Recall: on a compact Riemannian manifold \((M, g)\) there is the Hodge Laplacian

\[ \Delta = d d^* + d^* d : \Omega^k \to \Omega^k \]

We want to prove the Regularity Theorem and the Compactness Theorem about \(\Delta\).

We have established "local versions" about elliptic differential operators on \(\mathbb{R}^n\) with periodic coefficients.

Fix \(p \in M\), choose local coordinate chart

\[ \tilde{x} : U \to \mathbb{R}^n, \quad p \in U. \]

We can assume \(\tilde{x}(U) = \mathbb{R}^n\). Then using local coord \(k\)-forms on \(U\)

\[ \{k\text{-forms on } U\} \cong \{\mathbb{R}^m\text{-valued functions on } \mathbb{R}^n\} \]

where \(m = \binom{n}{k}\).

Regard \(\mathbb{R}^m \cong \mathbb{C}^m\).

Also \(\Delta\) is transformed to a 2nd order diff.
operator $\Delta : C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^m \to C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^m$

which also extends to $C^m$-valued functions (not necessarily have periodic coefficients).

**Exercise** Show that in this local coord. system

$$\begin{pmatrix} \psi_1 \\ \psi_m \end{pmatrix} = \begin{pmatrix} -\Delta_{\text{std}} \psi_1 \\ \psi_m \end{pmatrix} + L' \begin{pmatrix} \psi_1 \\ \psi_m \end{pmatrix}$$

where $L'$ is a differential operator of order $\leq 1$.

**principal symbol** $|\beta|^2 I_{m \times m}$ invertible when $\beta \neq 0$.

$\Rightarrow \Delta$ is elliptic.

```
2\pi
\{ \begin{array}{c}
\text{a figure}
\end{array} \}
\begin{array}{c}
Q \subset \mathbb{R}^n
\end{array}
```

**Remark** The $L^2$-norm on forms and $L^2$-norm on
$C^m$-valued functions are not identical but always comparable.

**Exercise** There exists $C > 0$ such that for any $\alpha \in \Omega^k$ whose support is in $\xi^{-1}(Q)$, one has

$$C^{-1} \| \alpha \|_{\infty} \leq \| \alpha \|_{L^2(\mathbb{R}^n)} \leq C \| \alpha \|_{\infty}.$$  
(Remember $H_0 \cong L^2(Q) \otimes C^m$).

II. **Compactness Theorem.**

**Theorem** Suppose $\{\alpha_n\}$ is a sequence in $\Omega^k$ such that

$$\sup_n \| \alpha_n \|_{L^2} < \infty, \quad \sup_n \| \Delta \alpha_n \|_{L^2} < \infty.$$  

Then a subsequence of $\{\alpha_n\}$ is Cauchy w.r.t. $L^2$.

**Lemma** For each $p \in M$, there exists a neighborhood $U_p \subset M$ of $p$ such that for any $f \in C^\infty(U_p)$ the sequence $\{f \alpha_n\}$ has a Cauchy subsequence.
Lemma ⇒ Theorem

Proof. B.c. \( M \) is compact, one finds \( p, \ldots, p_N \in M \) and corresponding nbhds \( U_1, \ldots, U_N \) covering \( M \). Then choose a subordinate partition of unity \( p_1, \ldots, p_N \).

Then by Lemma, there is a subsequence (still indexed by \( n \)) such that \( \{p_i \cdot \alpha_n\}_n \) is a Cauchy sequence for \( i = 1, \ldots, N \). Then

\[
\|x_n - \alpha_m\|_{L^2}^2 = \| \sum_{i=1}^N p_i (\alpha_n - \alpha_m) \|_{L^2}^2 \\
\leq \sum_{i=1}^N \| p_i \cdot \alpha_n - p_i \cdot \alpha_m \|_{L^2}^2.
\]

Proof of Lemma. Choose a local chart \( \bar{\zeta} : U \to \mathbb{R}^n \) around \( p \). \( \bar{\zeta}(p) = 0 \).

Denote \( U' := \bar{\zeta}^{-1} (\text{int } Q) \), \( U'' := \bar{\zeta}^{-1} (\text{int } \frac{1}{\delta} Q) \)
Take $\phi \in C^\infty (\Omega')$. Then we can identify with a $C^m$-valued function supported in int $\Omega$.
So we can consider its Sobolev norms.

**Claim 1.** $\| \phi \Delta n \|_0$ is uniformly bounded.

\[ \| \phi \Delta n \|_0 \leq \text{const.} \| \phi \Delta n \|_{L^2} \leq \text{const.} \| \Delta n \|_{L^2} \]

**Claim 2.** $\| \Delta (\phi \Delta n) \|_{-1}$ is uniformly bounded.

\[ \| u \|_{-1}^2 := \sum_{3 \in \mathbb{Z}^n} (1 + |3|^2)^{s} |u_3|^2 \]

\[ \Delta (\phi \Delta n) = (\Delta \circ \phi) (\Delta n) \]

\[ = (\phi \circ \Delta) (\Delta n) + (\Delta \circ \phi - \phi \circ \Delta) (\Delta n) \]

\[ = (\phi \circ \Delta) (\Delta n) + (\Delta \circ \phi - \phi \circ \Delta) (\Delta n) \]

(i) $\| \phi (\Delta n) \|_{-1} \leq \| \Delta n \|_0 \leq \text{const.} \| \Delta n \|_{L^2}$

(ii) $(\Delta \circ \phi - \phi \circ \Delta)$ in 1st order with coefficients in $\Omega$, so we can regard it as a periodic
operator.

\[ \| (\Delta \cdot \varphi - \varphi \cdot \Delta) (\phi_n) \|_{-1} \leq \text{const.} \| \phi_n \|_0. \]

By the elliptic estimate,

\[ \| \varphi \phi_n \|_1 \leq C \left( \| \nabla \phi_n \|_1 + \| \Delta (\phi \phi_n) \|_{-1} \right) \]

is uniformly bounded. Then use compact embedding \( \| H_1 \hookrightarrow \| H_0 \), we see that a subsequence (still indexed by \( n \)) s.t.

\( \phi_n \) converges in \( H_0 \). B.c. \( 0 \)-norm \( \varphi \) is comparable to \( L^2 \)-norm on functions supported in \( Q \), \( \phi \phi_n \) is a Cauchy seq w.r.t. the \( L^2 \)-norm. \( \square \)

III. Regularity Theorem

**Theorem.** Suppose a bounded linear functional \( T : \mathbb{R}^k \to \mathbb{R} \) is a weak solution to
$\Delta \alpha = \omega \in \Omega^k$

i.e. $T(\Delta \beta) = \langle \beta, \omega \rangle$, $\forall \beta \in \Omega^k$.

Then there exists a classical solution $\alpha \in \Omega^k$ s.t. $T(\beta) = \langle \beta, \alpha \rangle$.

Proof. \red{$\exists : U_q \xrightarrow{\sim} \mathbb{R}^n$, $q \in M$.}

So every smooth function $f \in C^\infty(\mathbb{R}^n) \otimes \mathbb{C}^m$ can be regarded as a $k$-form with complex coefficients on $M$.

(Naturally extend $T : \Omega^k \to \mathbb{R}$ to $T : \Omega^k \otimes \mathbb{C} \to \mathbb{C}$).

So $T$ defines a linear functional $l : C^\infty(\mathbb{R}^n) \otimes \mathbb{C}^m \to \mathbb{C}$.

Lemma. For $p \in \mathbb{R}^n$, $\exists$ a nbhd $W_p \subset \mathbb{R}^n$ of $p$ and a periodic smooth $u_p \in \mathbb{P}$ such that
\[ l(t) = \langle t, W_p \rangle \quad \forall \ t \in C^\infty(W_p) \otimes C^m \]

(means the functional is actually a real k-form and locally equal to \( W_p \).)

Lemma \implies Theorem. skipped

Proof of lemma: The weak solution property implies
\[ l(\Delta t) = \langle t, w \rangle, \quad \forall \ t \in C^\infty(\mathbb{R}^n) \otimes C^m \]
(notice that \( w \) is only smooth but not periodic).

Given \( p \in \mathbb{R}^n \). Let \( Q_p \subseteq \mathbb{R}^n \) be the \( 2\pi \)-cube centered at \( p \).

Claim: \( \exists C > 0 \) s.t. for
all \( \varphi \in C^\infty(Q_p) \otimes \mathbb{C}^m \),

\[ |l(\varphi)| \leq C \|\varphi\|_0. \]

On the other hand, b.c. \( C^\infty(Q_p) \otimes \mathbb{C}^m \) is dense in \( L^2(Q_p) \otimes \mathbb{C}^m \), this claim means \( l \) extends to a bounded linear functional on the Hilbert space \( L^2(Q_p) \). By Riesz Representation Theorem, \( l \) corresponds to a vector of \( L^2(Q_p) \), say \( (l_3) \in L^2(Q_p) \).

So \( l(u) = \sum_3 u_3 \cdot l_3 \), \( \forall u = (u_3) \in L^2(Q_p) \).

The form \( \omega \in \Omega^k \), restricted to \( Q_p \) is \( L^2 \)-integrable, so \( \omega \) also gives a vector \( (\omega_3) \in L^2(Q_p) \).

Now, remember \( T \) is a weak solution to \( \Delta d = \omega \).

Then for functions \( \varphi \in C^\infty(Q_p) \otimes \mathbb{C}^m \),

\[ |l(\varphi)| = |T(\varphi)| \leq C \|\varphi\|_{L^2(Q_p)} \leq \text{const.} \|\varphi\|_0. \]
\[ l(\Delta \phi) = \langle \phi, \omega \rangle. \]  
inner prod. in \( H_0 = L^2 \).

And also, \( \Delta = -\left( \frac{\partial^2}{\partial x^1} \right)^2 - \cdots - \left( \frac{\partial^2}{\partial x^n} \right)^2 + \text{lower order terms} \)  
(\text{ignore lower order terms})

\[
\sum_3 |3|^{-2} \phi_3 \cdot \omega_3 = \sum_3 \phi_3 \cdot \omega_3
\]

B.c. such \( \phi \in C^\infty(Q_\rho) \otimes C^m \) dense in \( H_0 \).

\[
|3|^{-2} \omega_3, \ \forall \ 3 \in \mathbb{Z}^n
\]

Fourier series of \( -\Delta \text{std.} \ l \in H_0 \subset H_0 \).

So as vectors in \( H_{-2} \), \( \Delta \Theta = \omega \)

\[ \Theta \]

\[ H_{-2} \]
By Regularity, because $\Delta l \in \mathcal{H}_0$, $\Rightarrow \ell \in \mathcal{H}_2$

\[\[\begin{array}{c}
O_1 \\
V_p \\
O_2 \\
O_p
\end{array}\]

$V_p \leq \ldots \leq O_n \leq O_{n-1} \leq \ldots \leq O_1 \leq O_p$.

Choose a cut-off function $p_k$ supported in $O_k$ which is identically 1 in $O_{k+1}$.

$$\Delta (p_k \ell) = p_k (\Delta \ell) + (\Delta \circ p_k - p_k \circ \Delta) (\ell)$$

$$\| \ell \| = p, \omega + (\Delta \circ p_k - p_k \circ \Delta) (\ell)$$

$$\text{1st order } 1H_2, \text{ alls.}$$

$\rho, l \in 1H_3, 1H_5 (\text{alls.}), \text{ etc.} \quad \ell \in 1H_1$.

Inductively, one can show $p_k \ell \in 1H_{k+2}$. 
If a function is in $H^1$ for all $s$, then it is smooth. So in $V_p$, $l$ is a smooth function.

So $\forall \varphi \in C^\infty (V_p) \otimes C^m$,

\[ l(\varphi) = \langle \varphi, l \rangle_0. \]