Quiz 7 (Notes, books, and calculators are not authorized). Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $n \in \mathbb{N}$, $n \geq 1$, and consider the vector space $V = \text{span}\{\sin(t), \dots, \sin(nt)\}$ over $\mathbb{R}$. Recall that $\langle f, q\rangle := \int_{-\pi}^{\pi} f(t)g(t)\text{d}t$ is an inner product in V for any $n \geq 1$, and the set $\{\sin(t), \dots, \sin(nt)\}$ is orthogonal. We also use the notation $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$ for the associated norm.

(a) Prove (with a simple argument) that $|\int_{-\pi}^{\pi}(4-\sin(t))(2+3\sin(3t))\text{d}t| \leq \left(\int_{-\pi}^{\pi}(4-\sin(t))^2\text{d}t\right)^{\frac{1}{2}}\left(\int_{-\pi}^{\pi}(2+3\sin(3t))^2\text{d}t\right)^{\frac{1}{2}}$.

Let $f(t) = 4 - \sin(t)$ and $g(t) = 2 + 3\sin(3t)$. The Cauchy-Schwarz inequality implies that

$$|\langle f, g\rangle| \leq \|f\|\|g\|,$$

that is to say

$$|\int_{-\pi}^{\pi}(4-\sin(t))(2+3\sin(3t))\text{d}t| \leq \left(\int_{-\pi}^{\pi}(4-\sin(t))^2\text{d}t\right)^{\frac{1}{2}}\left(\int_{-\pi}^{\pi}(2+3\sin(3t))^2\text{d}t\right)^{\frac{1}{2}}.$$

This is exactly the inequality in question.

(b) Prove (with a simple argument) that $\int_{-\pi}^{\pi}(\sin(t)+2\sin(2t)+\ldots+n\sin(nt))^2\text{d}t = \pi(1+2^2+\ldots+n^2)$. (*Hint:* $\int_{-\pi}^{\pi}(\sin(kt))^2 dt = \pi$ *for any* $k \in \mathbb{N}$, $k \geq 1$).

Since the set $\{\sin(t), \dots, \sin(nt)\}$ is orthogonal, the set $\{\sin(t), 2\sin(2(t) \dots, n\sin(nt)\}$ is also orthogonal. Then the Pythagorean theorem implies that

$$\begin{aligned}\|\sin(t)+2\sin(2t)+\ldots+n\sin(nt)\|^2 &= \|\sin(t)\|^2 + \|2\sin(2t)\|^2 + \ldots + \|n\sin(nt)\|^2 \\ &= \|\sin(t)\|^2 + 2^2\|\sin(2t)\|^2 + \ldots + n^2\|\sin(nt)\|^2 = \pi(1+2^2+\ldots+n^2).\end{aligned}$$

(c) Consider the subspace U of V spanned by $v_1 = \sin(t)+\sin(2t)+\sin(3t)+\sin(4t)$ and $v_2 = \sin(t)+\sin(2t)+2\sin(3t)+4\sin(4t)$. Find an orthogonal basis of U.

We apply the Gram-Schmidt process:

$$w_1 = v_1 = \sin(t) + \sin(2t) + \sin(3t) + \sin(4t)$$

then

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2}w_1.$$

Here we have

$$\langle v_2, w_1\rangle = \|\sin(t)\|^2 + \|\sin(2t)\|^2 + 2\|\sin(3t)|^2 + 4\|\sin(4t)\|^2 = \pi(1+1+2+4) = 8\pi$$

and

$$\|w_1\|^2 = \|\sin(t)\|^2 + \|\sin(2t)\|^2 + \|\sin(3t)|^2 + \|\sin(4t)\|^2 = 4\pi.$$

Hence

$$\begin{aligned}w_2 &= \sin(t) + \sin(2t) + 2\sin(3t) + 4\sin(4t) - \frac{8\pi}{4\pi}(\sin(t) + \sin(2t) + \sin(3t) + \sin(4t)) \\ &= -\sin(t) - \sin(2t) + 2\sin(4t).\end{aligned}$$

The set $\{w_1, w_2\}$ is orthogonal; hence it is linearly independent. This set is a basis of U.

Question 2: Consider the following basis of $\mathbb{P}_3(t)$, $S := \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$, and the following basis of $\mathbb{P}_1(X,Y)$, $S' := \{1, X, Y\}$. Let $F : \mathbb{P}_3(t) \longrightarrow \mathbb{P}_1(X,Y)$ be the linear mapping defined by $F(p) = p(0) + p(1)X + p(2)Y$. ($\mathbb{P}_3(t)$ is the vector space over $\mathbb{R}$ of the univariate polynomials of degree at most 1. $\mathbb{P}_1(X,Y)$ is the vector space over $\mathbb{R}$ of the bivariate polynomials of total degree at most 1.)

(a) Find the matrix of F relative to the bases S and S'.

We compute the coordinate vectors of $F(1)$, $F(1+t)$, $F(1+t+t^2)$ and $F(1+t+t^2+t^3)$ relative to S'.

$$F(1) = 1 + X + Y.$$

$$F(1+t) = 1 + 2X + 3Y.$$

$$F(1+t+t^2) = 1 + 3X + 7Y.$$

$$F(1+t+t^2+t^3) = 1 + 4X + 15Y.$$

This means that

$$[F]_{SS'} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 7 & 15 \end{bmatrix}.$$

Question 3: Let V and W be two finite-dimensional vector spaces over a field K, and let $m = \dim(V)$, $n = \dim(W)$. Assume that $n < m$. Let $T : V \longrightarrow W$ be a linear mapping. Give a lower bound on the dimension of $\mathrm{Ker}(T)$). Can T be injective? (Justify carefully your answer).

Using the Rank-Nullity theorem we have that

$$\mathsf{dim}(\mathsf{Ker}(T)) + \mathsf{dim}(\mathsf{Im}(T)) = m.$$

Since we also have $\mathsf{dim}(\mathsf{Im}(T)) \leq n$, this implies that

$$\mathsf{dim}(\mathsf{Ker}(T)) = m - \mathsf{dim}(\mathsf{Im}(T)) \geq m - n.$$

In conclusion the dimension of Ker(T)) is at least $m - n$. T cannot be injective since $m - n > 0$.