Suppose that we are given a mesh $\mathcal{T}_h$, i.e., a reference cell $\hat{K}$ and a collection of geometric maps $\{T_K\}_{K \in \mathcal{T}_h}$ such that $T_K : \hat{K} \to K$ as described in Chapter 3.

Our first goal in this chapter is to generate a finite element $(K, P, \Sigma)$ in each cell $K \in \mathcal{T}_h$ from a reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$. We will see that this task entails introducing a new ingredient, namely an isomorphism to transform functions defined in $K$ to functions defined in $\hat{K}$. This construction provides the cornerstone for the analysis of the interpolation error to be performed in Chapter 8. The second important result presented in this chapter identifies how the standard differential operators are transformed by the geometric map. These results are of fundamental importance in the rest of the book.

### 7.1 Geometric maps

In this book, we consider meshes where each cell $K$ is constructed from a fixed reference cell, say $\hat{K}$, and a geometric map $T_K : \hat{K} \to T_K(\hat{K}) = K$. We assume that $\hat{K}$ is a polyhedron (see Definition B.8) and we are going to construct $T_K$ so that the vertices, edges, and faces of $K$ are the images by $T_K$ of the vertices, edges, and faces of $\hat{K}$. For this purpose we introduce a scalar-valued Lagrange finite element $(\hat{K}, \hat{P}_\text{geo}, \hat{\Sigma}_\text{geo})$ with nodes $\{\hat{z}_i\}_{i \in N_\text{geo}}$ and shape functions $\{\hat{\psi}_i\}_{i \in N_\text{geo}}$ spanning the real vector space $\hat{P}_\text{geo}$ where $n_{\text{geo}} = \text{card}(\hat{\Sigma}_{\text{geo}})$. We assume for simplicity that all the mesh cells $K \in \mathcal{T}_h$ are generated from the same geometric reference finite element $(\hat{K}, \hat{P}_\text{geo}, \hat{\Sigma}_\text{geo})$.

It is standard to assume that $\hat{P}_\text{geo}$ is a space of $d$-variate polynomials and that there is an integer $k_{\text{geo}} \geq 1$ so that

$$\mathbb{P}_{k_{\text{geo}},d} \subset \hat{P}_\text{geo} \subset C^\infty(\hat{K}).$$

(7.1)
Definition 7.1 (Geometric element). The triple \((\widehat{K}, \widehat{P}_{geo}, \widehat{\Sigma}_{geo})\) is called the geometric reference finite element, \(\{\widehat{z}_i\}_{i \in \mathcal{N}_{geo}}\) the geometric reference nodes, and \(\{\widehat{\psi}_i\}_{i \in \mathcal{N}_{geo}}\) the geometric reference shape functions.

A mesh is a data structure produced by a mesh generator. This data structure always contains a list of points called geometric nodes or mesh vertices, say \(\mathcal{V}_h := \{z_1, \ldots, z_{N_{geo}}\}\), and assuming that the mesh is composed of \(N_{el}\) elements, the mesh generator provides a connectivity array relating nodes and cells. The connectivity array is a map \(j : \{1, \ldots, n_{geo}\} \times \{1, \ldots, N_{el}\} \rightarrow \{1, \ldots, N_{geo}\}\) such that the collection of points \(\{z_j(1, m), \ldots, z_j(n_{geo}, m)\}\) belong to the cell \(K_m, m \in \{1:N_{el}\}\). We then define the geometric map \(T_{K_m}\) by setting
\[
T_{K_m}(\widehat{x}) = \sum_{i \in \mathcal{N}_{geo}} z_j(i, m) \widehat{\psi}_i(\widehat{x}), \quad \forall \widehat{x} \in \widehat{K}.
\] (7.2)

Note that \(T_{K_m}(\widehat{z}_i) = z_j(i, m)\) for all \(i \in \mathcal{N}_{geo}\). In the book we sometimes abuse the notation and write \(z_j(i, K)\) or \(z_{K,i}\), i.e., we replace the enumeration index \(m\) by a generic cell \(K \in \mathcal{T}_h\).

![Fig. 7.1. From left to right and from top to bottom: P\(_1\) transformation of a triangle; P\(_2\) transformation of a triangle; P\(_1\) transformation of a square; Q\(_1\) transformation of a square.](image)

Example 7.2 (Generation of triangles and quadrangles). Figure 7.1 presents four examples in dimension 2: A geometric map based on the \(P_1\) (resp., \(P_2\)) Lagrange finite element maps a reference triangle to a triangle (resp., curved triangle). A geometric map based on the \(P_1\) (resp., \(Q_1\)) Lagrange finite element maps a reference square to a parallelogram (resp., general quadrangle).

We assume that the \(n_{geo}\)-tuples associated with the cells are chosen so that all the geometric maps are diffeomorphisms. Contrary to what is sometimes done in the literature, we do not require the Jacobian determinant of \(T_K\) to have a particular sign. The Jacobian determinant could be positive in one cell and negative in another.
7.2 Differential calculus

In this section we investigate how the geometric map $T_K : \hat{K} \to K$ transforms the usual differential operators ($\nabla$, $\nabla \cdot$, and $\nabla \times$) as well as normal and tangent vectors in $\hat{K}$. These results are of fundamental importance. For further reading, see, e.g., Marsden and Hughes [351, pp. 116-119], Ciarlet [143, p. 39], Monk [358, §3.9], Rognes et al. [404, p. 4134].

**Lemma 7.3 (Differential operators).** Let $\phi \in C^1(K; \mathbb{R})$ and $v \in C^1(K; \mathbb{R}^d)$ be a differentiable scalar field and a differentiable vector field on $K$, respectively. Then,

$$\nabla (\phi \circ T_K)(\hat{x}) = \mathbb{J}_K(\hat{x})^T (\nabla \phi)(T_K(\hat{x})), \quad (7.3a)$$

$$\nabla \cdot (\det(\mathbb{J}_K)^{-1}(v \circ T_K))(\hat{x}) = \det(\mathbb{J}_K(\hat{x})) (\nabla \cdot v)(T_K(\hat{x})), \quad (7.3b)$$

$$\nabla \times (\mathbb{J}_K^T(v \circ T_K))(\hat{x}) = \det(\mathbb{J}_K(\hat{x})) \mathbb{J}_K^{-1}(\hat{x}) (\nabla \times v)(T_K(\hat{x})). \quad (7.3c)$$

**Proof.** Recall that the usual convention consists of identifying vectors in $\mathbb{R}^d$ with column vectors. The geometric map $T_K : \hat{K} \to K$ is thus identified with the column vector with components $(T_K)_i$, for all $i \in \{1:d\}$, and the Jacobian of $T_K$ is identified with the Jacobian matrix with entries $(\mathbb{J}_K)_{ij} = \partial_j(T_K)_i$ for all $i, j \in \{1:d\}$, where $i$ is the row index and $j$ the column index. With this convention, the link between the Jacobian of $T_K$ and its Fréchet derivative (see Definition B.2.1) is that $\mathbb{D}T_K(h) = \mathbb{J}_K(\hat{x})h$ for all $h \in \mathbb{R}^d$.

1. Using Lemma B.13 (chain rule) with $n = 1$ yields

$$D(\phi \circ T_K)(\hat{x})(h) = D\phi(T_K(\hat{x}))(\mathbb{D}T_K(x)(h)) = D\phi(T_K(\hat{x}))(\mathbb{J}_K(\hat{x})h).$$

We denote by $(\cdot, \cdot)_{\mathbb{R}^d}$ the Euclidean scalar product in $\mathbb{R}^d$. Using the gradient to represent the Fréchet derivative yields (7.3a) since

$$(\nabla (\phi \circ T_K)(\hat{x}), h)_{\mathbb{R}^d} = D(\phi \circ T_K)(\hat{x})(h) = D\phi(T_K(\hat{x}))(\mathbb{J}_K(\hat{x})h)$$

$$= (\nabla \phi(T_K(\hat{x})), \mathbb{J}_K(\hat{x})h)_{\mathbb{R}^d} = (\mathbb{J}_K(\hat{x})^T \nabla \phi(T_K(\hat{x})), h)_{\mathbb{R}^d}.$$ 

(2) The identity (7.3b) is deduced from (7.3a) by duality. Let $q \in C^\infty_0(K; \mathbb{R})$ be a smooth scalar-valued function compactly supported on $K$. Then, integrating by parts and using (7.3a), we infer that

$$\int_{\hat{K}} (\nabla \cdot v)(T_K(\hat{x}))q(T_K(\hat{x})) \det(\mathbb{J}_K(\hat{x})) d\hat{x} = \int_K (\nabla \cdot v)(x)q(x) dx$$

$$= - \int_K (v, \nabla q)_{\mathbb{R}^d}(x) dx = - \int_K (v, \nabla q)_{\mathbb{R}^d}(T_K(\hat{x})) \det(\mathbb{J}_K(\hat{x})) d\hat{x}$$

$$= - \int_{\hat{K}} (v \circ T_K, \mathbb{J}_K^{-T}(\nabla q \circ T_K))_{\mathbb{R}^d}(\hat{x}) \det(\mathbb{J}_K(\hat{x})) d\hat{x}$$

$$= - \int_{\hat{K}} (\det(\mathbb{J}_K)^{-1}(v \circ T_K), \nabla(q \circ T_K))_{\mathbb{R}^d}(\hat{x}) d\hat{x}$$

$$= \int_{\hat{K}} \nabla \cdot(\det(\mathbb{J}_K)^{-1}(v \circ T_K))(\hat{x})q(T_K(\hat{x})) d\hat{x},$$
which proves (7.3b) since $q$ is arbitrary.

(3) We prove (7.3c) in $\mathbb{R}^3$. Let $\varepsilon$ be the Levi-Civita symbol ($\varepsilon_{ijk} = 0$ if at least two indices take the same value, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, and $\varepsilon_{132} = \varepsilon_{213} = -1$). Recall that $\det(J_K) = \varepsilon_{ijk}(J_K)_{ij}(J_K)_{2j}(J_K)_{3k} = \varepsilon_{ijk}(J_K)_{ij}(J_K)_{2j}(J_K)_{3k}$ and $(\nabla \times \psi)_i = \varepsilon_{ijk} \partial_j \psi_k$ with the Einstein convention on the summation of repeated indices. Let us compute $\nabla \times (J_K^{-1}(v \circ T_K))$:

$$
(J_K \nabla \times (J_K^{-1}(v \circ T_K)))_i = (J_K)_{ij} \varepsilon_{jkl} \partial_k((J_K^{-1})_{lm}(v_m \circ T_K))_l
= (J_K)_{ij} \varepsilon_{jkl} \partial_k((J_K^{-1})_{lm}(v_m \circ T_K))
= (J_K)_{ij} \varepsilon_{jkl} \partial_k((J_K^{-1})_{ml}(v_m \circ T_K)) + (J_K)_{ij} \varepsilon_{jkl} \partial_k(v_m \circ T_K).
$$

Let $I_1$ and $I_2$ be the two terms on the right-hand side of the above equality. Upon observing that $\partial_k(J_K)_{ml} = \partial_k(T_K)_m = \partial_k(J_K)_{ml} = \partial_k(J_K)_{nk}$, we infer that $I_1 = (J_K)_{ij} \varepsilon_{jkl} \partial_k((J_K^{-1})_{ml}(v_m \circ T_K)) = 0$. Noticing that $\varepsilon_{jkl}(J_K)_{ij}(J_K)_{nk} = \varepsilon_{inm} \det(J_K)$, $I_2$ is computed as follows:

$$
I_2 = (J_K)_{ij} \varepsilon_{jkl} \partial_k((J_K^{-1})_{ml}(v_m \circ T_K)(J_K)_{nk}
= \varepsilon_{jkl}(J_K)_{ij}(J_K)_{nk}(J_K)_{ml}((J_K^{-1})_{ml}(v_m \circ T_K))
= \varepsilon_{inm} \det(J_K)((J_K^{-1})_{ml}(v_m \circ T_K) = \det(J_K)(\nabla \times (v \circ T_K)_i).
\qedhere
$$

Remark 7.4 (Piola transformations). Let set $\psi^R_K(v) := v \circ T_K$, $\psi^L_K(v) := \det(J_K) J_K^{-1}(v \circ T_K)$, $\psi^C_K(v) := J_K^{-1}(v \circ T_K)$, and $\psi^L_K(\phi) := \det(J_K)(\phi \circ T_K)$ (the superscript refers to the identity operator), then Lemma 7.3 shows that the following holds for all $\hat{x} \in \hat{K}$:

$$
\nabla(\psi^R_K(\phi)) = \psi^R_K(\nabla \phi), \quad \nabla \times (\psi^L_K(v)) = \psi^L_K(\nabla \times v), \quad \nabla \cdot (\psi^C_K(v)) = \psi^C_K(\nabla \cdot v).
$$

The transformations $\psi^R_K, \psi^C_K, \psi^L_K, \psi^L_K$ will play an important role in the rest of the book.

Lemma 7.3 is useful to understand how $T_K$ transforms normal and tangent vectors on $\partial K$. The following important result will be used in Chapters 10 and 11 in the construction of vector-valued finite elements.

Lemma 7.5 (Normal and tangent vectors). Let $\hat{n}(\hat{x}), \hat{t}(\hat{x})$ be the outward unit normal vector and a unit tangent vector on $\partial K$ at $\hat{x}$, respectively. Then the following holds:

(i) The vector $J_K^{-1}\hat{n}(\hat{x})$ is an outward normal vector on $\partial K$.

(ii) The vector $J_K\hat{t}(\hat{x})$ is a tangent vector on $\partial K$; in particular, if $\hat{t}$ is tangent to the edge $E$ of $K$, $J_K\hat{t}$ is tangent to the edge $E = T_K(E)$ of $K$. 

\[\]
Proof. (1) Let \( \hat{F} \) be a face of \( \hat{K} \) and let \( \hat{x} \in \hat{F} \) in an open neighborhood in \( \hat{F} \) (to make sure that the normal vector at \( \hat{x} \) is singled-valued). Let \( \hat{n} \) be the outward unit normal at \( \hat{x} \). Let \( \hat{\psi} \) be the signed distance function to \( \hat{F} \), assumed to be negative inside \( \hat{K} \). Then \( \nabla \hat{\psi}(\hat{x}) = \hat{n}(\hat{x}) \). Define \( \psi(x) = \hat{\psi}(T_K^{-1}(x)) \). The normal at \( x \) is parallel to \( \nabla \psi \) since \( \psi \) is constant (equal to zero) over \( F = T_K(\hat{F}) \). Using (7.3a), we conclude that \( n \) is parallel to \( J_K^T \hat{n} \) and points outward \( K \).

(2) Item (ii) follows from (i) since \( (J_K^{-1} \hat{n}, J_K \hat{t})_{\ell^2(\mathbb{R}^d)} = 0 \). Assume that \( \hat{t} \) is parallel to the edge \( \hat{E} := \hat{F}_i \cap \hat{F}_j \), then \( J_K \hat{t} \) is tangent to both \( F_i := T_K(\hat{F}_i) \) and \( F_j := T_K(\hat{F}_j) \), hence \( J_K \hat{t} \) is parallel to \( F_i \cap F_j = E = T_K(\hat{E}) \).

Finally, we investigate how the geometric map \( T_K \) transforms surface and line measures. We denote by \( \| \cdot \|_{\ell^2(\mathbb{R}^d)} \) the Euclidean norm in \( \mathbb{R}^d \).

**Lemma 7.6 (Surface and line measures).**

(i) The surface measures on \( \partial K \) at \( x \) and on \( \partial \hat{K} \) at \( \hat{x} \) are transformed as follows: \( ds = | \det(J_K) | \| J_K^T \hat{n} \|_{\ell^2(\mathbb{R}^d)} ds, \ d\hat{s} = | \det(J_K) | \| J_K^T \hat{n} \|_{\ell^2(\mathbb{R}^d)} ds \).

(ii) Let \( E \) be an edge of \( K \). The line measures on \( E \) at \( x \) and on \( \hat{E} \) at \( \hat{x} \) are transformed as follows: \( dl = \| J_K \hat{t} \|_{\ell^2(\mathbb{R}^d)} dl, \ d\hat{l} = \| J_K^{-1} \hat{n} \|_{\ell^2(\mathbb{R}^d)} dl \).

**Proof.** (1) Let \( \hat{F} \) be a face of \( \hat{K} \) (recall that \( \hat{K} \) is a polyhedron). Let \( F = T_K(\hat{F}) \) and observe that \( n \) is smooth on \( F \). Let \( p \in C_0^\infty(F; \mathbb{R}) \) and let \( v \in C^\infty(K; \mathbb{R}^d) \) be such that \( v \cdot n \big|_F = p \) and \( v \cdot n \big|_{\partial K \setminus F} = 0 \) (this construction is always possible since \( p \) is compactly supported in \( F \) and so vanishes near \( \partial F \) where \( n \) is multivalued). Set \( \hat{p} = p \circ T_K \) and \( \hat{v} = \det(J_K) J_K^{-1} (v \circ T_K) \). Using (7.3b) together with \( \hat{n} = J_K^T (n \circ T_K) \| J_K^T \hat{n} \|_{\ell^2(\mathbb{R}^d)} \), we infer that

\[
\int_F p(x) \, ds = \int_{\partial K} (v \cdot n)(x) \, ds = \int_{\partial \hat{K}} (\nabla \cdot v)(\hat{x}) \, d\hat{s}
= \int_{\partial \hat{K}} (\nabla \hat{\psi})(\hat{x}) \, d\hat{s} = \int_{\partial \hat{K}} (\hat{v} \cdot \hat{n})(\hat{x}) \, d\hat{s}
= \int_{\partial \hat{K}} \left\{ (J_K^{-1} (v \circ T_K)) \cdot (J_K^T (n \circ T_K)) \| J_K^T \hat{n} \|_{\ell^2(\mathbb{R}^d)} \det(J_K) \right\} (\hat{x}) \, d\hat{s}
= \int_{\partial \hat{K}} \left\{ (v \cdot T_K) \cdot (n \circ T_K) \| J_K^{-1} \hat{n} \|_{\ell^2(\mathbb{R}^d)} \det(J_K) \right\} (\hat{x}) \, d\hat{s}
= \int_{\hat{F}} \hat{p}(T_K(\hat{x})) \left\{ \| J_K^{-1} \hat{n} \|_{\ell^2(\mathbb{R}^d)} \det(J_K) \right\} (\hat{x}) \, d\hat{s}.
\]

(2) For item (ii), see Exercise 7.3. \( \square \)
7.3 Finite element generation

Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a fixed a finite element. We show in this section how to generate a finite element on each cell $K = T_K(\hat{K})$ from $(\hat{K}, \hat{P}, \hat{\Sigma})$. The finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ should not be confused with the geometric finite element introduced in §7.1: The element $(\hat{K}, \hat{P}_{\text{geo}}, \hat{\Sigma}_{\text{geo}})$ is used for geometric purposes only, whereas $(\hat{K}, \hat{P}, \hat{\Sigma})$ will be used to interpolate $\mathbb{R}^q$-valued functions (for some integer $q \geq 1$). The number of shape functions and degrees of freedom in $(\hat{K}, \hat{P}, \hat{\Sigma})$ is denoted $n_{\text{sh}}$, and we define the integer set $N := \{1 : n_{\text{sh}}\}$.

Definition 7.7 (Reference element). $(\hat{K}, \hat{P}, \hat{\Sigma})$ is called the reference finite element, and, with obvious notation, $\{\hat{\sigma}_i\}_{i \in N}$ and $\{\hat{\theta}_i\}_{i \in N}$ are called the reference degrees of freedom and the reference shape functions, respectively.

Recalling Definition 4.6, we also assume that we have at hand a Banach space $V(\hat{K}) \subset L^1(\hat{K}; \mathbb{R}^d)$ such that $\hat{P} \subset V(\hat{K})$ and such that the linear forms in $\hat{\Sigma}$ can be extended to $\mathcal{L}(V(\hat{K}); \mathbb{R})$. Let $I_{\hat{K}} : V(\hat{K}) \to \hat{P}$ be the interpolation operator associated with $(\hat{K}, \hat{P}, \hat{\Sigma})$ defined as follows, see (4.7):

$$I_{\hat{K}}(\hat{v})(\hat{x}) = \sum_{i \in N} \hat{\sigma}_i(\hat{v})\hat{\theta}_i(\hat{x}), \quad \forall \hat{x} \in \hat{K}. \quad (7.5)$$

$I_{\hat{K}}$ is called the reference interpolation operator.

Since our goal is to generate a finite element $(K, P, \Sigma)$ on $K$ and to build an interpolation operator $I_K$ acting on functions defined on $K$, we introduce a counterpart of the space $V(\hat{K})$ for those functions, say $V(K)$. The new ingredient we need for the construction is a map

$$\psi_K : V(K) \to V(\hat{K}), \quad (7.6)$$
and for the purpose of analysis, we want this map to be a bounded isomorphism. A simple way of devising this map is to consider the pullback by the geometric map defined as $\psi_K(v) = v \circ T_K$ for all $v \in V(K)$. We will see in §7.4 that this definition is well-suited to nodal and modal finite elements. However, we will also see that this definition is not adequate when considering vector-valued functions for which the tangential or the normal component at the boundary of $K$ plays a specific role. This is the reason why we use an abstract notation for the functional map $\psi_K$.

Proposition 7.8 (Finite element generation). Let $K = T_K(\hat{K})$. Assume that there exists a Banach space $V(K)$ and a bounded isomorphism $\psi_K \in \mathcal{L}(V(K); V(\hat{K}))$. Then, setting

$$P := \psi_K^{-1}(\hat{P}) = \{ p = \psi_K^{-1}(\hat{p}) \mid \hat{p} \in \hat{P}\}, \quad (7.7a)$$
$$\Sigma := \{ \sigma_{K,i} \mid i \in N \text{ s.t. } \sigma_{K,i} = \hat{\sigma}_i \circ \psi_K \}, \quad (7.7b)$$

the triple $(K, P, \Sigma)$ is a finite element.
Proof. We apply Remark 4.4. Since \( \psi_K \) is bijective, \( \dim(P) = \dim(\hat{P}) = n_{sh} \). Moreover, a function \( p \in P \) such that \( \sigma_{K,i}(p) = 0 \) for all \( i \in \mathcal{N} \) is such that \( \hat{\sigma}_i(\psi_K(p)) = 0 \) for all \( i \in \mathcal{N} \), so that \( \psi_K(p) = 0 \) by the unisolvency property of the reference finite element; hence, local degrees of freedom are called \( \hat{\sigma}_i(\psi_K(p)) \) for all \( i \in \mathcal{N} \). Finally, the linear forms \( \sigma_{K,i} \) are in \( \mathcal{L}(V(K); \mathbb{R}) \) since \( |\sigma_{K,i}(v)| \leq \|\hat{\sigma}_i\|_{\mathcal{L}(V(K); \mathbb{R})}\|\psi_K\|_{\mathcal{L}(V(K); \mathbb{R})}\|v\|_{V(K)} \), for all \( v \in V(K) \).

Note that even if \( \hat{P} \) in (7.7) is a polynomial space of some degree, the space \( P = \psi_K^{-1}(\hat{P}) \) may not be a polynomial space. The linear forms \( \{\sigma_{K,i}\}_{i \in \mathcal{N}} \) are called local degrees of freedom. Moreover, the functions \( \theta_{K,i} \) such that \( \theta_{K,i} = \psi_K^{-1}(\hat{\theta}_i) \) for all \( i \in \mathcal{N} \) satisfy

\[
\sigma_{K,i}(\theta_{K,j}) = \hat{\sigma}_i(\psi_K(\theta_{K,j})) = \hat{\sigma}_i(\hat{\theta}_j) = \delta_{ij} \tag{7.8}
\]

and are called local shape functions. The local interpolation operator \( I_K : V(K) \to P \), with domain \( V(K) \) and codomain \( P \), acts as follows:

\[
I_K(v)(x) = \sum_{i \in \mathcal{N}} \sigma_{K,i}(v) \theta_{K,i}(x), \quad \forall x \in K. \tag{7.9}
\]

The following proposition plays a key role in the analysis of the interpolation error (see for instance the proof of Theorem 8.12).

**Proposition 7.9 (Commuting diagram).** We have \( I_K = \psi_K^{-1} \circ I_{\hat{K}} \circ \psi_K \), i.e., the following diagram commutes:

\[
\begin{array}{ccc}
V(K) & \xrightarrow{\psi_K} & V(\hat{K}) \\
| & I_K & | \\
P & \psi_K & \hat{P} \\
\end{array}
\]

As a result, \( P \) is pointwise invariant under \( I_K \), i.e., \( I_K p = p \) for all \( p \in P \).

**Proof.** Let \( v \in V(K) \). The definition (7.7) of \( (K, P, \Sigma) \) implies that

\[
I_{\hat{K}}(\psi_K(v)) = \sum_{i \in \mathcal{N}} \hat{\sigma}_i(\psi_K(v)) \hat{\theta}_i = \sum_{i \in \mathcal{N}} \sigma_{K,i}(v) \psi_K(\theta_{K,i}) = \psi_K(I_K(v)),
\]

owing to the linearity of \( \psi_K \). The pointwise invariance of \( P \) under \( I_K \) is a direct consequence of the commuting property and that of \( \hat{P} \) under \( I_{\hat{K}} \).

**Definition 7.10 (Iso/subparametric interpolation).** Let \( (\hat{K}, \hat{P}_{geo}, \hat{\Sigma}_{geo}) \) be the geometric reference finite element. Let \( (K, P, \Sigma) \) be the reference finite element. When \( \hat{P} = (\hat{P}_{geo})^q \) for some \( q \geq 1 \), the interpolation is said to be isoparametric, whereas it is said to be subparametric whenever \( (\hat{P}_{geo})^q \subsetneq \hat{P} \). The most common example of subparametric interpolation is \( \mathbb{P}_1 \subset \hat{P}_{geo} \neq \mathbb{P}_2 \subset \hat{P} \) (for \( m = 1 \)).
7.4 Examples

7.4.1 Lagrange finite elements

Let \((\hat{K}, \hat{P}, \hat{\Sigma})\) be a Lagrange finite element with \(V(\hat{K}) = C^0(\hat{K}; \mathbb{R}^q)\). Set \(V(K) = C^0(K; \mathbb{R}^q)\). The map \(\psi_K^\hat{\cdot} : V(K) \to V(\hat{K})\) such that

\[
\psi_K^\hat{\cdot}(v) := v \circ T_K,
\]

is an isomorphism in \(\mathcal{L}(V(K); V(\hat{K}))\). This map is called the pullback by the geometric map \(T_K\). The superscript refers to the gradient operator, see in particular Remark 7.4. The finite element \((K, P, \Sigma)\) constructed in Proposition 7.8 using \(\psi_K^\hat{\cdot}\) is also a Lagrange finite element. Indeed, denoting by \(\{\hat{a}_i\}_{i \in \mathcal{N}}\) the Lagrange nodes of \((\hat{K}, \hat{P}, \hat{\Sigma})\), we observe that \(\sigma_i(v) = \hat{\sigma}_i(\psi_K^\hat{\cdot}(v)) = \psi_K^\hat{\cdot}(v)(\hat{a}_i) = (v \circ T_K)(\hat{a}_i)\). Thus, upon setting \(a_{K,i} = T_K(\hat{a}_i)\) for all \(i \in \mathcal{N}\), the above equality shows that \(\{a_{K,i}\}_{i \in \mathcal{N}}\) are the Lagrange nodes of \((K, P, \Sigma)\). The Lagrange interpolant acts as follows:

\[
I_K^L(v)(x) = \sum_{i \in \mathcal{N}} v(a_{K,i}) \theta_{K,i}(x), \quad \forall x \in K.
\]

Note that even if \(\hat{P}\) is a polynomial space, \(P = \{\hat{p} \circ T_K^{-1}, \hat{p} \in \hat{P}\}\) is not necessarily a polynomial space of the same degree unless \(T_K\) is affine.

7.4.2 Modal finite elements

Let \((\hat{K}, \hat{P}, \hat{\Sigma})\) be the modal finite element introduced in §4.4.2; recall that the degrees of freedom are such that \(\hat{\sigma}_i(\hat{p}) = \frac{1}{|\hat{K}|} \int_{\hat{K}} \hat{\zeta}_i \hat{p} \, d\hat{x}\) for all \(\hat{p} \in \hat{P}\) and all \(i \in \mathcal{N}\), where \(\{\hat{\zeta}_i\}_{i \in \mathcal{N}}\) is a basis of \(\hat{P}\). The domain of the interpolation operator is \(V(\hat{K}) = L^1(\hat{K})\). Set \(V(K) = L^1(K)\), and observe that the map \(\text{map}Kg\) defined in (7.10) is an isomorphism in \(\mathcal{L}(V(K); V(\hat{K}))\). The finite element \((K, P, \Sigma)\) constructed in Proposition 7.8 using \(\psi_K^\hat{\cdot}\) is a modal finite element. Assuming that the map \(T_K\) is affine, its degrees of freedom are

\[
\sigma_{K,i}(p) = \hat{\sigma}_i(\psi_K^\hat{\cdot}(p)) = \frac{1}{|K|} \int_{\hat{K}} (p \circ T_K) \hat{\zeta}_i \, d\hat{x}
\]

\[
= \frac{1}{|K|} \int_{\hat{K}} (p \circ T_K)(\zeta_{K,i} \circ T_K) \, d\hat{x} = \frac{1}{|K|} \int_{K} \zeta_{K,i} p \, dx,
\]

for all \(p \in P\) and all \(i \in \mathcal{N}\), where we have set \(\zeta_{K,i} = \hat{\zeta}_i \circ T_K^{-1}\).

7.4.3 Piola transformations

The simple choice (7.10) for the map \(\psi_K\) is not suitable when working with vector-valued finite elements where the tangential and normal components
play specific roles. For instance, let \( \hat{K} \) be the triangle with vertices \((0, 0), (1, 0), \) and \((0, 1)\), see Figure (7.2). Let \( K \) be the image of \( \hat{K} \) by the geometric map \( T_K \) defined as the rotation of center \((0, 0)\) and of angle \( \frac{\pi}{2} \). Let \( \hat{F}_1 \) (resp., \( \hat{F}_2 \)) be the edge of \( \hat{K} \) corresponding to \( x_2 = 0 \) (resp., \( x_1 = 0 \)), and let \( F_1 \) and \( F_2 \) be the images of \( \hat{F}_1 \) and \( \hat{F}_2 \) by \( T_K \), respectively. Consider the constant vector-valued function \( v(x, y) = (1, 0) \) (v is invariant under the pullback by \( T_K \)). Then, \( v \) is tangent to \( F_2 \) but \( v \circ T_K \) is normal to \( \hat{F}_2 \), whereas \( v \) is normal to \( F_1 \) but \( v \circ T_K \) is tangent to \( \hat{F}_1 \).

In the above context, it is important to use the so-called Piola transformations, namely the contravariant Piola transformation

\[
\psi^d_K(v) := \det(J_K)J_K^{-1}(v \circ T_K) \tag{7.12}
\]

which is important to preserve the normal component, and the covariant Piola transformation

\[
\psi^c_K(v) := J_K^T(v \circ T_K) \tag{7.13}
\]

which is important to preserve the tangential component. Indeed, returning to the above example, one can verify that \( \psi^d_K(v) = (0, -1) \) which is tangent to \( \hat{F}_2 \), while \( \psi^d_K(v) = (0, -1) \) which is normal to \( \hat{F}_1 \). The superscripts in (7.12)-(7.13) refer to the divergence and curl operators, respectively; see again Remark 7.4. Finally, for later use, we define \( \psi_i^I_K(\phi) := \det(J_K)(\phi \circ T_K) \) (the superscript refers to the identity operator).

**Exercises**

**Exercise 7.1 (Canonical hybrid element).** Consider an affine geometric map \( T_K \) and the pullback by \( T_K \) for \( \psi_K \). Let \((\hat{K}, \hat{P}, \hat{Σ})\) be the canonical hybrid element of §6.4. Verify that Proposition 7.8 generates the canonical hybrid element in \( K \). Write the degrees of freedom.

**Exercise 7.2 (Piola identity).** Defining the divergence of a second-order tensor field \( K \) to be the vector field with entries \( \sum_{j=1}^d \partial_j K_{ij} \), \( i \in \{1:d\} \), show that \( \nabla \cdot (\det(J_K)J_K^{-T}) = 0 \) in \( \mathbb{R}^d \). (This is an important property in elasticity theory; see Ciarlet [143, p. 37-40].)

**Exercise 7.3 (Line measure).** Prove Lemma 7.6(ii). (Hint: use (7.3c).)

**Exercise 7.4 (Surface measure).** Let \( T_F = T_K|_{\mathcal{F}} : \hat{F} \rightarrow F \). Let \( \mathcal{J}_F(\hat{x}) \in \mathbb{R}^{d \times (d-1)} \) be the Jacobian matrix representing the (Fréchet) derivative \( DT_F(\hat{x}) \).
Let $g_F(\hat{x}) = (J_F(\hat{x}))^T J_F(\hat{x}) \in \mathbb{R}^{(d-1) \times (d-1)}$ be the surface metric tensor for all $\hat{x} \in \hat{F}$. Prove that $\sqrt{\det(g_F(\hat{x}))} = \det(J_K) \|J_K^T \hat{n}\|_{L^2(\mathbb{R}^d)}$. (Hint: ??)

Exercise 7.5 (Sobolev spaces). Prove that $\psi^S_K$ is a bounded isomorphism from $H^1(K)$ onto $H^1(\hat{K})$, that $\psi^c_K$ is a bounded isomorphism from $H(\text{curl}; K)$ onto $H(\text{curl}; \hat{K})$, and that $\psi^d_K$ is a bounded isomorphism from $H(\text{div}; K)$ onto $H(\text{div}; \hat{K})$. 