Recall that $H$ denotes a separable Hilbert space. An operator $K: H \to H$ is called "of finite rank" if $\dim \text{Im} K < \infty$.

**Lemma.** If $K$ has finite rank, then $\text{Id} + K: H \to H$ is Fredholm and $\text{Index}(\text{Id} + K) = 0$.

**Proof.** Consider the diagram

```
\begin{array}{ccc}
O & \downarrow & O \\
\downarrow & & \downarrow \\
\text{Im} K & \xrightarrow{\text{Id} + K} & \text{Im} K \\
\downarrow & & \downarrow \\
H & \xrightarrow{\text{Id} + K} & H \\
\downarrow & & \downarrow \\
H/\text{Im} K & \xrightarrow{\text{Id} + K} & H/\text{Im} K \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\text{Id} + K} & 0
\end{array}
```

On the upper level, the index is zero; on the lower level, we can show that the operator is invertible, hence has index zero. Then by the lemma discussed last time, the assertion is proved. □

So far we used purely algebraic arguments. Now we have to use some analytic tool to proceed.

**Exercise**

a) If $K: H \to H$ has finite rank, then $K$ maps the unit ball $\{ v \in H \mid \| v \| \leq 1 \}$ to a relatively compact set.

b) If $H$ is infinite-dimensional, then the unit ball is not compact.
Defn. An operator \( K : H \to H \) is called "compact" if it maps the unit ball \( H \) to a relatively compact subset.

Basically, compact operators are limits of finite-rank operators.

Recall \( \mathcal{B} = \mathcal{B}_H = \{ \text{bounded linear operators } T : H \to H \} \).

"Operator norm" \( \| T \| = \inf \left\{ \frac{\| T(w) \|}{\| w \|} : \| w \| = 1 \right\} \)

A sequence of operators \( T_i \in \mathcal{B} \) is said to converge to \( T_\infty \in \mathcal{B} \) in "operator norm" if
\[
\lim_{i \to \infty} \| T_i - T_\infty \| = 0.
\]

\( \mathcal{B} \) is a Banach algebra. Basically, as a Banach space with an algebra structure given by compositions of operators, and \( \| T \circ U \| \leq \| T \| \cdot \| U \| \).

Let \( K \subseteq \mathcal{B} \) be the subset of compact operators.

**Theorem**

a) \( K \) is a 2-sided ideal of \( \mathcal{B} \).

b) \( K \) is closed in operator norm.

c) \( K \) is the closure of finite-rank operators

**Proof.**

a) If \( T \) bounded, \( K \) compact, then \( T \circ K \), \( K \circ T \) compact by definition.
(b) Suppose \( T \in \mathcal{K} \). Given a bounded sequence \( u_i \in H \), we want to show \( T(u_i) \) has a convergent subsequence. Given \( n > 0 \), \( \exists i_n \in \mathbb{N} \) such that \( \| K_{i_n} - T \| \leq \frac{1}{n} \). Moreover, there is a subsequence \( u_i^{(m)} \) such that \( K_{i_n} u_i^{(m)} \) converges. We can choose \( u_i^{(m)} \) inductively such that \( u_i^{(m+1)} \) is a subsequence of \( u_i^{(m)} \). Then using the diagonal argument consider \( (Tu_i^{(m)})_m \). We have

\[
\| Tu_i^{(m)} - Tu_i^{(m)} \| \leq \| Tu_i^{(m)} - K_{i_n} u_i^{(m)} \| + \| K_{i_n} u_i^{(m)} - K_{i_n} u_i^{(m)} \|
\]

+ \( \| K_{i_n} u_i^{(m)} - Tu_i^{(m)} \| \) which can be arbitrarily small.

(c) Let \( K \) be a compact operator. Choose a complete orthonormal basis \( e_1, e_2, \ldots \) and define \( Q_n \colon H \to H \) by

\[
Q_n \left( \sum_{i=1}^{\infty} a_i e_i \right) = \sum_{i=1}^{n} a_i e_i.
\]

Then \( Q_n \) has finite rank. So is \( Q_n K \). We want to show \( Q_n K \to K \). We only need to prove \( \forall \varepsilon > 0 \), for \( n \) sufficiently large, \( \| (Q_n K - K)(u) \| < \varepsilon \) for all \( u \in B_1 \). Suppose it is not the case, then \( \exists \varepsilon > 0 \) and \( Q_n \) and \( u_i \in B_1 \), s.t.

\[
\| Q_n K(u_i) - K(u_i) \| > \varepsilon. \]

However, as \( K \) compact, there is
a subsequence (still indexed by $i$) such that

$$K u_i \to v \in H.$$ 

Then

$$\|Q_n K(u_i) - K(u_i)\| \leq \|Q_n (v) - v\| + \|Q_n K(v) - Q_n K(u_i)\|$$

$$+ \|v - K(u_i)\|$$

which can be arbitrarily small. Contradiction. Hence

$$\lim_{n \to \infty} Q_n K \to K \text{ in operator norm.}$$

\[\square\]

Example: Integral Kernel.

Consider functions on an interval $[a, b]$. Consider a function $G(x, y)$ with $x, y \in [a, b]$ and define

$$K : L^2([a, b]) \to L^2([a, b])$$

$$K(u)(x) = \int_a^b G(x, y) u(y) \, dy.$$ 

Suppose first $G(x, y) = \sum_{i=1}^n f_i(x) \overline{g_i(y)}$, $f_i, g_i \in L^2$.

Then

$$K(u) = \sum_{i=1}^n f_i(x) \int_a^b u(y) \overline{g_i(y)} \, dy$$

$$= \sum_{i=1}^n f_i(x) \langle u, g_i \rangle \in \text{span }\{ f_1, \ldots, f_n \}$$

has finite rank.
Now suppose \( G(x,y) \in C^0([a,b] \times [a,b]) \), we can approximate \( G(x,y) \) by \( G_j(x,y) \) which \( \lim \Sigma f_i g_j \) in \( L^2 \). Then in operator norm, we have \( K = \lim K_j \). Hence \( K \) is a compact operator.

\( G(x,y) \) is called the "kernel" of the operator.

**Exercise.** \( \|K\| \leq \|G\|_{L^2([a,b] \times [a,b])} \)

- \( K_1 \circ K_2 \) has kernel \( G(x,y) = \int_a^b G_1(x,z) G_2(z,y) \, dz \)