Lecture 17
Dirac Operators

Recall: given a Riemannian manifold \((M,g)\), even-dimensional and oriented, one has the Clifford bundle \(C(TM)\). We can consider a bundle of Clifford modules \(E \to M\) which is a complex \(\mathbb{Z}_2\)-graded vector bundle with a Clifford action:

\[
\forall \nu \in T_x M \quad c(\nu) : E^\pm_x \to E^\mp_x \text{ satisfying }
\]

\[
c(\nu) c(w) + c(w) c(\nu) = -2 \langle \nu, w \rangle \text{id}.
\]

Main Example: \(E = \bigwedge^* T^* M \otimes \mathbb{C}, \quad E^\pm = \bigwedge^{\text{even/odd}} T^* M\)

A connection \(\nabla^E : \Gamma(E) \to \Gamma(T^* M \otimes E)\) is called a Clifford connection if \(\forall\) tangent vector \(X\) and \(a \in C^\infty(TM)\)

\[
[\nabla^E_X, c(a)] = c(\nabla^TM_X a).
\]

Define \(D : \Gamma(E^\pm) \to \Gamma(E^\mp)\) by

\[
D(s) = \sum_{i=1}^n c(e_i) \nabla^E_{e_i} s \quad \text{where } e_1, \ldots, e_n \text{ is a local orthonormal basis of } TM.
\]

In the main example, we have shown in previous lectures that the connection \(\nabla^\bigwedge T^* M\) induced from the Levi-Civita
connection is a Clifford connection and the induced Dirac operator is \( D = d + d^* \) on \( \Omega^2(M) \).

We consider the other important example. Suppose first we have an oriented Riemannian manifold. This means we have a principal \( SO(n) \)-bundle \( SO(TM) \to TM \) consisting of oriented orthonormal frames. Remember that there is a double cover

\[
\text{Spin}(n) \xrightarrow{p} SO(n)
\]

which is nontrivial when \( n \geq 3 \). We may or may not be able to lift \( SO(TM) \) to a \( \text{Spin}(n) \)-bundle.

**Fact:** \( SO(TM) \) can be lifted to a \( \text{Spin}(n) \)-bundle.

(Which means we can cover \( M \) by \( U_\alpha \) and find \( \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Spin}(n) \) satisfying cocycle condition s.t.

\[
\tilde{g}_{\alpha\beta} = p(\tilde{g}_{\alpha\beta}) : U_\alpha \cap U_\beta \to SO(n)
\]

is the transition function for the bundle \( SO(TM) \).

If and only if \( w_2(TM) = 0 \in H^2(M; \mathbb{Z}_2) \).

\[
0 \to H^1(M; \mathbb{Z}_2) \to H^1(M; \text{Spin}(n)) \to H^1(M; SO(n)) \to H^2(M; \mathbb{Z})
\]
This is the spin condition. If \( \exists \) such a lift, then we can cook up the bundle
\[
S = \text{Spin}(TM) \times \text{Spin}(n) \cong S(\mathbb{R}^n)
\]
spinor representation.

Notice that \( C(TM) = \text{Spin}(TM) \times \text{Spin}(n) \) and one has (Notice that \( \text{Spin}(TM) \) is not the union of \( \text{Spin}(T_x M) \)) an action of \( C(TM) \) on \( S \), hence the spinor bundle is a Clifford module.

Now if we have a Levi-Civita connection \( \nabla^M \), it lifts to a connection on the principal bundle (which becomes a tensor) and then lifts to a connection on \( S \). It is still a Clifford connection, hence we can define a Dirac operator
\[
D : \Gamma(S^+) \to \Gamma(S^-)
\]

Consider the Signature operator. Recall on any oriented Riemannian manifold \((M,g)\) we have a Hodge star \( * : \Omega^k \to \Omega^{n-k} \) s.t. \( *(e_1 \wedge \ldots \wedge e_k) = e_k \wedge \ldots \wedge e_1 \)
and hence \( \ast^2 = (-1)^{k(n-k)} \). Now change this operator by
\[ \star : \Omega^k \otimes \mathbb{C} \to \Omega^{n-k} \otimes \mathbb{C}, \quad \star = (-1)^{\frac{k(k+1)}{2} + \frac{n}{2}}. \]

Then \[ \star^2 = (-1)^{\frac{k(k+1)}{2}} \cdot (-1)^{\frac{n}{2}} \cdot i^n \star^2 = 1 \mathrm{d}. \] Hence we obtain a \( \mathbb{Z}_2 \)-grading on \( \bigwedge^* \mathbb{C} \otimes \mathbb{C} \) by \( \pm 1 \) eigen subbundle.

By linear algebra, we get another Clifford module. When \( n = 4k \), \( \star \) is a real operator. Now we have the signature operator \( D : \Omega_\text{self-dual} \to \Omega_\text{anti-self-dual} \)

**Lemma.** Index \( (D |_{\Omega_\text{self-dual}}) = 0 \quad \alpha_i \pm \ast \alpha_i \)

\[ \text{Index} \left( D \bigg|_{\Omega_\text{self-dual}} \right) = \text{Signature} \left( M \right) \quad \text{deg} = \frac{n}{2} \quad \text{harmonic} \]

There are also the examples of \( \bar{\alpha} + \bar{\alpha}^\ast \) on Kähler manifolds. We do not discuss them here.

Now let's consider the corresponding Laplacian. Consider \( D^2 : \Gamma(\Theta^\pm) \to \Gamma(\Theta^\pm) \)

Suppose the geometry is flat. Then

\[ D^2 = \sum_{i=1}^{n} c(e_i) \nabla^\Theta e_i \sum_{j=1}^{n} c(e_j) \nabla^\Theta e_j = \sum_{i,j} c(e_i) c(e_j) \nabla^\Theta e_i \nabla^\Theta e_j \]

\[ + \sum_{i,j} c(e_i) c(\nabla^\Theta e_i e_j) \nabla^\Theta e_j = \Delta \Theta. \] Here
recall $\Delta^E$ is defined by

$$\Gamma(E) \to \Gamma(\bigotimes^{\ast} E) \to \Gamma(\bigotimes^{\bigotimes} E \bigotimes^{\bigotimes} E)$$

Thus $\Gamma(E)$. So the square of a Dirac operator is the Laplacian associated to the Clifford connection.