Gauged linear $\sigma$-model and gauged Witten equation
(joint work with Gang Tian)

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Motivation
Let $Q : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial of degree $y$. Associated to $Q$ there are two interesting geometric theories:

(i) The nonlinear $\sigma$-model of $\overline{X}_Q \subseteq \mathbb{P}^{N-1}$ (Gromov-Witten)

based on the analysis of $\mathcal{M}(\Sigma) = \{ u : \Sigma \to \overline{X}_Q \mid \overline{\partial} u = 0 \}$

Mathematical fundations: Ruan, Ruan-Tian, Fukaya-Oh, Li-Tian

(ii) Landau-Ginzburg model of the singularity $(\mathbb{C}^n, Q)$

based on the analysis of $\mathcal{W}(\Sigma) = \{ u : \Sigma \to \mathbb{C}^n \mid \overline{\partial} u + \nabla Q(u) = 0 \}$

Mathematical fundation: Fan-Jarvis-Ruan (2013), following Witten's idea (1993)
In superstring theory, physicists care about the NLSM of Calabi-Yau manifolds. When \( \deg Q = N \), \( X_Q \) is CY. Physicists discovered the so-called Landau-Ginzburg / Calabi-Yau correspondence between the two above theories, long before the mathematical theories rigorously constructed. The LG/CY correspondence remains mysterious to mathematicians.

A mathematically more accessible approach towards the LG/CY correspondence was introduced by Witten (1993). His theory is now referred to as the gauged linear \( \sigma \)-model.

**The idea of GLSM.** Consider \( \mathbb{C}^{N+1} = \mathbb{C} \times \mathbb{C} \) and a new potential \( W(x;p) = p(Q(x)) \). \( W \) is invariant under a \( \mathbb{C}^* \)-action on \( \mathbb{C}^{N+1} \) given by

\[
\xi(x_1, \ldots, x_N, p) = (\xi x_1, \ldots, \xi x_N, \xi^{-1} p) \quad (r = \deg Q).
\]

This action is Hamiltonian, with moment map \( \mu = |x_1|^2 + \cdots + |x_N|^2 - |p|^2 - 2 \rho \cdot p \).

"GLSM" is a gauge theory with a superpotential \( W \).

**Classical vacuum:** \( (\mu^{-1}(0) \cap \text{Crit} W) / S^1 \)

\[
\theta W = p \Rightarrow \frac{\partial W}{\partial p} = Q \quad \Rightarrow \quad \tau > 0, \ |x_1| \neq 0, \ \text{so} \quad p = 0 \quad Q = 0 \quad X_Q
\]

\( \tau < 0 \Rightarrow p \neq 0, \ x_1 = \cdots = x_n = 0 \) (\( Q \) is nondegenerate)
The variation of $\mathcal{I}$ should relate the Landau-Grinzhburg theory with the Calabi-Yau theory.

Our project is to give a mathematical construction of the gauged linear $\sigma$-model, then study its dependence on certain parameters (including $\mathcal{I}$); we hope eventually the LG/CY correspondence can be understood.

**Gauged Witten Equation**

The first step is to set up a good elliptic PDE over a Riemann surface. Indeed we can work under a more general situation.

Let $(X,\omega, J)$ be a noncompact Kähler manifold and $W : X \to \mathbb{C}$ be a holomorphic function. Suppose we have a reductive Lie group $G = K^c$ acting on $X$ such that $W$ is equivariant with respect to a character $\rho : G \to \mathbb{C}^\times$, i.e. $W(gx) = \rho(g) W(x)$.

Let $\Sigma$ be a Riemann surface. A "$W$-structure" over $\Sigma$ is a pair $(P, \Phi)$, where $P \to \Sigma$ is a holomorphic principal $G$-bundle and $\Phi$ is an isomorphism $\phi : P \times_{\rho} \mathbb{C} \to K^c_{\Sigma}$.

This allows us to lift $W$ to the fibre bundle $Y = P \times X$. Choose a local coordinate $z$ on $\Sigma$ and a frame $e$ of $P$ such that $\phi([e, 1]) = dz$. ($[e, 1] \in P \times \mathbb{C}$)
Then define $W_Y([p,x]) = W(x) \, dz$ \, ($[p,x] \in P \times G, X = Y$).

Easy to check that $W_Y \in \Gamma(Y, \pi_Y^* K_\Sigma)$ is well-defined and holomorphic.

Suppose $G = K^e$ and $K$ is a compact Lie group. Choose a $K$-reduction of $P$. It induces a Hermitian metric on $T^* Y$. (Depending on the Kähler metric on $X_1$ and the $K$-action being Hamiltonian.)

The vertical differential of $W_Y$ is $dW_Y \in \Gamma(Y, (T^* Y)^* \otimes \pi_Y^* K_\Sigma)$.

Dualize it using the Hermitian metric of $T^* Y$, we obtain the vertical gradient

$$\nabla W_Y \in \Gamma(Y, T^* Y \otimes \pi_Y^* \Omega^{0,1}_{\Sigma})$$

The "Witten equation" for sections $u \in \Gamma(Y)$ is

$$\overline{\partial} u + \nabla W_Y(u) = 0$$

Here since $Y$ is holomorphic, $\overline{\partial} u \in \Gamma(\Sigma, \text{Hom}^{0,1}(T\Sigma, u^* T^* Y))$

$$\cong \Omega^{0,1}(u^* T^* Y)$$

$$\nabla W_Y(u) = u^* \nabla W_Y \in \Omega^{0,1}(u^* T^* Y)$$

Note that in writing down the Witten equation, we need to choose a $W$-structure and a $K$-reduction of $P$.

$W$-structures exist in a moduli and two $W$-structures $(P, \phi)$ and $(P', \phi')$ are equivalent if there is an isomorphism...
\[ \Phi : p \rightarrow p', \text{ such that the following diagram commutes. } \]

\[ \begin{array}{c}
\begin{array}{c}
p \times_p C \\
\downarrow p \\
p' \times_{p'} C
\end{array} \\
\Phi \\
\Phi'
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
C \\
\downarrow 1
\end{array}
\end{array} \]

In a simpler situation, let \( G = G_1 \times C^* \) and \( p : G \rightarrow C^* \) is induced from a character \( p : C^* \rightarrow C^* \) (characterized by an integer \( \gamma \)), then a \( W \)-structure on \( \Sigma \) consists of an arbitrary \( G_1 \)-bundle \( p_i : \Sigma \rightarrow \Sigma \) together with an \( r \)-spin structure \( (L, \Phi) \). That is, \( L \rightarrow \Sigma \) is a holomorphic line bundle and \( \Phi \) is an isomorphism \( \Phi : L^\otimes r \rightarrow K_{\Sigma} \).

In particular, when \( G_1 \) is a finite group \( \Gamma \), \( W = Q : C^N \rightarrow C \) is homogeneous, then the Witten equation is the one considered by Fan-Jarvis-Ruan.

What is the gauged Witten equation? From now on we restrict to the splitting case, i.e. \( G = C^* \times G_1 \) and \( G_1 \otimes = K_{\Sigma}^C \).

Instead of considering holomorphic \( G_1 \)-bundle, we consider smooth \( K_i \)-bundles with connections. The connection \( A \) is allowed to
 vary, we also allow the reduction on the $C^*$ part, i.e., a Hermitian metric $H_0$ on $L$, to vary. Now for any smooth $K_i$-bundle $Q_i \to \Sigma$ and an $r$-th root $L \to \Sigma$, for any $K_i$-connection $A_i \in \mathcal{A}(Q_i)$ and Hermitian metric $H_0$ on $L$, we can form $\nabla^{H_0} W_L \in \Gamma(Y, T^* Y \otimes \pi_Y^*(\Omega^2))$, and the Witten equation for $u \in \Gamma(Y)$ (here $Y = (Q_i \times L)_{K_i \times \Sigma} \times X$)

$$\bar{\partial} A u + \nabla^{H_0} W_L(u) = 0.$$ 

To control the behavior of the variables $A_i$ and $H_0$, we borrow the idea of symplectic vortex equations. Choose a volume form $\omega \in \Omega^2(\Sigma)$. We can write down the "vortex equation"

$$\ast F_{A_i, H_0} + \mu_{H_0}(u) = 0.$$ 

Here $F_{A_i, H_0} \in \Omega^2(\Sigma, \text{Ad}(\text{Lie} K))$ is the curvature form of the $K_i$-connection $A_i$ and the Chern connection of $H_0$; $\ast$ is the Hodge-star induced from the volume form $\omega$. On the other hand, $\mu : X \to \text{Lie}(K) \ast \cong \text{Lie} K$ and $\mu_{H_0}(u) \in \Omega^0(\Sigma, \text{Ad}(\text{Lie} K))$ is well-defined. The gauged Witten equation is

$$\sum \bar{\partial} A u + \nabla^{H_0} W_L(u) = 0$$

$$\ast F_{A_i, H_0} + \mu_{H_0}(u) = 0$$

(gauge invariant under $G(\mathbb{R}) \times C^0(\Sigma, S^1)$)
Special case: When $W = 0$, we don't need $W$-structures and the setting can be extended to Hamiltonian $K$-manifolds $(X, \omega, \mu)$ with $K$-invariant almost complex structure $J$. This is the case of symplectic vortex equation. (introduced by Cieliebak-Epstein-Salamon and I. Mundet)

**Energy functional.** The symplectic vortex equation is a natural equation in the sense that solutions are minimizers of the Yang-Mills-Higgs functional on $(A, u)$, given by

$$YMH(A, u) = \frac{1}{2} \left( \| d_A W \|_L^2 + \| \mu(u) \|_L^2 + \| F_A \|_L^2 \right).$$

For gauged Witten equation, solutions are minimizers of the following energy functional:

$$E(A, u) = \frac{1}{2} \left( \| d_{A_h, H_0} W \|_L^2 + \| \mu_{H_0}(u) \|_L^2 + \| F_{A_h, H_0} \|_L^2 \right) + \| \nabla H_0 W(u) \|_L^2$$

We could continue working on the general situation but let us restrict to the linear case.

Let $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ be homogeneous of degree $r$. Let $X = \mathbb{C}^{n+1}$
Let \( W(x_1, \ldots, x_N, p) = p^l \Omega(x_1, \ldots, x_N) \) be as previously discussed.

Now \( G = \mathbb{C}^* \times \mathbb{C}^* \), which acts on \( \mathbb{C}^{N+1} \) as

\[
(\xi_0, \xi_1) (x_1, \ldots, x_N, p) = (\xi_0 x_1, \ldots, \xi_0 x_N, \xi_1 p)
\]

This action is Hamiltonian, with moment map

\[
\mu(x_1, \ldots, x_N, p) = \begin{pmatrix}
\sum_{i=1}^{N} |x_i|^2 - \tau_0 \\
\sum_{i=1}^{N} |x_i|^2 - r |p|^2 - \tau_1
\end{pmatrix}
\]

\( \tau_0, \tau_1 \in \mathbb{R} \).

The parameter \( \tau_1 \) will be responsible for the wall-crossing

\[ W \text{ is invariant under the second } \mathbb{C}^* \text{--action but homogeneous under the first } \mathbb{C}^* \text{--action. } \]

\[ P(\xi_0, \xi_1) = \xi_0 \in \mathbb{C}^* \]

\[ W(gx) = P(g) W(x) \]

A \( W \)-structure is an \( r \)-spin structure \( (L_0, \phi) \), \( L_0 \xrightarrow{\phi} K \Sigma \) together with an arbitrary holomorphic line bundle \( L_1 \).

We consider \( H_0 \) : Hermitian metrics on \( L_0 \) and consider \( L_1 \) as a Hermitian line bundle with arbitrary unitary connections. \( (U, H_0, A_1) \)

\[
\{ \begin{array}{l}
\bar{\partial}_{A_1} U + \nabla_{H_0} W(U) = 0 \\
\ast F_{H_0, A_1} + \mu_{H_0}(U) = 0
\end{array} \}
\]

\( U \in \Gamma(L_0 \otimes L_1 \oplus \cdots \oplus L_0 \otimes L_1 \oplus L_1 \oplus L_1^{-1}) \)
All the above are working with a general Riemann surface. For compact Riemann surfaces, the above setting is not enough. Motivated from Fan-Jarvis-Ruan's work, we consider compact Riemann surfaces $\Sigma$, with marked points $p_1, \ldots, p_k$. Then the $r$-th root $\mathcal{O}(L_0)$ of $K_\Sigma$ is allowed to have orbifold structures at $p_1, \ldots, p_k$, and $K_\Sigma$ should be replaced by
\[ K_{\text{log}} = K_\Sigma \otimes \mathcal{O}(p_1) \otimes \ldots \otimes \mathcal{O}(p_k) \]
Since $\phi: \mathcal{O}(L_0) \subseteq K_{\text{log}}$, the local group of $L_0$ near $p_i$ must be $\mathbb{Z}_r$ (or its subgroup). So locally, $L_0$ is identified with $\mathbb{D} \times \mathbb{C}/\mathbb{Z}_r$ with $\mathbb{Z}_r$-action given by $\xi(x, t) = (\xi x, \xi^m t)$.

The above structures gives an "$r$-spin curve" $(\Sigma, p_1, \ldots, p_k, L_0, \phi)$.

Perturbations. When there is an orbifold point whose $m_i = 0$, the Witten equation looks like the Floer type equation:
\[ \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \nabla W(u) = 0 \]

In general $W$ has degenerate critical points, so the linearization of the Witten equation or the gauged Witten equation is not naturally a Fredholm operator.
Therefore we need to perturb the equation, not to achieve transversality but to achieve Fredholmness.
(There are other reasons why we need to perturb.)

A perturbation is given as follows. Near the cylindrical ends at $P_i$ with cylindrical coordinates $(s,t)$, consider

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t}\right) + \Delta W(u) + \beta(s) \Delta F(u) = 0.$$ 

Here $\beta(s)$ is a cut-off function supported near $s = \pm \infty$ and $F : X \to \mathbb{C}$ is a holomorphic function such that $W + F : X \to \mathbb{C}$ is a holomorphic Morse function.

If $W = pQ$, then we choose $F = -ap + \sum_{i=1}^{N} b_i x_i$ and

$$W + F = p(Q - a) + \sum_{i=1}^{N} b_i x_i. \quad \text{(Lefschetz pencil.)}$$

Thm (Tian-X, 2014) In the above setting, for solutions to the perturbed gauged Witten equation, we have:

1. Asymptotic behavior: $m \to 0 \Rightarrow U(s,t) \to \mathbb{R}^3 \times \mathbb{C}; \ m = 0 \Rightarrow U(s,t) \to \mathbb{C}^4 + (W + F)$

2. Fredholm theory

3. Compactly (fixing an $\mathfrak{g}$-spin curve)
Invariant. How to define (formally) an invariant out of the moduli space of solutions:

\[ \mathcal{Z}_Q = \bigoplus_{k=0}^{r-1} \mathcal{H}^{(k)}_Q \]

\[ \mathcal{H}_Q^{(k)} = \bigoplus_{i=0}^{r-1} e^{k} \mathcal{H}_Q^{(i)}; \]

\[ \mathcal{H}_Q^{(0)} \cong \text{PH}^{N-2}(X_Q) \]

If we fix an r-spin curve \( C = (\Sigma, p_1, \ldots, p_k; L_0, \Phi) \) with monodromies \( \gamma_i = \exp(\frac{2\pi i m_i}{r}) \), \( m_i \in \{0, 1, \ldots, r-1\} \), then the invariant is a multilinear function (for \( d \in \mathbb{Z} \))

\[ \langle \cdots \rangle^d_c : \mathcal{H}_Q^{(m_1)} \otimes \cdots \otimes \mathcal{H}_Q^{(m_k)} \rightarrow \mathbb{Q} \]

Example. When \( k = 3 \), suppose \( m_1 \neq 0, m_2 \neq 0, m_3 = 0 \). Choose \( \Theta \in \mathcal{H}_Q^{(0)} \cong \text{PH}^{N-2}(X_Q) \cong H^{N-1}(Q^d(a) \setminus \mathbb{Q}) \mathbb{Z}_r \). Then

\[ \langle e^{m_1}, e^{m_2}, \Theta \rangle_c^d = \sum_{x \in \text{Crit}(W+F)} \# A^\Theta_{x} \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right) \]

Here \( \# \) is the virtual counting of solutions to the perturbed gauged Witten equation whose asymptotic at \( p_3 \) is given by \( K \). It is nonzero only when the virtual dimension is zero. \( A^\Theta_{x} \) is a topological intersection number in \( Q^d(a) \) b.c. \( K \) gives a cycle \( H_{N}^d(\Omega^d(a), \infty) \).