omega)$, we have the associated Hamiltonian vector field $X_H$ on $X$ defined by

$$\omega(X_H, \cdot) = dH.$$
Let $X$ be a compact $2n$-dimensional manifold. A symplectic form is a closed 2-form $\omega \in \Omega^2(X)$ such that $\omega^n$ is a volume form.

For any smooth function $H$ on a symplectic manifold $(X, \omega)$, we have the associated Hamiltonian vector field $X_H$ on $X$ defined by

$$\omega(X_H, \cdot) = dH.$$ 

Suppose we have a time-dependent family of Hamiltonian functions $H_t$, $t \in S^1 \cong \mathbb{R}/\mathbb{Z}$, then we have a 1-periodic family of Hamiltonian vector fields $X_t = X_{H_t}$. The associated Hamiltonian flow is a 1-parameter family of diffeomorphisms $\phi_t$ defined by

$$\frac{d\phi_t(x)}{dt} = X_t(\phi_t(x)), \quad t \in \mathbb{R}.$$
For the time one map $\phi_1 : X \to X$, its fixed point set corresponds to the 1-periodic orbits of the flow, i.e. solutions to

$$\dot{x}(t) = X_t(x(t)), \quad x : S^1 \to X \implies \phi_1(x(0)) = x(0).$$
For the time one map \( \phi_1 : X \to X \), its fixed point set corresponds to the 1-periodic orbits of the flow, i.e. solutions to

\[
\dot{x}(t) = X_t(x(t)), \quad x : S^1 \to X \implies \phi_1(x(0)) = x(0).
\]

The fixed point \( p = x(0) \) is nondegenerate if

\[
\det (d\phi_1(p) - \text{Id}_{T_pX}) \neq 0.
\]
For the time one map $\phi_1 : X \to X$, its fixed point set corresponds to the 1-periodic orbits of the flow, i.e. solutions to

$$\dot{x}(t) = X_t(x(t)), \quad x : S^1 \to X \implies \phi_1(x(0)) = x(0).$$

The fixed point $p = x(0)$ is nondegenerate if

$$\det (d\phi_1(p) - \text{Id}_{T_pX}) \neq 0.$$ 

**Conjecture (Arnold conjecture)**

*For generic nondegenerate Hamiltonian $H : X \times S^1 \to \mathbb{R}$, we have that*

$$\#\text{Fix} \phi_1 \geq \text{rank} H_*(X).$$
The most successful approach is to define Floer homology $HF(X, H)$. 

Guangbo Xu

Gauged Hamiltonian Floer Homology
The most successful approach is to define Floer homology $HF(X, H)$.

It is defined as the homology of a chain complex $(CF_\ast(X, H), \delta)$, which is generated by contractible 1-periodic orbits of $H$. 
The most successful approach is to define Floer homology $HF(X, H)$.

It is defined as the homology of a chain complex $(CF_*(X, H), \delta)$, which is generated by contractible 1-periodic orbits of $H$.

One can show that $HF(X, H)$ is isomorphic to $H(X)$ (over an appropriate coefficient ring), so Arnold conjecture is proven.
For simplicity we assume that \((X, \omega)\) is symplectically aspherical, which means

\[ [\omega]|_{\pi_2(X)} = 0. \]
For simplicity we assume that \((X, \omega)\) is symplectically aspherical, which means

\[ [\omega]|_{\pi_2(X)} = 0. \]

We consider the infinite dimensional manifold

\[ \mathcal{P} := LX = \{ x : S^1 \to X \mid [x] = 0 \in \pi_1(X) \} \]

on which we define the symplectic action functional \(A_H : \mathcal{P} \to \mathbb{R} : \)

\[ A_H(x) = -\int_{\mathbb{D}^2} u^* \omega - \int_{S^1} H_t(x(t)) dt, \quad u : \mathbb{D}^2 \to X \text{ extends } x. \]
For simplicity we assume that \((X, \omega)\) is *symplectically aspherical*, which means
\[
[\omega]|_{\pi_2(X)} = 0.
\]

We consider the infinite dimensional manifold
\[
P := LX = \{ x : S^1 \to X \mid [x] = 0 \in \pi_1(X) \}
\]
on which we define the *symplectic action functional* \(A_H : P \to \mathbb{R}:
\]
\[
A_H(x) = - \int_{D^2} u^* \omega - \int_{S^1} H_t(x(t)) dt, \quad u : D^2 \to X \text{ extends } x.
\]

Formally, we define \(HF(X, H)\) as \(HM(P, A_H)\).
It is easy to see that

$$\text{Crit} \mathcal{A}_H = \{ x \in \mathcal{P} : x'(t) = X_t(x(t)) \}.$$
It is easy to see that

\[ \text{Crit} \mathcal{A}_H = \{ x \in \mathcal{P} : x'(t) = X_t(x(t)) \} . \]

The nondegenerate condition somehow means \( \mathcal{A}_H \) is a “Morse function”. We have a Conley-Zehnder index

\[ \text{ind} : \text{Crit} \mathcal{A}_H \rightarrow \mathbb{Z} \]

which plays the role as the Morse index.
It is easy to see that

$$\text{Crit} \mathcal{A}_H = \{ x \in \mathcal{P} : x'(t) = X_t(x(t)) \}.$$ 

The nondegenerate condition somehow means $\mathcal{A}_H$ is a “Morse function”. We have a Conley-Zehnder index

$$\text{ind} : \text{Crit} \mathcal{A}_H \to \mathbb{Z}$$

which plays the role as the Morse index.

We can take a compatible almost complex structure $J$ on $X$, so that $g(\cdot, \cdot) := \omega(\cdot, J \cdot)$ is a metric. It induces an $L^2$-metric on $\mathcal{P}$. 
The negative gradient flow of $A_H$ is the PDE

$$\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_t(u(s, t)) \right) = 0, \quad u : \mathbb{R} \times S^1 \to X.$$
The negative gradient flow of $A_H$ is the PDE

$$\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_t(u(s, t)) \right) = 0, \quad u : \mathbb{R} \times S^1 \to X.$$  

So Floer adopted Gromov’s technique on pseudoholomorphic curves to study the moduli space of solutions. For $x_{\pm} \in \text{Crit} A_H$,

$$\widetilde{M}(x_-, x_+) := \left\{ \text{solutions} \mid \lim_{s \to \pm \infty} u(s, t) = x_{\pm}(t) \right\}.$$
The negative gradient flow of $A_H$ is the PDE

$$\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_t(u(s, t)) \right) = 0, \quad u : \mathbb{R} \times S^1 \to X.$$ 

So Floer adopted Gromov’s technique on pseudoholomorphic curves to study the moduli space of solutions. For $x_\pm \in \text{Crit} A_H$,

$$\tilde{M}(x_-, x_+) := \left\{ \text{solutions} \mid \lim_{s \to \pm \infty} u(s, t) = x_\pm(t) \right\}.$$

**Proposition**

*For generic $J$, $\tilde{M}(x_-, x_+)$ is a smooth, oriented manifold, with dimension equal to $\text{ind}(x_-) - \text{ind}(x_+)$.***
Floer’s equation has a translation symmetry. So we define

\[ \mathcal{M}(x_-, x_+) := \frac{\widetilde{\mathcal{M}}(x_-, x_+)}{\mathbb{R}}. \]
Floer’s equation has a translation symmetry. So we define

\[ \mathcal{M}(x_-, x_+) := \widetilde{\mathcal{M}}(x_-, x_+)/\mathbb{R}. \]

In particular, for generic \( J \),

\[ \text{ind}(x_-) - \text{ind}(x_+) = 1 \implies \# \mathcal{M}(x_-, x_+) < \infty. \]
Floer’s equation has a translation symmetry. So we define

\[ M(x_-, x_+) := \tilde{M}(x_-, x_+)/\mathbb{R}. \]

In particular, for generic \( J \),

\[ \text{ind}(x_-) - \text{ind}(x_+) = 1 \implies \#M(x_-, x_+) < \infty. \]

Due to Floer-Hofer, we can define a system of “consistent orientations” on all moduli spaces. So each isolated trajectory has a sign.
Floer’s equation has a translation symmetry. So we define

\[ \mathcal{M}(x_-, x_+) := \widetilde{\mathcal{M}}(x_-, x_+)/\mathbb{R}. \]

In particular, for generic \( J \),

\[ \text{ind}(x_-) - \text{ind}(x_+) = 1 \implies \# \mathcal{M}(x_-, x_+) < \infty. \]

Due to Floer-Hofer, we can define a system of “consistent orientations” on all moduli spaces. So each isolated trajectory has a sign.

We define the Floer chain complex as

\[
\mathcal{C}F_*(X, H; \mathbb{Z}) := \bigoplus_{x \in \text{Crit} A_H} \mathbb{Z}\langle x \rangle, \quad \partial_{H,J} : \mathcal{C}F_* \to \mathcal{C}F_*-1,
\]

where the boundary operator is defined by counting isolated trajectories.
Floer homology:

**Theorem (Floer, 1980s)**

For generic $H$ and $J$, $HF(X, H; \mathbb{Z})$ is defined as the homology of the chain complex $(CF_*(X, H), \partial_{H,J})$. It is (canonically) independent of the choice of $(H, J)$. Moreover, there is a canonical isomorphism:

$$HF(X, H; \mathbb{Z}) \simeq H(X; \mathbb{Z}).$$
**Theorem (Floer, 1980s)**

For generic $H$ and $J$, $HF(X, H; \mathbb{Z})$ is defined as the homology of the chain complex $(CF_*(X, H), \partial_{H,J})$. It is (canonically) independent of the choice of $(H, J)$. Moreover, there is a canonical isomorphism:

$$HF(X, H; \mathbb{Z}) \simeq H(X; \mathbb{Z}).$$

This implies the Arnold conjecture for symplectically aspherical $(X, \omega)$. 
II. Transversality issue for general compact symplectic manifolds
The key point in proving that $\partial_{H,J}^2 = 0$ and the independence of $HF$ in $(H, J)$ is the compactness of the moduli space. If $(X, \omega)$ is not aspherical, then there may exist $J$-holomorphic spheres and the bubbling also causes noncompactness of the moduli spaces.
The key point in proving that $\partial^2_{H,J} = 0$ and the independence of $HF$ in $(H, J)$ is the compactness of the moduli space. If $(X, \omega)$ is not aspherical, then there may exist $J$-holomorphic spheres and the bubbling also causes noncompactness of the moduli spaces.

People expected that the moduli space of trajectories with nontrivial bubble trees attached only occupy a subset of the compactified moduli space with codimension at least 2, so it doesn’t affect the counting.
The key point in proving that $\partial^2_{H,J} = 0$ and the independence of $HF$ in $(H,J)$ is the compactness of the moduli space. If $(X,\omega)$ is not aspherical, then there may exist $J$-holomorphic spheres and the bubbling also causes noncompactness of the moduli spaces.

People expected that the moduli space of trajectories with nontrivial bubble trees attached only occupy a subset of the compactified moduli space with codimension at least 2, so it doesn’t affect the counting.

This idea works up to “semi-positive” symplectic manifolds, which means that for generic almost complex structure $J$, there is no $J$-holomorphic spheres $u : S^2 \to X$ such that

$$c_1(TX) \cdot u_*[S^2] < 0.$$
Theorem (Hofer-Salamon, Ono)

If \((X, \omega)\) is semi-positive, then the Floer homology group \(HF(X)\) is defined and is isomorphic to the homology of \(X\) (with some extended coefficient ring).
**Theorem (Hofer-Salamon, Ono)**

If \((X, \omega)\) is semi-positive, then the Floer homology group \(HF(X)\) is defined and is isomorphic to the homology of \(X\) (with some extended coefficient ring).

The extended coefficient ring, called the Novikov ring, is typically a ring of formal sums “Laurent series”

\[
\Lambda_R := \left\{ \sum a_\nu T^\nu \mid a_\nu \in R, \lim \nu = \infty \right\}, \quad R = \mathbb{Z}, \mathbb{Z}_2, \text{ or } \mathbb{Q}.
\]
Theorem (Hofer-Salamon, Ono)

If \((X, \omega)\) is semi-positive, then the Floer homology group \(HF(X)\) is defined and is isomorphic to the homology of \(X\) (with some extended coefficient ring).

The extended coefficient ring, called the Novikov ring, is typically a ring of formal sums “Laurent series”

\[\Lambda_R := \left\{ \sum a_\nu T^\nu \mid a_\nu \in R, \lim_\nu = \infty \right\}, \ R = \mathbb{Z}, \mathbb{Z}_2, \text{ or } \mathbb{Q}.\]

For general \((X, \omega)\) the existence of negative spheres cannot be excluded. In such cases, the moduli space is more degenerate and doesn’t have the right dimension.
The resolution of this issue is called the virtual technique. In the Floer theory setting, there are the following versions:
The resolution of this issue is called the virtual technique. In the Floer theory setting, there are the following versions:

\[
\begin{align*}
\text{virtual technique} : \\
&\text{Liu – Tian : virtual cycle} \\
&\text{Fukaya – Ono : Kuranishi structure} \\
&\text{Hofer – Wysocki – Zehnder : polyfolds}
\end{align*}
\]
The resolution of this issue is called the virtual technique. In the Floer theory setting, there are the following versions:

\[
\text{virtual technique} : \begin{cases} 
\text{Liu – Tian : virtual cycle} \\
\text{Fukaya – Ono : Kuranishi structure} \\
\text{Hofer – Wysocki – Zehnder : polyfolds}
\end{cases}
\]

**Theorem (Fukaya-Ono, Liu-Tian, 1990s)**

For a general compact symplectic manifold, $HF(X)$ is defined as a module over $\Lambda_Q$. Moreover, it is canonically isomorphic to $H_*(X; \Lambda_Q)$. 
III. Vortex Floer homology for a Hamiltonian $G$-manifold
Let $G$ be a compact Lie group. A Hamiltonian $G$-manifold is a symplectic manifold $(X,\omega)$ with a smooth $G$-action, such that there exists a moment map $\mu : X \rightarrow g^*$, i.e.,

$$\forall \xi \in g, \ \omega(X_\xi, \cdot) = d(\mu \cdot \xi).$$
Let $G$ be a compact Lie group. A Hamiltonian $G$-manifold is a symplectic manifold $(X, \omega)$ with a smooth $G$-action, such that there exists a moment map $\mu : X \to \mathfrak{g}^*$, i.e.,

$$\forall \xi \in \mathfrak{g}, \; \omega(X_\xi, \cdot) = d(\mu \cdot \xi).$$

If $0 \in \mathfrak{g}^*$ is a regular value of $\mu$, $\overline{X} := \mu^{-1}(0)/G$ is a symplectic orbifold, called the symplectic quotient (downstairs).
Let $G$ be a compact Lie group. A Hamiltonian $G$-manifold is a symplectic manifold $(X, \omega)$ with a smooth $G$-action, such that there exists a moment map $\mu : X \to g^*$, i.e.,

$$\forall \xi \in g, \ \omega(X_\xi, \cdot) = d(\mu \cdot \xi).$$

If $0 \in g^*$ is a regular value of $\mu$, $\overline{X} := \mu^{-1}(0)/G$ is a symplectic orbifold, called the symplectic quotient (downstairs).

Example: $X = \mathbb{C}^{n+1}$, $G = S^1$ acts by $e^{i\theta} \cdot \vec{z} = e^{i\theta} \vec{z}$. $\mu = \frac{1}{2} (|\vec{z}|^2 - 1)$ is a moment map, and $\overline{X} = \mathbb{C} \mathbb{P}^n$. 
Let $G$ be a compact Lie group. A Hamiltonian $G$-manifold is a symplectic manifold $(X,\omega)$ with a smooth $G$-action, such that there exists a moment map $\mu : X \to g^*$, i.e.,

$$\forall \xi \in g, \omega(X_\xi, \cdot) = d(\mu \cdot \xi).$$

If $0 \in g^*$ is a regular value of $\mu$, $\overline{X} := \mu^{-1}(0)/G$ is a symplectic orbifold, called the symplectic quotient (downstairs).

Example: $X = \mathbb{C}^{n+1}$, $G = S^1$ acts by $e^{i\theta} \cdot \vec{z} = e^{i\theta} \vec{z}$. $\mu = \frac{1}{2} (|\vec{z}|^2 - 1)$ is a moment map, and $\overline{X} = \mathbb{C}\mathbb{P}^n$.

Example: toric manifolds are symplectic quotients of $\mathbb{C}^N$ by $\mathbb{T}^k$, which may have negative holomorphic spheres.
The vortex Floer homology for a Hamiltonian $G$-manifold $(X, \mu)$ (proposed by Cieliebak-Gaio-Salamon in 2000)
The vortex Floer homology for a Hamiltonian $G$-manifold $(X, \mu)$ (proposed by Cieliebak-Gaio-Salamon in 2000)

1. We will define $VHF(X, \mu)$ for symplectic aspherical $X$ as an alternative to $HF(\overline{X})$ (for smooth quotient $\overline{X}$).
The vortex Floer homology for a Hamiltonian $G$-manifold $(X, \mu)$ (proposed by Cieliebak-Gaio-Salamon in 2000)

1. We will define $VHF(X, \mu)$ for symplectic aspherical $X$ as an alternative to $HF(X)$ (for smooth quotient $\overline{X}$).
2. When defining $VHF(X, \mu)$, we can bypass the virtual techniques, which is indispensable when defining $HF(\overline{X})$. 
The vortex Floer homology for a Hamiltonian $G$-manifold $(X, \mu)$ (proposed by Cieliebak-Gaio-Salamon in 2000)

1. We will define $VHF(X, \mu)$ for symplectic aspherical $X$ as an alternative to $HF(\overline{X})$ (for smooth quotient $\overline{X}$).

2. When defining $VHF(X, \mu)$, we can bypass the virtual techniques, which is indispensable when defining $HF(\overline{X})$.

3. $VHF(X, \mu)$ can be defined over $\mathbb{Z}$ because the moduli spaces have better structures (manifolds).
The vortex Floer homology for a Hamiltonian $G$-manifold $(X, \mu)$ (proposed by Cieliebak-Gaio-Salamon in 2000)

1. We will define $VHF(X, \mu)$ for symplectic aspherical $X$ as an alternative to $HF(X)$ (for smooth quotient $\overline{X}$).
2. When defining $VHF(X, \mu)$, we can bypass the virtual techniques, which is indispensable when defining $HF(\overline{X})$.
3. $VHF(X, \mu)$ can be defined over $\mathbb{Z}$ because the moduli spaces have better structures (manifolds).
4. It is the idea of “gauged linear $\sigma$-model vs. nonlinear $\sigma$-model” in physics.
The vortex Floer homology for a Hamiltonian $G$-manifold $(X, \mu)$ (proposed by Cieliebak-Gaio-Salamon in 2000)

1. We will define $VHF(X, \mu)$ for symplectic aspherical $X$ as an alternative to $HF(X)$ (for smooth quotient $\overline{X}$).
2. When defining $VHF(X, \mu)$, we can bypass the virtual techniques, which is indispensable when defining $HF(\overline{X})$.
3. $VHF(X, \mu)$ can be defined over $\mathbb{Z}$ because the moduli spaces have better structures (manifolds).
4. It is the idea of “gauged linear $\sigma$-model vs. nonlinear $\sigma$-model” in physics.
5. It is also a Morse theory for a Lagrange multiplier type function.
Formally, the loop group $LG$ has Lie algebra $Lg$. The loop space $LX$ formally has a symplectic structure, and the $LG$-action on $LX$ has a moment map

$$\tilde{\mu} : LX \to (Lg)^*, \ x \mapsto \mu \circ x \in Lg^*.$$ 

The loop space $\overline{LX}$ is formally the symplectic quotient $\tilde{\mu}^{-1}(0)/LG$. 
Formally, the loop group $LG$ has Lie algebra $Lg$. The loop space $LX$ formally has a symplectic structure, and the $LG$-action on $LX$ has a moment map

$$\tilde{\mu} : LX \to (Lg)^*, \ x \mapsto \mu \circ x \in Lg^*.$$  

The loop space $LX$ is formally the symplectic quotient $\tilde{\mu}^{-1}(0)/LG$.  

For a $G$-invariant Hamiltonian $H_t : X \to \mathbb{R}$, we define the action

$$A_H : x \mapsto -\iint u^*\omega - \int_{S^1} H_t(x(t))\ dt.$$
Formally, the loop group $LG$ has Lie algebra $Lg$. The loop space $LX$ formally has a symplectic structure, and the $LG$-action on $LX$ has a moment map

$$\tilde{\mu} : LX \to (Lg)^*, \; x \mapsto \mu \circ x \in Lg^*.$$ 

The loop space $LX$ is formally the symplectic quotient $\tilde{\mu}^{-1}(0)/LG$.

For a $G$-invariant Hamiltonian $H_t : X \to \mathbb{R}$, we define the action

$$A_H : x \mapsto -\iint u^*\omega - \int_{S^1} H_t(x(t))dt.$$ 

The Lagrange multiplier for $(A_H, \tilde{\mu})$ is: for every loop $(x, \eta)$ in $X \times g$:

$$\tilde{A}_H(x, \eta) = A_H(x) + \langle \tilde{\mu}(x), \eta \rangle$$

$$= -\tilde{A}_H(x, \eta) + \int_{S^1} \langle \mu(x(t)), \eta(t) \rangle dt.$$
Formally, the vortex Floer homology is defined as

\[ \text{VHF}(X, \mu) \sim H_{\mathbb{L}G}^L \left( L(X \times \mathfrak{g}), \tilde{A}_H \right). \]
Formally, the vortex Floer homology is defined as

$$VHF(X, \mu) \sim HM^G \left( L(X \times g), \tilde{A}_H \right).$$

We can see that the critical points of $\tilde{A}_H$ are solutions $(x(t), \eta(t))$ to

$$x'(t) + X_{\eta(t)}(x(t)) = X_t(x(t)), \mu(x(t)) \equiv 0.$$
Formally, the vortex Floer homology is defined as

\[ VHF(X, \mu) \sim HM^G \left( L(X \times \mathfrak{g}), \tilde{A}_H \right). \]

We can see that the critical points of $\tilde{A}_H$ are solutions $(x(t), \eta(t))$ to

\[ x'(t) + X_{\eta(t)}(x(t)) = X_t(x(t)), \mu(x(t)) \equiv 0. \]

Via the projection $\mu^{-1}(0) \rightarrow \overline{X}$, they correspond to 1-periodic orbits of the induced Hamiltonian $\overline{H}_t$ on $\overline{X}$. 
Formally, the vortex Floer homology is defined as

\[ VHF(X, \mu) \sim HM^{LG} \left( L(X \times \mathfrak{g}), \tilde{A}_H \right). \]

We can see that the critical points of \( \tilde{A}_H \) are solutions \((x(t), \eta(t))\) to

\[ x'(t) + X_{\eta(t)}(x(t)) = X_t(x(t)), \quad \mu(x(t)) \equiv 0. \]

Via the projection \( \mu^{-1}(0) \to X \), they correspond to 1-periodic orbits of the induced Hamiltonian \( H_t \) on \( X \).

We can assume that all 1-periodic orbits of the induced Hamiltonian \( H_t \) are nondegenerate. Then critical points of \( \tilde{A}_H \) upstairs are isolated modulo \( LG \)-action.
Now we consider the equivariant negative gradient flow of $\tilde{A}_H$. Choosing a $G$-invariant almost complex structure $J$, trajectories are solutions $(u, \psi) : \mathbb{R} \times S^1 \to X \times g$ to the following equation:

$$\begin{cases} \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} + X_\psi(u) - X_t(u)\right) = 0 \\ \frac{\partial \psi}{\partial s} + \mu(u) = 0 \end{cases}.$$
Now we consider the equivariant negative gradient flow of $\tilde{A}_H$. Choosing a $G$-invariant almost complex structure $J$, trajectories are solutions $(u, \psi) : \mathbb{R} \times S^1 \to X \times g$ to the following equation

$$\begin{cases}
\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} + X_\psi(u) - X_t(u) \right) = 0 \\
\frac{\partial \psi}{\partial s} + \mu(u) = 0
\end{cases}.$$

It has an $\mathbb{R}$-invariance as well as an $LG$-invariance.
Now we consider the equivariant negative gradient flow of $\tilde{A}_H$. Choosing a $G$-invariant almost complex structure $J$, trajectories are solutions $(u, \psi) : \mathbb{R} \times S^1 \to X \times g$ to the following equation

\[
\begin{cases}
  \frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} + X_\psi(u) - X_t(u) \right) = 0 \\
  \frac{\partial \psi}{\partial s} + \mu(u) = 0
\end{cases}
\]

It has an $\mathbb{R}$-invariance as well as an $LG$-invariance.

It is a (perturbed) symplectic vortex equation, which was discovered independently by Cieliebak-Gaio-Salamon and Mundet.
The above equation is actually the vortex equation in temporal gauge, which is similar to the temporal gauge in other gauge-theoretic Floer theories, like instanton Floer theory or Seiberg-Witten Floer theory.
The above equation is actually the vortex equation in *temporal gauge*, which is similar to the temporal gauge in other gauge-theoretic Floer theories, like *instanton Floer theory* or *Seiberg-Witten Floer theory*.

Now fixing two $LG$-orbits $\tau_{\pm} \in \text{Crit}\tilde{\mathcal{A}}_H/LG$, we consider the moduli space of solutions

$$\mathcal{M}(\tau_-, \tau_+; H, J) := \left\{ \text{solutions } (u, \Psi) : \lim_{s \to \pm \infty} (u, \Psi) = \tau_{\pm} \right\}/LG.$$
The above equation is actually the vortex equation in temporal gauge, which is similar to the temporal gauge in other gauge-theoretic Floer theories, like instanton Floer theory or Seiberg-Witten Floer theory.

Now fixing two $LG$-orbits $\mathfrak{r}_\pm \in \text{Crit} \tilde{A}_H/LG$, we consider the moduli space of solutions

$$
\mathcal{M} (\mathfrak{r}_-, \mathfrak{r}_+; H, J) := \left\{ \text{solutions } (u, \Psi) : \lim_{s \to \pm \infty} (u, \Psi) = \mathfrak{r}_\pm \right\} / LG.
$$

**Proposition (G.X., 2013)**

Fix certain $H$. Then for generic $G$-invariant almost complex structure $J$, for any pair $\mathfrak{r}_\pm$, $\mathcal{M} (\mathfrak{r}_-, \mathfrak{r}_+; H, J)$ is a smooth manifold and

$$
\dim \mathcal{M} (\mathfrak{r}_-, \mathfrak{r}_+; H, J) = \text{ind}(\mathfrak{r}_-) - \text{ind}(\mathfrak{r}_+).
$$
Moreover, the bubbling is also the only issue about the compactness of vortex equation. So we assume that $X$ is \textit{aspherical} to exclude bubbling. The moduli space is compact up to breaking.
Moreover, the bubbling is also the only issue about the compactness of vortex equation. So we assume that $X$ is aspherical to exclude bubbling. The moduli space is compact up to breaking.

We can define the Floer chain complex as

$$VCF_\ast(X, \mu; H) := \bigoplus_{\tilde{r} \in \text{Crit.}\tilde{A}_H} \Lambda \langle \tilde{r} \rangle, \quad \Rightarrow \delta_H, J : VCF_\ast \to VCF_{\ast-1}$$
Moreover, the bubbling is also the only issue about the compactness of vortex equation. So we assume that $X$ is \textit{aspherical} to exclude bubbling. The moduli space is compact up to breaking.

We we can define the \textbf{Floer chain complex} as

$$VCF_\ast(X, \mu; H) := \bigoplus_{\varphi \in \text{Crit} \tilde{\mathcal{A}}_H} \Lambda \langle \varphi \rangle, \quad \Longrightarrow \delta_{H,J} : VCF_\ast \rightarrow VCF_{\ast-1}$$

\textbf{Theorem (G.X., 2013)}

\textit{For generic $J$, the vortex Floer homology $VHF_\ast(X, \mu; H)$ is defined as the homology of $(VCF_\ast(X, \mu; H), \delta_{H,J})$ and it is independent of the choice of generic $H_t$ and $J$.}
The remaining issue is to prove that $VHF(X, \mu; H)$ is indeed isomorphic to $HF(\overline{X})$ in some sense.
The remaining issue is to prove that $VHF(X, \mu; H)$ is indeed isomorphic to $HF(\overline{X})$ in some sense.

Remember that in general the latter can only be defined over rationals, and we have isomorphism

$$HF(\overline{X}; \Lambda_\mathbb{Q}) \simeq H(\overline{X}) \otimes \Lambda_\mathbb{Q}.$$
The remaining issue is to prove that $VHF(X, \mu; H)$ is indeed isomorphic to $HF(\overline{X})$ in some sense.

Remember that in general the latter can only be defined over rationals, and we have isomorphism

$$HF(\overline{X}; \Lambda_{\mathbb{Q}}) \simeq H(\overline{X}) \otimes \Lambda_{\mathbb{Q}}.$$  

Indeed we expect to prove that there is an isomorphism

$$VHF(X, \mu; \Lambda_{\mathbb{Z}}) \simeq H(\overline{X}) \otimes \Lambda_{\mathbb{Z}}.$$
The remaining issue is to prove that $VHF(X, \mu; H)$ is indeed isomorphic to $HF(\overline{X})$ in some sense.

Remember that in general the latter can only be defined over rationals, and we have isomorphism

$$HF(\overline{X}; \Lambda_{\mathbb{Q}}) \simeq H(\overline{X}) \otimes \Lambda_{\mathbb{Q}}.$$ 

Indeed we expect to prove that there is an isomorphism

$$VHF(X, \mu; \Lambda_{\mathbb{Z}}) \simeq H(\overline{X}) \otimes \Lambda_{\mathbb{Z}}.$$ 

So the vortex Floer homology indeed refines the ordinary Hamiltonian Floer homology of $\overline{X}$. 
A more interesting and important question is about the \textit{adiabatic limits} of the vortex equation.
A more interesting and important question is about the adiabatic limits of the vortex equation.

\[
\begin{aligned}
\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} + X_\psi - X_t \right) &= 0; \\
\frac{\partial \psi}{\partial s} + \lambda^2 \mu(u) &= 0.
\end{aligned}
\]
A more interesting and important question is about the adiabatic limits of the vortex equation.

\[
\begin{cases}
\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} + X_\psi - X_t \right) = 0; \\
\frac{\partial \psi}{\partial s} + \lambda^2 \mu(u) = 0.
\end{cases}
\]

Let \( \lambda \to \infty \), we should be able to recover the Floer homology of the quotient \( \overline{X} \).
A more interesting and important question is about the adiabatic limits of the vortex equation.

\[
\begin{cases}
\frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} + X_\psi - X_t \right) = 0; \\
\frac{\partial \psi}{\partial s} + \lambda^2 \mu(u) = 0.
\end{cases}
\]

Let \( \lambda \to \infty \), we should be able to recover the Floer homology of the quotient \( \overline{X} \).

In the Gromov-Witten setting, there is a correspondence between the Gromov-Witten invariants downstairs and an “equivariant” Gromov-Witten invariants of upstairs. (cf. Gaio-Salamon 2005)
On the other hand, let $\lambda \to 0$, we see flat connections appear.
On the other hand, let $\lambda \to 0$, we see flat connections appear.

In physics level, there is the “ultra-violet” limit of this theory, which is related to the intersection theory on the moduli space of flat connections/parabolic bundles.
On the other hand, let $\lambda \to 0$, we see flat connections appear.

In physics level, there is the “ultra-violet” limit of this theory, which is related to the intersection theory on the moduli space of flat connections/parabolic bundles.

The variation of $\lambda$ gives a “cobordism” of two theories. Witten used this idea to show that the Gromov-Witten invariants of complex Grassmannian $G(k, N)$ (which is the symplectic quotient of $\mathbb{C}^{kN}$ by $U(k)$) are equivalent to the “Verlinde algebra”. A mathematical proof by direct calculation was given by Agnihotri.
On the other hand, let $\lambda \to 0$, we see flat connections appear.

In physics level, there is the “ultra-violet” limit of this theory, which is related to the intersection theory on the moduli space of flat connections/parabolic bundles.

The variation of $\lambda$ gives a “cobordism” of two theories. Witten used this idea to show that the Gromov-Witten invariants of complex Grassmannian $G(k, N)$ (which is the symplectic quotient of $\mathbb{C}^{kN}$ by $U(k)$) are equivalent to the “Verlinde algebra”. A mathematical proof by direct calculation was given by Agnihotri.

However, we expect a geometric proof which can realize Witten’s idea in the mathematical level.
Thank you!