Part IV, Chapter 18

Weak formulation of model problems

We consider in this chapter a few simple PDEs posed over a domain $D$ in $\mathbb{R}^d$. Our goal is to reformulate these problems in weak forms using the important notion of *test functions*. We show that, in general, there are many ways to write weak formulations. The choice can be guided, e.g., by the regularity of the data and the quantities of interest (e.g., the solution or its gradient). Weak formulation are the starting point for building finite element approximations. The scope of this chapter is restricted to time-independent problems for simplicity.

### 18.1 A second-order PDE

#### 18.1.1 Model problem

Let $D$ be a domain in $\mathbb{R}^d$ and consider $f : D \to \mathbb{R}$. The problem we want to solve consists of finding a function $u : D \to \mathbb{R}$ with some appropriate regularity such that

$$-\Delta u = f \quad \text{in } D \quad u = 0 \quad \text{on } \partial D,$$

where we recall that, in Cartesian coordinates, the Laplace operator acts on functions as follows:

$$\Delta u := \nabla \cdot \nabla u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}.$$  \hspace{1cm} (18.2)

The PDE $-\Delta u = f$ in $D$ is called the *Poisson equation* (the PDE is called the *Laplace equation* when $f = 0$). The Laplace operator is ubiquitous in physics. For instance it is the prototypical operator encountered in the modelling of physical processes involving diffusion. Simplified settings for application include heat transfer (where $u$ is the temperature and $f$ is the heat source), mass transfer (where $u$ is the concentration of a species and $f$...
is the mass source), porous media flow (where \( u \) is the hydraulic head and \( f \) is the mass source), electrostatics (where \( u \) is the electrostatic potential and \( f \) is the charge density), and static equilibria of membranes (where \( u \) is the transverse membrane displacement and \( f \) is the transverse load).

The condition enforced on \( \partial D \) in (18.1) is called a boundary condition. In the context of the above models, this condition means that the temperature (the concentration, the electrostatic potential, or the transverse membrane displacement) is fixed at a constant value on \( \partial D \), and without loss of generality we can assume that this constant is zero. A condition prescribing the value of the solution at the boundary is called a Dirichlet condition, and when the prescribed value is zero, the condition is called an homogeneous Dirichlet condition. Other boundary conditions can be prescribed for the Poisson equation; those are reviewed in Chapter 24 in the more general context of second-order elliptic PDEs.

To sum up, (18.1) is the Poisson problem with an homogeneous Dirichlet condition. We now present three weak formulations of (18.1).

### 18.1.2 First weak formulation

We derive a weak formulation for (18.1) by proceeding formally. Consider an arbitrary test function \( \varphi \in C^\infty_0(D) \), where \( C^\infty_0(D) \) is the space of infinitely differentiable functions compactly supported in \( D \). As a first step, we multiply the PDE in (18.1) by \( \varphi \) and integrate over \( D \) to obtain

\[
- \int_D (\Delta u) \varphi \, dx = \int_D f \varphi \, dx.
\]  

Equation (18.3) is equivalent to the PDE in (18.1) if \( \Delta u \) is smooth enough (e.g., integrable over \( D \)); indeed, if an integrable function \( g \) satisfies \( \int_D g \varphi \, dx = 0 \) for all \( \varphi \in C^\infty_0(D) \), then \( g = 0 \) a.e. in \( D \).

As a second step, we use the **divergence formula** stating that, for a smooth vector-valued function \( \Phi \),

\[
\int_D \nabla \cdot \Phi \, dx = \int_{\partial D} \Phi \cdot n \, ds,
\]  

where \( n \) is the outward unit normal to \( D \). We apply this formula to the function \( \Phi = w \nabla v \), where \( v \) and \( w \) are two scalar-valued, smooth functions. Since \( \nabla \Phi = \nabla w \cdot \nabla v + w \Delta v \), we infer that

\[
- \int_D (\Delta v) w \, dx = \int_D \nabla v \cdot \nabla w \, dx - \int_{\partial D} (n \cdot \nabla v) w \, ds.
\]  

This is **Green’s formula**, which is a very useful tool to derive weak formulations of PDEs involving the Laplace operator. This formula is valid, for instance, if \( v \in C^2(D) \cap C^1(\overline{D}) \) and \( w \in C^1(D) \cap C^0(\overline{D}) \). The formula can be extended to functions in the usual Sobolev spaces. In particular, it remains valid for
all $w \in H^1(D)$ and all $v \in H^2(D)$, and even for all $v \in H^1(D)$ such that $\Delta v \in L^2(D)$ in which case the boundary integral in (18.5) is understood in a weak sense; see Corollary B.114. We apply Green’s formula to the functions $v := u$ and $w := \varphi$, assuming enough regularity for $u$. Since $\varphi$ vanishes at the boundary, we transform (18.3) into

$$
\int_D \nabla u \cdot \nabla \varphi \, dx = \int_D f \varphi \, dx, \quad \forall \varphi \in C_0^\infty(D). \tag{18.6}
$$

We now recast (18.6) into a functional framework. Let us take $f \in L^2(D)$. We observe that a natural solution space is

$$H^1(D) = \{ v \in L^2(D) \mid \nabla v \in L^2(D) \}. \tag{18.7}$$

Recall from Proposition B.48 that $H^1(D)$ is a Hilbert space when equipped with the inner product $(u, v)_{H^1(D)} = \int_D uv \, dx + \int_D \nabla u \cdot \nabla v \, dx$ with associated norm $\|v\|_{H^1(D)} = (\int_D v^2 \, dx + \int_D \|\nabla v\|_2^2 \, dx)^{1/2}$, where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. Furthermore, in order to account for the boundary condition in (18.1), we consider the subspace spanned by those functions in $H^1(D)$ that vanish at the boundary. It turns out that this space is $H^1_0(D)$; see Theorem B.106. Finally, we can extend the space of test functions in (18.6) to the closure of $C_0^\infty(D)$ in $H^1(D)$, which is by definition $H^1_0(D)$ (see Definition B.62). Indeed, for any test function $w \in H^1_0(D)$, there is a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C_0^\infty(D)$ converging to $w$ in $H^1_0(D)$. Then, we can pass to the limit in (18.6) where $\varphi_n$ is the test function. To sum up, a weak formulation of the Poisson problem with homogeneous Dirichlet condition is as follows:

$$\begin{cases}
\text{Find } u \in H^1_0(D) \text{ such that } \\
\int_D \nabla u \cdot \nabla w \, dx = \int_D f \, w \, dx, \quad \forall w \in H^1_0(D). \tag{18.8}
\end{cases}$$

A function $u$ solving (18.8) is called a weak solution.

We now investigate whether a weak solution of (18.8) satisfies the PDE and the boundary condition in (18.1). In what follows, we identify $L^2(D)$ with its dual space $L^2(D)'$ so that, using the notation $H^{-1}(D) = H^1_0(D)'$, we are in the situation where $H^1_0(D) \hookrightarrow L^2(D) \equiv L^2(D)' \hookrightarrow H^{-1}(D)$ with dense (and continuous) embeddings. Notice that if $v \in H^1(D)$ (that is to say $\nabla v \in L^2(D)$), then $\Delta v = \nabla \cdot (\nabla v) \in H^{-1}(D)$; see Example B.70.

**Proposition 18.1 (Weak solution).** Assume that $u$ solves (18.8) with $f \in L^2(D)$. Then, the PDE in (18.1) holds a.e. in $D$, and the boundary condition holds a.e. in $\partial D$.

**Proof.** Let $u$ be a weak solution. Then, $\Delta u \in H^{-1}(D)$. Consider an arbitrary function $\varphi \in C_0^\infty(D)$ in (18.8). Since $f \in L^2(D)$, we infer that

$$\langle -\Delta u, \varphi \rangle_{H^1_0(D), H^{-1}(D)} = \int_D \nabla u \cdot \nabla \varphi \, dx = \int_D f \varphi \, dx \leq C \|\varphi\|_{L^2(D)} \quad \text{with } C = \|f\|_{L^2(D)}.$$

Invoking the density of $C_0^\infty(D)$ in $L^2(D)$ (see Theorem B.36), we
infer that $-\Delta u$ defines a bounded linear form on $L^2(D)$ with Riesz–Fréchet representative equal to $f$. Owing to the identification $L^2(D) \equiv L^2(D)'$, we can write $-\Delta u = f$. Hence, the PDE holds a.e. in $D$. Finally, since $u \in H^1_0(D)$, $u$ vanishes a.e. in $\partial D$ owing to the Trace Theorem B.106.

The crucial advantage of the weak formulation (18.8) with respect to the original formulation (18.1) is that, as we will see in the next chapter, there exists powerful tools that will allow us to assert the existence and uniqueness of weak solutions. Incidentally, note that uniqueness is not a trivial property in spaces larger than $H^1(D)$. For instance, one can construct domains in which this property does not hold in $L^2(D)$; see Exercise 18.1.

18.1.3 Second weak formulation

To derive our second formulation, we introduce the vector-valued function $\sigma = -\nabla u$. To avoid notational collisions, we use the letter $p$ instead of $u$ to denote the scalar-valued dependent function, and we use the symbol $u$ to denote the dependent pair $(\sigma, p)$. In many applications, $p$ plays the role of a potential and $\sigma$ plays the role of a diffusive flux.

Since $\sigma = -\nabla p$ and $-\Delta p = f$, we obtain $\nabla \cdot \sigma = f$. Therefore, the model problem is now written as follows:

$$\sigma + \nabla p = 0 \text{ in } D, \quad \nabla \cdot \sigma = f \text{ in } D, \quad p = 0 \text{ on } \partial D. \quad (18.9)$$

This is the mixed formulation of the original problem (18.1). The PDEs in (18.9) are often called Darcy’s equations (in the context of porous media flows, $p$ is the hydraulic head and $\sigma$ the filtration velocity).

We multiply the first PDE in (18.9) by a vector-valued test function $\tau$ and integrate over $D$ to obtain

$$\int_D \sigma \cdot \tau \, dx + \int_D \nabla p \cdot \tau \, dx = 0. \quad (18.10)$$

We multiply the second PDE in (18.9) by a scalar-valued test function $q$ and integrate over $D$ to obtain

$$\int_D (\nabla \cdot \sigma)q \, dx = \int_D fq \, dx. \quad (18.11)$$

No integration by parts are performed in this approach.

We can now specify a functional framework. We consider $H^1(D)$ as solution space for $p$ (so that $\nabla p \in L^2(D)$ and $p \in L^2(D)$), and $H(\text{div}; D)$ as solution space for $\sigma$ (so that $\nabla \cdot \sigma \in L^2(D)$ and $\sigma \in L^2(D)$). Moreover, we enforce the boundary condition explicitly by restricting $p$ to be in the space $H^1_0(D)$. With this setting, the test function $\tau$ can be taken in $L^2(D)$ and the test function $q$ in $L^2(D)$. To sum up, we obtain the following weak formulation:
Part IV. Galerkin Approximation

\begin{align}
\left\{ \begin{array}{l}
\text{Find } u := (\sigma, p) \in V \text{ such that } \\
\int_D (\sigma \cdot \tau + \nabla p \cdot \tau + (\nabla \cdot \sigma) q) \, dx = \int_D f q \, dx, \quad \forall w := (\tau, q) \in W, 
\end{array} \right. \tag{18.12}
\end{align}

with functional spaces \( V := H(\text{div}; D) \times H^1_0(D) \) and \( W := L^2(D) \times L^2(D) \).

Note that the space where the solution is expected to be (trial space) differs from the space where the test functions are taken (test space).

**Proposition 18.2 (Weak solution).** Assume that \( u \) solves (18.12) with \( f \in L^2(D) \). Then, the PDEs in (18.9) hold a.e. in \( D \), and the boundary condition holds a.e. in \( \partial D \).

**Proof.** Left as an exercise. \( \square \)

### 18.1.4 Third weak formulation

We start again with the mixed formulation (18.9), and we now perform an integration by parts on the term involving \( \nabla \cdot \sigma \). Proceeding formally, we obtain

\begin{align}
- \int_D \sigma \cdot \nabla q \, dx + \int_{\partial D} (n \cdot \sigma) q \, ds = \int_D f q \, dx. \tag{18.13}
\end{align}

We take the test function \( q \) in \( H^1(D) \) for the first integral to make sense; moreover, to eliminate the boundary integral, we restrict \( q \) to be in the space \( H^1_0(D) \). Now, the diffusive flux \( \sigma \) can be taken in \( L^2(D) \). To sum up, we obtain the following weak formulation:

\begin{align}
\left\{ \begin{array}{l}
\text{Find } u := (\sigma, p) \in V \text{ such that } \\
\int_D (\sigma \cdot \tau + \nabla p \cdot \tau + \sigma \cdot \nabla q) \, dx = -\int_D f q \, dx, \quad \forall w := (\tau, q) \in V,
\end{array} \right. \tag{18.14}
\end{align}

with the same functional space \( V := L^2(D) \times H^1_0(D) \) for the solution and the test functions. The change of sign on the right-hand side has been introduced to make the left-hand side symmetric with respect to \((\sigma, p)\) and \((\tau, q)\).

**Proposition 18.3.** Let \( u \) solve (18.14) with \( f \in L^2(D) \). Then, the PDEs in (18.9) hold a.e. in \( D \), and the boundary condition holds a.e. in \( \partial D \).

**Proof.** Left as an exercise. \( \square \)

### 18.2 A first-order PDE\( \diamondsuit \)

#### 18.2.1 Model problem

For simplicity, we consider a one-dimensional model problem. Let \( D := (0, 1) \) and let \( f : D \to \mathbb{R} \) be a smooth function. The problem we want to solve consists of finding a function \( u : D \to \mathbb{R} \) such that

\begin{align}
\left\{ \begin{array}{l}
\text{Find } u := (\sigma, p) \in V \text{ such that } \\
\int_D (\sigma \cdot \tau + \nabla p \cdot \tau + \sigma \cdot \nabla q) \, dx = -\int_D f q \, dx, \quad \forall w := (\tau, q) \in V,
\end{array} \right. \tag{18.14}
\end{align}

with the same functional space \( V := L^2(D) \times H^1_0(D) \) for the solution and the test functions. The change of sign on the right-hand side has been introduced to make the left-hand side symmetric with respect to \((\sigma, p)\) and \((\tau, q)\).

**Proposition 18.3.** Let \( u \) solve (18.14) with \( f \in L^2(D) \). Then, the PDEs in (18.9) hold a.e. in \( D \), and the boundary condition holds a.e. in \( \partial D \).

**Proof.** Left as an exercise. \( \square \)
Chapter 18. Weak formulation of model problems

\[ u' = f \quad \text{in} \ D, \quad u(0) = 0. \]  \hspace{1cm} (18.15)

Formally, the solution to this problem is given by

\[ u(x) = \int_0^x f(t) \, dt, \quad \forall x \in D. \]  \hspace{1cm} (18.16)

To give a precise mathematical meaning to this statement, we assume that \( f \in L^1(D) \) and we introduce the Sobolev space

\[ W^{1,1}(D) = \{ v \in L^1(D) \mid v' \in L^1(D) \}, \]  \hspace{1cm} (18.17)

with the derivative understood in the distribution sense; see Definition B.47.

**Lemma 18.4 (Solution in \( W^{1,1}(D) \)).** If \( f \in L^1(D) \), the problem (18.15) has a unique solution in \( W^{1,1}(D) \) which is given by (18.16).

**Proof.** (1) Defining the function \( u \) by the formula (18.16) is meaningful since \( f \in L^1(D) \) (recall that the integral is defined in the Lebesgue sense). Let us show that \( u \in C^0([0,1]) \). Let \( x \in [0,1] \) and let \( (x_n)_{n \in \mathbb{N}} \) be a sequence converging to \( x \) in \([0,1]\). The following equalities hold:

\[ \int_0^x f(t) \, dt - \int_0^{x_n} f(t) \, dt = \int_{x_n}^x f(t) \, dt = \int_D \chi_{[x_n,x]}(t) f(t) \, dt, \]

where \( \chi_{[x_n,x]} \) denotes the characteristic function of the interval \([x_n,x]\). Since \( \chi_{[x_n,x]} f \to 0 \) and \( |\chi_{[x_n,x]} f| \leq |f| \) a.e. in \( D \), Lebesgue’s Dominated Convergence Theorem B.20 implies that \( u(x_n) \to u(x) \). This shows that \( u \in C^0([0,1]) \).

(2) Let us now prove that the function \( u \) solves (18.15). One can verify (see Exercise 18.5) that \( u' = f \) holds in the distribution sense. Identifying functions in \( L^1(D) \) with distributions, \( u' = f \) holds in \( L^1(D) \). Moreover, since \( u \in C^0([0,1]) \), the boundary condition \( u(0) = 0 \) is meaningful.

(3) Uniqueness of the solution is a consequence of Lemma B.50. \( \square \)

We now present two possible mathematical settings for the weak formulation of problem (18.15).

### 18.2.2 Formulation in \( L^1(D) \)

Since \( f \in L^1(D) \) and \( u \in W^{1,1}(D) \) with \( u(0) = 0 \), a natural weak formulation is obtained by just multiplying the PDE in (18.15) by a test function \( w \) and integrating over \( D \), yielding

\[ \int_D u' w \, dt = \int_D f w \, dt. \]  \hspace{1cm} (18.18)

This equality is meaningful for \( w \in W^{\infty} := L^\infty(D) \). Moreover, the boundary condition \( u(0) = 0 \) can be explicitly enforced by considering the solution
space \( V^{(1)} := \{ v \in W^{1,1}(D) \mid v(0) = 0 \} \). Thus, we obtain the following weak formulation:

\[
\begin{aligned}
\text{Find } u \in V^{(1)} \text{ such that } \\
\int_D u' w \, dt = \int_D f \, w \, dt, \quad \forall w \in W^{(\infty)}. 
\end{aligned}
\] (18.19)

Solving first-order PDEs using \( L^1 \)-based approximation techniques has been introduced by Lavery [326, 327] and further explored by Guermond [263]; see also Guermond and Popov [265] and references therein. In the literature however, the dominant viewpoint consists of using \( L^2 \)-based formulations. This leads us to consider a second weak formulation where the source term \( f \) has slightly more regularity (\( f \in L^2(D) \) rather than just \( f \in L^1(D) \)), thereby allowing us to work in a Hilbertian setting.

### 18.2.3 Formulation in \( L^2(D) \)

Assume \( f \in L^2(D) \). Since \( L^2(D) \subset L^1(D) \), \( f \in L^1(D) \), and we can still apply Lemma 18.4. Moreover, the unique solution \( u \), which is given by (18.16), is in \( H^1(D) \). Let us verify this property directly. Owing to the Cauchy–Schwarz inequality and Fubini’s Theorem, we infer that

\[
\int_0^1 |u(x)|^2 \, dx = \int_0^1 \left( \int_0^x f(t) \, dt \right)^2 \, dx \leq \int_0^1 \left( \int_0^x |f(t)|^2 \, dt \right) \, dx \\
\leq \int_0^1 \left( \int_t^1 |f(t)|^2 \, dt \right) \, dx \\
= \int_0^1 \left( \int_t^1 |f(t)|^2 \, dt \right) \, dx = \int_0^1 (1-t)|f(t)|^2 \, dt \leq \int_0^1 |f(t)|^2 \, dt,
\]

that is to say, \( \|u\|_{L^2(D)} \leq \|f\|_{L^2(D)} \). Moreover, \( \|u'\|_{L^2(D)} = \|f\|_{L^2(D)} \). Hence, \( u \in H^1(D) \). We can then restrict the test functions to the Hilbert space \( W^{(2)} := L^2(D) \) and use the Hilbert space \( V^{(2)} := \{ v \in H^1(D) \mid v(0) = 0 \} \) as solution space. We obtain the following weak formulation:

\[
\begin{aligned}
\text{Find } u \in V^{(2)} \text{ such that } \\
\int_D u' w \, dt = \int_D f \, w \, dt, \quad \forall w \in W^{(2)}. 
\end{aligned}
\] (18.20)

Observe that the key change with respect to (18.19) is in the choice of the trial and test spaces.

### 18.3 Towards an abstract model problem

Observe that all the above weak formulations fit into the following model problem:
Find \( u \in V \) such that
\[
\begin{aligned}
a(u, w) &= \ell(w), \\
\forall w \in W,
\end{aligned}
\] (18.21)
where \( V \) and \( W \) are vector spaces whose elements are scalar- or vector-valued functions defined over \( D \). \( V \) is called the trial space or solution space, and \( W \) is called the test space. Moreover,

(i) \( a \) is a bilinear form on \( V \times W \), i.e., \( a \) is a map from \( V \times W \) to \( \mathbb{R} \) which is linear in each argument separately (that is, the map \( V \ni v \mapsto a(v, w) \in \mathbb{R} \) is linear for all \( w \in W \), and the map \( W \ni w \mapsto a(v, w) \) is linear for all \( v \in V \)).

(ii) \( \ell \) is a linear form on \( W \), i.e., \( \text{form} \) is a linear map from \( W \to \mathbb{R} \).

The above model problem is the starting point of the next chapter where we introduce a mathematical setting to study (18.21).

Remark 18.5 (Bilinearity of \( a \)). The linearity of \( a \) with respect to its second argument directly results from the weak formulation, whereas the linearity with respect to its first argument is a consequence of the linearity of the model problem itself. Observe that bilinear forms and linear forms on \( V \times W \) are different objects. For instance, the action of a linear form on the pair \((v, 0) \in V \times W\) is not necessarily zero, whereas \( a(v, 0) = 0 \).

Remark 18.6 (Role of test functions on right-hand side). In the weak formulation (18.19), the role of test functions is somewhat different than in the above model problem. Indeed, since \( L^\infty(D) \) is the dual space of \( L^1(D) \) (the reverse is not true), the test function \( w \in L^\infty(D) \) acts on the function \( f \in L^1(D) \). Although this alternative viewpoint is not often considered in the literature, it actually allows for a more general setting regarding well-posedness; we return to this point in §19.3.2. This distinction is not relevant for model problems set in a Hilbertian framework.

Remark 18.7 (Complex-valued functions). We have considered so far only real-valued functions and thus we have worked with vector spaces over \( \mathbb{R} \). Let us now consider complex-valued problems and vector spaces over \( \mathbb{C} \). Consider for instance the PDE \( iu - \Delta u = f \) in \( D \) with \( f : D \to \mathbb{C} \) and an homogeneous Dirichlet condition at the boundary. The functional setting is \( V = W = H^1_0(D; \mathbb{C}) \) equipped with the norm \( \| \cdot \|_{H^1(D; \mathbb{C})} \). It turns out that it is more convenient to use the complex conjugate of the test function in the weak problem. This leads us to introduce the map \( a : V \times V \to \mathbb{C} \) such that
\[
a(u, w) = \int_D iu\overline{w} \, dx + \int_D \nabla u \cdot \nabla \overline{w} \, dx,
\]
and to the map \( \ell : V \to \mathbb{C} \) such that
\[
\ell(w) = \int_D f \overline{w} \, dx.
\]
The reason for using the complex conjugate is that it allows us to infer positivity properties on the real and imaginary parts of \( a(u, w) \) by taking \( w = u \) as the test function. The map \( \ell \) is called an \textit{antilinear} form, i.e., 
\[ \ell(\lambda v + \mu w) = \overline{\lambda} \ell(v) + \overline{\mu} \ell(w). \]
The map \( a \) is called a \textit{sesquilinear} form since \( a \) is linear with respect to its first argument and antilinear with respect to its second argument, i.e., \( a(\lambda v, w) = \lambda a(v, w) \) and \( a(v, \lambda w) = \overline{\lambda} a(v, w) \) for all \( \lambda \in \mathbb{C} \) and all \((v, w) \in V \times W\) (the prefix sesqui means one and a half).

\[ \blacksquare \]

### Exercises

**Exercise 18.1 ((Non)-uniqueness).** Consider the domain \( D \) whose definition in polar coordinates is \( D = \{(r, \theta) \mid 0 < r < 1, \frac{\pi}{2} < \theta < 0\} \) with \( \alpha \in (-1, -\frac{1}{2}) \). Let \( \partial D_1 = \{(r, \theta) \mid r = 1, \frac{\pi}{2} < \theta < 0\} \) and \( \partial D_2 = \partial D \setminus \partial D_1 \).

Consider the following problem: \(-\Delta u = 0 \) in \( D \), \( u = \sin(\alpha \theta) \) on \( \partial D_1 \), and \( u = 0 \) on \( \partial D_2 \).

(i) Let \( \varphi_1 = r^{\alpha} \sin(\alpha \theta) \) and \( \varphi_2 = r^{-\alpha} \sin(\alpha \theta) \). Prove that \( \varphi_1 \) and \( \varphi_2 \) solve the above problem. \( (\text{Hint:} \text{ In polar coordinates, } \Delta \varphi = \frac{1}{r^2} \partial_r(r \partial_r \varphi) + \frac{1}{r^2} \partial_{\theta \theta} \varphi. ) \)

(ii) Prove that \( \varphi_1 \) and \( \varphi_2 \) are in \( L^2(D) \) if \( \alpha \in (-1, -\frac{1}{2}) \).

(iii) Consider the following problem: Find \( u \in H^1(D) \) such that \( u = \sin(\alpha \theta) \) on \( \partial D_1 \), \( u = 0 \) on \( \partial D_2 \), and \( \int_D \nabla u \cdot \nabla v = 0 \) for all \( v \in H^1_0(D) \). Prove that \( \varphi_2 \) solves this problem, but \( \varphi_1 \) does not. Comment.

**Exercise 18.2 (Weak formulations).** Prove Propositions 18.2 and 18.3.

**Exercise 18.3 (Darcy).** Derive a another variation on (18.12) and (18.14) with the functional spaces \( V = W = H(\text{div}; D) \times L^2(D) \). \( (\text{Hint: use Theorem B.113.)} \) Derive yet another variation with the functional spaces \( V = L^2(D) \times L^2(D) \) and \( W = H(\text{div}; D) \times H^1(D) \).

**Exercise 18.4 (Variational formulation).** Prove that \( u \) solves (18.8) if and only if \( u \) minimizes over \( H^1_0(D) \) the (so-called) energy functional

\[ \mathcal{E}(v) = \frac{1}{2} \int_D |\nabla v|^2 \, dx - \int_D f v \, dx. \]

\( (\text{Hint: show first that } \mathcal{E}(v + tw) = \mathcal{E}(v) + t \left\{ \int_D \nabla v \cdot \nabla w \, dx - \int_D f w \, dx \right\} + \frac{1}{2} t^2 \int_D |w|^2 \, dx \text{ for all } v, w \in H^1_0(D) \text{ and all } t \in \mathbb{R}. ) \)

**Exercise 18.5 (Derivative of primitive).** Let \( D = (0, 1) \). Prove that the following holds in the distribution sense for all \( f \) in \( L^1(D) \):

\[ \partial_x \left( \int_0^x f \, ds \right) = f. \]  

\( (\text{Hint: use the density result from Theorem B.36 and Lebesgue’s Dominated Convergence Theorem.}) \)
Exercise 18.6 (Biharmonic problem). Let $D$ be an open, bounded, set in $\mathbb{R}^d$ with smooth boundary. Derive a weak formulation for the biharmonic problem

$$
\Delta(\Delta u) = f \text{ in } D, \quad u = \partial_n u = 0 \text{ on } \partial D,
$$

with $f \in L^2(D)$. (Hint: use Theorem B.110.)

Exercise 18.7 (Weak and classical derivatives). Let $k \in \mathbb{N}$, $k \geq 1$, and let $v \in C^k(D)$. Prove that, up to the order $k$, the weak derivatives and the classical derivatives of $v$ coincide.
Solution to exercises

Exercise 18.1 ((Non)-uniqueness).

(i) Direct verification.
(ii) Since $\alpha < 0$, $\varphi_2 \in L^\infty(D)$, while $\varphi_1$ is in $L^2(D)$ if $2\alpha + 1 > -1$, i.e., if $\alpha > -1$.
(iii) Let us verify that $\varphi_2 \in H^1(D)$; the condition is $2(-\alpha - 1) + 1 > -1$, i.e., $\alpha < 0$, which is indeed satisfied. The same argument shows that $\varphi_1 \notin H^1(D)$. Hence, we can observe that by going from $L^2(D)$ to the smaller space $H^1(D)$, the above non-uniqueness of the solution disappears.

Exercise 18.2 (Weak formulations). Consider Proposition 18.2. Taking the test function $(\tau, 0)$ with $\tau$ arbitrary in $C_0^\infty(D)$ shows that $\sigma + \nabla p = 0$ a.e. in $D$. Take next the test function $(0, q)$ with $q$ arbitrary in $C_0^\infty(D)$ to infer that $\nabla \cdot \sigma = f$ a.e. in $D$. The boundary condition is explicitly enforced in the space for $p$. Consider now Proposition 18.3. The PDE $\sigma + \nabla p = 0$ a.e. in $D$ is recovered as before. Take next the test function $(0, q)$ with $q$ arbitrary in $C_0^\infty(D)$ to infer that $\sigma$ has a weak divergence in $L^2(D)$ and that $\nabla \cdot \sigma = f$ a.e. in $D$. The boundary condition on $p$ is explicitly enforced.

Exercise 18.3 (Darcy). Another variation with the functional spaces $V = W = H(\text{div}; D) \times L^2(D)$ is

$$\left\{ \begin{array}{l}
\text{Find } u := (\sigma, p) \in V \text{ such that } \\
\int_D (\sigma \cdot \tau - p \nabla \cdot \tau - q \nabla \cdot \sigma) \, dx = -\int_D f q \, dx, \quad \forall w := (\tau, q) \in W.
\end{array} \right.$$ 

Another variation with the functional spaces $V = L^2(D) \times L^2(D)$ and $W = H(\text{div}; D) \times H^1(D)$ is

$$\left\{ \begin{array}{l}
\text{Find } u := (\sigma, p) \in V \text{ such that } \\
\int_D (\sigma \cdot \tau - p \nabla \cdot \tau - \sigma \nabla q) \, dx = \int_D f q \, dx, \quad \forall w := (\tau, q) \in W.
\end{array} \right.$$ 

Exercise 18.4 (Variational formulation). The expanded formula for $\mathcal{E}(v + tw)$ is established by developing the various terms and re-ordering them as zeroth-, first- and second-order terms in $w$. Let $u$ solve (18.8). Then, taking $v = u$ and $t = 1$ in the expanded formula leads to $\mathcal{E}(u + w) \geq \mathcal{E}(u)$ for all $w \in H^1_0(D)$. This implies that $u$ minimizes $\mathcal{E}$ over $H^1_0(D)$. Conversely, assume that $u$ minimizes $\mathcal{E}$ over $H^1_0(D)$. Let $w \in H^1_0(D)$. The right-hand side of the expanded formula is a second-order polynomial in $t$ that is minimal at $t = 0$. Hence, the derivative of this polynomial vanishes at $t = 0$, which amounts to $\int_D \nabla v \cdot \nabla w \, dx - \int_D f w \, dx = 0$. Since $w$ is arbitrary in $H^1_0(D)$, $u$ solves (18.8).

Exercise 18.5 (Biharmonic problem). Theorem B.110 shows that $V := H^1_0(D) = \{ v \in H^2(D) \mid v = \partial n v = 0 \text{ on } \partial D \}$. Multiplying by a test function $v \in C_0^\infty(D)$, integrating over $D$, and using twice Green’s formula along with the boundary conditions leads to the following weak formulation:
Find \( u \in V \) such that
\[
\int_D \Delta u \Delta w \, dx = \int_D f w \, dx, \quad \forall w \in V.
\]

If \( u \) solves the above weak formulation, taking \( w \in C_0^\infty(D) \) shows that \( \Delta u \) has a weak Laplacian in \( L^2(D) \), and since \( f \in L^2(D) \), we infer that \( \Delta(\Delta u) = f \) a.e. in \( D \). The boundary conditions on \( u \) are explicitly enforced in \( V \).

**Exercise 18.6 (Derivative of primitive).** We use a density argument. Since \( C_0^\infty(D) \) is dense in \( L^1(D) \), there is a sequence \( (f_n)_{n \in \mathbb{N}} \) in \( C_0^\infty(D) \) that converges to \( f \) in \( L^1(D) \) and such that \( \|f_n\|_{L^1(D)} \leq 2\|f\|_{L^1(D)} \). Let \( \phi \in C_0^\infty(D) \). It is clear that \( \int_0^1 f_n \phi \, ds - \int_0^1 f \phi \, ds \leq (\sup_{x \in D} |\phi(x)|) \int_0^1 |f_n - f| \, ds \to 0 \). Likewise, \( (\int_0^x f_n \, ds)\phi'(x) \to (\int_0^x f \, ds)\phi'(x) \) a.e. in \( D \), and \( |(\int_0^x f_n \, ds)\phi'(x)| \leq 2|\phi'(x)| \int_0^1 |f| \, ds \). Lebesgue’s Dominated Convergence Theorem implies that \( \int_0^1 (\int_0^x f_n \, ds)\phi'(x) \, dx \to 0 \). Passing to the limit in the relation
\[
\int_0^1 \left( \int_0^x f_n \, ds \right) \phi'(x) \, dx = - \int_0^1 f_n(x)\phi(x) \, dx,
\]
yields
\[
\int_0^1 \left( \int_0^x f \, ds \right) \phi'(x) \, dx = - \int_0^1 f(x)\phi(x) \, dx,
\]
for all \( \phi \in C_0^\infty(D) \). This shows that (18.22) holds in the distribution sense.

**Exercise 18.7 (Weak and classical derivatives).** Let \( \alpha \in \mathbb{N}^d \) be a multi-index of length \( |\alpha| \leq k \). Let \( (\partial^\alpha v)_{cl}, (\partial^\alpha v)_{wk} \) denote the classical and weak derivatives, respectively; then, for all \( \varphi \in C_0^\infty(D) \),
\[
\langle (\partial^\alpha v)_{cl}, \varphi \rangle = \int_D (\partial^\alpha v)_{cl} \varphi \, dx \quad \text{(since } (\partial^\alpha v)_{cl} \in L^1_{loc}(D)) \nabla = (-1)^{|\alpha|} \int_D v \partial^\alpha \varphi \, dx \quad \text{(integration by parts)}
\]
\[
= (-1)^{|\alpha|} \langle v, \partial^\alpha \varphi \rangle = \langle (\partial^\alpha f)_{wk}, \varphi \rangle \quad \text{(since } v \in L^1_{loc}(D)) \text{.}
\]
There are no boundary terms when integrating by parts since \( (\partial^\beta \varphi)_{|\partial D} = 0 \) for any multi-index \( \beta \).