Part III, Chapter 12

Local inverse and functional inequalities

Inverse inequalities rely on the fact that all the norms are equivalent in finite-dimensional normed vector spaces, e.g., in the local (polynomial) space P_K generated from the reference finite element. The term 'inverse' refers to the fact that high-order Sobolev (semi)norms are bounded by lower-order (semi)norms, but the constants involved in these estimates either tend to zero or to infinity as the mesh size goes to zero. Our purpose is then to study how the norm-equivalence constants depend on the local meshsize and the polynomial degree of the reference finite element. We also derive some local functional inequalities valid in infinite-dimensional spaces. All of these inequalities are of fundamental importance. In the whole chapter, we consider the same setting as in Chapter 11, i.e., $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is the reference finite element, $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape regular sequence of affine meshes, $\boldsymbol{T}_K : \widehat{K} \to K$ is the geometric mapping for every mesh cell $K \in \mathcal{T}_h$, and the local finite element (K, P_K, Σ_K) is generated by using the transformation $\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K)$ with $\mathbb{A}_K \in \mathbb{R}^{q\times q}$ s.t. $\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2} \le c$ (which follows from (11.12) and the regularity of the mesh sequence).

12.1 Inverse inequalities in cells

Lemma 12.1 (Bound on Sobolev seminorm). *Let $l \in \mathbb{N}$ be s.t. $\widehat{P} \subset W^{l,\infty}(\widehat{K};\mathbb{R}^q)$. There is c s.t. for every integer $m \in \{0{:}l\}$, all $p, r \in [1,\infty]$, all $v \in P_K$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, the following holds true:*

$$|v|_{W^{l,p}(K;\mathbb{R}^q)} \le c\, h_K^{m-l+d(\frac{1}{p}-\frac{1}{r})} |v|_{W^{m,r}(K;\mathbb{R}^q)}. \tag{12.1}$$

Proof. (1) Since all the norms in the finite-dimensional space $\widehat{P}$ are equivalent, there exists $\widehat{c}$, only depending on $\widehat{K}$ and l, such that $\|\widehat{v}\|_{W^{l,\infty}(\widehat{K};\mathbb{R}^q)} \le \widehat{c}\,\|\widehat{v}\|_{L^1(\widehat{K};\mathbb{R}^q)}$ for all $\widehat{v} \in \widehat{P}$, which in turn means that for all $p, r \in [1,\infty]$,

$$\|\widehat{v}\|_{W^{l,p}(\widehat{K};\mathbb{R}^q)} \le \widehat{c}\,\|\widehat{v}\|_{L^r(\widehat{K};\mathbb{R}^q)}, \qquad \forall \widehat{v} \in \widehat{P}. \tag{12.2}$$

(2) Let now $v \in P_K$. Since $P_K := \psi_K^{-1}(\widehat{P})$, $\widehat{v} := \psi_K(v)$ is in $\widehat{P}$. Let $j \in \{0{:}l\}$. Using Lemma 11.7, (12.2), the assumption $\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2} \le c$, and the regularity of the mesh sequence implies that (the value of c changes at each occurrence)

$$\begin{aligned}
|v|_{W^{j,p}(K;\mathbb{R}^q)} &\le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^{j}|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,\|\widehat{v}\|_{W^{j,p}(\widehat{K};\mathbb{R}^q)} \\
&\le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^{j}|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,\|\widehat{v}\|_{L^{r}(\widehat{K};\mathbb{R}^q)} \\
&\le c\,\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^{j}|\det(\mathbb{J}_K)|^{\frac{1}{p}-\frac{1}{r}}\|v\|_{L^{r}(K;\mathbb{R}^q)} \\
&\le c\,h_K^{-j+d(\frac{1}{p}-\frac{1}{r})}\|v\|_{L^{r}(K;\mathbb{R}^q)}.
\end{aligned}$$

Taking $j = l$ proves (12.1) for $m = 0$.
(3) Let now $m \in \{0{:}l\}$. Let α be a multi-index of length l, i.e., $|\alpha| = l$. One can find two multi-indices β and γ such that $\alpha = \beta + \gamma$ with $|\gamma| = m$ and $|\beta| = l - m$. It follows from Step (2) that

$$\begin{aligned}
&\|\partial^{\alpha} v\|_{L^p(K;\mathbb{R}^q)} = \|\partial^{\beta}(\partial^{\gamma} v)\|_{L^p(K;\mathbb{R}^q)} \le |\partial^{\gamma} v|_{W^{l-m,p}(K;\mathbb{R}^q)} \\
&\le c\,h_K^{m-l+d(\frac{1}{p}-\frac{1}{r})}\|\partial^{\gamma} v\|_{L^r(K;\mathbb{R}^q)} \le c\,h_K^{m-l+d(\frac{1}{p}-\frac{1}{r})}|v|_{W^{m,r}(K;\mathbb{R}^q)},
\end{aligned}$$

which proves (12.1) for every integer $m \in \{0{:}l\}$. □

Remark 12.2 (Scale invariance). Inverse inequalities are invariant under a dilatation of K by a factor $\lambda > 0$. Indeed the left-hand side of (12.1) scales as $\lambda^{-l+\frac{d}{p}}$ and the right-hand side as $\lambda^{m-l+d(\frac{1}{p}-\frac{1}{q})}\lambda^{-m+\frac{d}{q}} = \lambda^{-l+\frac{d}{p}}$. This fact is useful to verify the correctness of the exponent of h_K in (12.1). □

Example 12.3 (Bound on gradient). Lemma 12.1 (with $l = 1$, $m = 0$) yields

$$\|\nabla v\|_{\boldsymbol{L}^p(K;\mathbb{R}^q)} \le c\,h_K^{-1}\|v\|_{L^p(K;\mathbb{R}^q)},$$

for all $p \in [1,\infty]$, all $v \in P_K$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$. □

Example 12.4 (Comparison of L^p-norms). Using $m = l = 0$, Lemma 12.1 yields

$$\|v\|_{L^p(K;\mathbb{R}^q)} \le c\,h_K^{d(\frac{1}{p}-\frac{1}{r})}\|v\|_{L^r(K;\mathbb{R}^q)}, \tag{12.3}$$

for all $p, r \in [1,\infty]$, all $v \in P_K$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$.

Proposition 12.5 (dof norm). *There is c s.t.*

$$c\,\|v\|_{L^p(K;\mathbb{R}^q)} \le |K|^{\frac{1}{p}}\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\Big(\max_{i\in\mathcal{N}}|\sigma_{K,i}(v)|\Big) \le c^{-1}\,\|v\|_{L^p(K;\mathbb{R}^q)}, \tag{12.4}$$

for all $p \in [1,\infty]$, all $v \in P_K$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$.

Proof. See Exercise 12.3. □

Example 12.6 (dof norm). For a Lagrange finite element with nodes $(\boldsymbol{a}_{K,i})_{i\in\mathcal{N}}$, $\|v\|_{L^p(K;\mathbb{R}^q)}$ is uniformly equivalent to

$$h_K^{\frac{d}{p}} \max_{i\in\mathcal{N}} \|v(\boldsymbol{a}_{K,i})\|_{\ell^2(\mathbb{R}^q)},$$

where $|K|^{\frac{1}{p}}$ has been replaced by $h_K^{\frac{d}{p}}$ owing to regularity of the mesh sequence. For the Raviart–Thomas $\mathbb{RT}_{k,d}$ element (see Chapter 14), inspection of the dofs shows that $\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)}$ is uniformly equivalent to

$$h_K^{\frac{1}{p}} \max_{F\in\mathcal{F}_K} \|\boldsymbol{v}\cdot\boldsymbol{n}_F\|_{L^p(F)} + \|\Pi_K^{k-1}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)},$$

where $\boldsymbol{n}_F$ is the unit normal vector orienting the face F of K, and Π_K^{k-1} is the $\boldsymbol{L}^2(K)$-orthogonal projection onto $\mathbb{P}_{k-1,d}$ $(k \geq 1)$. For the Nédélec $\mathbb{N}_{k,d}$ element (see Chapter 15), $\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)}$ is uniformly equivalent to

$$h_K^{\frac{2}{p}} \max_{E\in\mathcal{E}_K} \|\boldsymbol{v}\cdot\boldsymbol{\tau}_E\|_{L^p(E)} + h_K^{\frac{1}{p}} \max_{F\in\mathcal{F}_K} \|\Pi_K^{k-1}(\boldsymbol{v})\times\boldsymbol{n}_F\|_{\boldsymbol{L}^p(F)} + \|\Pi_K^{k-2}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)},$$

where $\boldsymbol{\tau}_E$ is the unit tangent vector orienting the edge E of K and Π_K^{k-2} is the $\boldsymbol{L}^2(K)$-orthogonal projection onto $\mathbb{P}_{k-2,d}$ $(k \geq 2)$. □

Sharp estimates of the constant c appearing in the above inverse inequalities can be important in various contexts. For instance the *hp*-finite element analysis requires to know how c behaves with respect to the polynomial degree; see, e.g., Schwab [169]. It turns out that estimating c in terms of the polynomial degree can be done in some particular cases. One of the earliest known inverse inequalities with a sharp estimate on c is the Markov inequality proved in the 1890s by Andrey Markov and Vladimir Markov for univariate polynomials over the interval $[-1, 1]$.

Lemma 12.7 (Markov inequality). *Let $k, l \in \mathbb{N}$ with $l \leq k$ and $k \geq 1$. The following holds true for every univariate polynomial $v \in \mathbb{P}_{k,1}$:*

$$\|v^{(l)}\|_{L^\infty(-1,1)} \leq C_{\infty,k,l}\|v\|_{L^\infty(-1,1)}, \tag{12.5}$$

with $C_{\infty,k,l} := \frac{k^2(k^2-1^2)\ldots(k^2-(l-1)^2)}{1\cdot3\ldots(2l-1)}$.

The case $l = 1$ in (12.5) gives $\|v'\|_{L^\infty(-1,1)} \leq C_{\infty,k}\|v\|_{L^\infty(-1,1)}$ with $C_{\infty,k} := k^2$. This type of result can be extended to the multivariate case in any dimension. In particular it is shown in Wilhelmsen [191] that

$$\|\nabla v\|_{\boldsymbol{L}^\infty(\widehat{K})} \leq \frac{4k^2}{\mathrm{width}(\widehat{K})}\|v\|_{L^\infty(\widehat{K})}, \qquad \forall v \in \mathbb{P}_{k,d}, \tag{12.6}$$

for all compact convex sets $\widehat{K}$ in $\mathbb{R}^d$ with non-empty interior, where width$(\widehat{K})$ is the width of $\widehat{K}$, i.e., the minimal distance between two parallel supporting hyperplanes of $\widehat{K}$; see also Kroó and Révész [124].

Results are also available for the L^2-Markov inequality in the univariate and multivariate cases; see Harari and Hughes [108], Schwab [169], Kroó [123], Özişik et al. [147]. For instance it is shown in [169, Thm. 4.76] that

$$\|v'\|_{L^2(-1,1)} \le C_{2,k}\|v\|_{L^2(-1,1)}, \qquad \forall v \in \mathbb{P}_{k,1}, \tag{12.7}$$

with $C_{2,k} := k((k+1)(k+\frac{1}{2}))^{\frac{1}{2}}$. Sharp estimates of the constant $C_{2,k}$ can be derived by computing the largest eigenvalue of the stiffness matrix $\mathcal{A}$ of order $(k+1)$ with entries $\mathcal{A}_{mn} := \int_{-1}^{1}(\tilde{L}_m)'(t)(\tilde{L}_n)'(t)\,\mathrm{d}t$ for all $m,n \in \{0{:}k\}$, where $\tilde{L}_m := \big(\frac{2m+1}{2}\big)^{\frac{1}{2}} L_m$, L_m being the Legendre polynomial from Definition 6.1, i.e., $\{\tilde{L}_m\}_{m\in\{0:k\}}$ is an L^2-orthonormal basis of $\mathbb{P}_{k,1}$. For instance it is found in Özişik et al. [147] that $C_{2,1} = 3$, $C_{2,2} = 15$, $C_{2,3} = \frac{45+\sqrt{1605}}{2}$, and $C_{2,4} = \frac{105+3\sqrt{805}}{2}$. The multivariate situation is slightly more complicated, but when $\widehat{K}$ is the unit triangle or the unit square, it is shown in [169] that

$$\|\nabla v\|_{\boldsymbol{L}^2(\widehat{K})} \le c\,k^2\|v\|_{L^2(\widehat{K})}, \qquad \forall v \in \mathbb{P}_{k,2}, \tag{12.8}$$

where c is uniform with respect to k. By numerically evaluating the largest eigenvalue of the stiffness matrix assembled from an L^2-orthonormal basis of $\mathbb{P}_{k,2}$ on the reference triangle $\widehat{K}$, it is shown in [147] that

$$\|\nabla v\|_{\boldsymbol{L}^2(K)} \le C_{2,k}\frac{|\partial K|}{|K|}\|v\|_{L^2(K)}, \qquad k \in \{1,2,3,4\}, \tag{12.9}$$

for every triangle K, with $C_{2,1} := \sqrt{6} \sim 2.449$, $C_{2,2} := 3\sqrt{\frac{5}{2}} \sim 4.743$, $C_{2,3} \sim 7.542$, and $C_{2,4} \sim 10.946$. Values of $C_{2,k}$ for tetrahedra with $k \in \{1{:}4\}$ are also given in [147].

12.2 Inverse inequalities on faces

Let $\mathcal{F}_K$ be the collection of the faces of a mesh cell $K \in \mathcal{T}_h$.

Lemma 12.8 (Discrete trace inequality). *Assume that $\widehat{P} \subset L^\infty(\widehat{K};\mathbb{R}^q)$. There is c s.t. the following holds true:*

$$\|v\|_{L^p(F;\mathbb{R}^q)} \le c\,h_K^{-\frac{1}{p}+d(\frac{1}{p}-\frac{1}{r})}\|v\|_{L^r(K;\mathbb{R}^q)}, \tag{12.10}$$

for all $p,r \in [1,\infty]$, all $v \in P_K$, all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_K$, and all $h \in \mathcal{H}$.

Proof. Let $\widehat{v} := \psi_K(v)$. Then $\|v\|_{L^p(F;\mathbb{R}^q)} \le \|\mathbb{A}_K^{-1}\|_{\ell^2} \left(\frac{|F|}{|\widehat{F}|}\right)^{\frac{1}{p}} \|\widehat{v}\|_{L^p(\widehat{F};\mathbb{R}^q)}$. Using norm equivalence in $\widehat{P}$, we infer that $\|\widehat{v}\|_{L^p(\widehat{F};\mathbb{R}^q)} \le \widehat{c}\|\widehat{v}\|_{L^p(\widehat{K};\mathbb{R}^q)}$. Hence, $\|v\|_{L^p(F;\mathbb{R}^q)} \le c'\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2} \left(\frac{|F|}{|\widehat{F}|}\frac{|\widehat{K}|}{|K|}\right)^{\frac{1}{p}} \|v\|_{L^p(K;\mathbb{R}^q)}$. The regularity of the mesh sequence yields (12.10) for $p = r$. The result for $r \neq p$ follows from (12.3). $\square$

Again, it can be important to have an accurate estimate of the constant c appearing in the discrete trace inequality (12.10). For instance this constant is invoked to determine a minimal threshold on the stability parameter that is used to enforce boundary conditions weakly in the boundary penalty method and the discontinuous Galerkin method for elliptic PDEs; see Chapters 37 and 38. It is indeed possible to estimate c in the Hilbertian setting (with $p = q = 2$), when K is a simplex or a cuboid. We start with the case of the cuboid; see Canuto and Quarteroni [55], Bernardi and Maday [22].

Lemma 12.9 (Discete trace inequality in cuboid). *Let K be a cuboid in $\mathbb{R}^d$ and let $F \in \mathcal{F}_K$. The following holds true for all $v \in \mathbb{Q}_{k,d}$:*

$$\|v\|_{L^2(F)} \le (k+1)|F|^{\frac{1}{2}}|K|^{-\frac{1}{2}}\|v\|_{L^2(K)}. \tag{12.11}$$

Proof. We first consider the reference hypercube $\widehat{K} = [-1,1]^d$ and the face $\widehat{F} := \{\widehat{x}_d = -1\}$. Recall the rescaled Legendre polynomials $\tilde{L}_m := \left(\frac{2m+1}{2}\right)^{\frac{1}{2}} L_m$, i.e., $\{\tilde{L}_m\}_{m\in\{0:k\}}$ is an L^2-orthonormal basis of $\mathbb{Q}_{k,1} = \mathbb{P}_{k,1}$. An L^2-orthonormal basis of $\mathbb{Q}_{k,d}$ is obtained by constructing the tensor product of this one-dimensional basis. Let $\widehat{v} \in \mathbb{Q}_{k,d}$ and write

$$\widehat{v}(\widehat{\boldsymbol{x}}) = \sum_{i_1=0}^{k} \dots \sum_{i_d=0}^{k} \widehat{v}_{i_1\dots i_d} \tilde{L}_{i_1}(\widehat{x}_1)\dots\tilde{L}_{i_d}(\widehat{x}_d).$$

Let $V \in \mathbb{R}^{(k+1)^d}$ be the coordinate vector of $\widehat{v}$ in this tensor-product basis. Using orthonormality we infer that

$$\int_{\widehat{F}} \widehat{v}(\widehat{\boldsymbol{x}})^2 \,\mathrm{d}\widehat{s} = V^{\mathsf{T}}\mathcal{T}V,$$

where the $(k+1)^d \times (k+1)^d$ symmetric matrix $\mathcal{T}$ is block-diagonal with $(k+1)^{d-1}$ diagonal blocks all equal to the rank-one matrix $\mathcal{U} := UU^{\mathsf{T}}$ where $U = (\tilde{L}_0(-1), \dots, \tilde{L}_k(-1))^{\mathsf{T}}$. As a result the largest eigenvalue of $\mathcal{T}$ is

$$\lambda_{\max}(\mathcal{T}) = \lambda_{\max}(\mathcal{U}) = \|U\|^2_{\ell^2(\mathbb{R}^{k+1})} = \sum_{m=0}^{k} \frac{2m+1}{2} = \frac{(k+1)^2}{2}.$$

Since $V^{\mathsf{T}}V = \|\widehat{v}\|^2_{L^2(\widehat{K})}$ by orthonormality of the basis, we infer that

$$\|\widehat{v}\|^2_{L^2(\widehat{F})} \le \lambda_{\max}(\mathcal{T})\|\widehat{v}\|^2_{L^2(\widehat{K})} = \tfrac{1}{2}(k+1)^2\|\widehat{v}\|^2_{L^2(\widehat{K})}.$$

Finally we obtain (12.11) by mapping the above estimate back to the cuboid K and by observing that $|\widehat{K}| = 2|\widehat{F}|$. □

Lemma 12.10 (Discete trace inequality in simplices). *Let K be a simplex in $\mathbb{R}^d$ and let $F \in \mathcal{F}_K$. The following holds true for all $v \in \mathbb{P}_{k,d}$:*

$$\|v\|_{L^2(F)} \le \big((k+1)(k+d)d^{-1}\big)^{\frac{1}{2}} |F|^{\frac{1}{2}}|K|^{-\frac{1}{2}}\|v\|_{L^2(K)}. \tag{12.12}$$

Proof. See Warburton and Hesthaven [187]. □

12.3 Functional inequalities in meshes

This section presents two important functional inequalities: the Poincaré–Steklov inequality for functions having zero mean-value over a given mesh cell and the multiplicative trace inequality for functions having a trace at the boundary of a mesh cell.

12.3.1 Poincaré–Steklov inequality in cells

Lemma 12.11 (Poincaré–Steklov). *Let $K \in \mathcal{T}_h$ and assume that K is a convex set. Then for all $v \in H^1(K)$ with $\underline{v}_K := \frac{1}{|K|}\int_K v\,\mathrm{d}x$, we have*

$$\|v - \underline{v}_K\|_{L^2(K)} \le \pi^{-1} h_K |v|_{H^1(K)}. \tag{12.13}$$

Proof. This is a paraphrase of Lemma 3.24. □

Lemma 12.12 (Fractional Poincaré–Steklov). *Let $p \in [1,\infty)$, $r \in (0,1)$, and let $K \in \mathcal{T}_h$. Then for all $v \in W^{r,p}(K)$ with $\underline{v}_K := \frac{1}{|K|}\int_K v\,\mathrm{d}x$, we have*

$$\|v - \underline{v}_K\|_{L^p(K)} \le h_K^s \left(\frac{h_K^d}{|K|}\right)^{\frac{1}{p}} |v|_{W^{r,p}(K)}. \tag{12.14}$$

Proof. This is a paraphrase of Lemma 3.26. □

Corollary 12.13 (Polynomial approximation). *Assume that the mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is shape regular. Let $k \in \mathbb{N}$. There is c s.t. the following holds true for every real numbers $r \in [0, k+1]$ and $p \in [1,\infty)$ if $r \notin \mathbb{N}$ or $p \in [1,\infty]$ if $r \in \mathbb{N}$, every integer $m \in \{0{:}\lfloor r\rfloor\}$ (here $\lfloor r\rfloor$ denotes the largest integer $n \in \mathbb{N}$ s.t. $n \le r$), all $v \in W^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$:*

$$\inf_{q\in\mathbb{P}_{k,d}} |v - q|_{W^{m,p}(K)} \le c\, h_K^{r-m} |v|_{W^{r,p}(K)}, \tag{12.15}$$

where the mesh cells are supposed to be convex sets if $r \ge 1$.

Proof. If $m = r$, there is nothing to prove, so let us assume that $m < r$. If $r \in (0,1)$, we have $m = 0$, and (12.15) follows from the fractional Poincaré–Steklov (12.14) and the regularity of the mesh sequence. If $r = 1$, we only need to consider the case $m = 0$ (since otherwise $m = 1 = r$), and (12.15) follows from the Poincaré–Steklov inequality (12.13) and the convexity of K. If $k = 0$, the proof is complete. Otherwise, $k \geq 1$ and let us assume now that $r > 1$. Let $\ell \in \mathbb{N}$ be s.t. $\ell := \lceil r \rceil - 1$ (here $\lceil r \rceil$ denotes the smallest integer $n \in \mathbb{N}$ s.t. $n \geq r$). Notice that we have $m \leq \ell \leq k$ and $1 \leq \ell$. The key idea is to take $q := \pi_\ell(v) \in \mathbb{P}_{\ell,d} \subset \mathbb{P}_{k,d}$ since $\ell \leq k$, where $\pi_\ell(v)$ is defined by $\int_K \partial^\alpha(v - \pi_\ell(v))\,\mathrm{d}x = 0$ for all $\alpha \in \mathbb{N}^d$ of length at most ℓ (see Exercise 11.8), and then to invoke the above Poincaré–Steklov inequalities in K. Since $\partial^\alpha(v-\pi_\ell(v))$ has zero mean-value on K for every multi-index $\alpha \in \mathbb{N}^d$ of length m with $0 \leq m \leq \ell-1$, repeated applications of the Poincaré–Steklov inequality (12.13) (and the convexity of K) imply that

$$|v - \pi_\ell(v)|_{W^{m,p}(K)} \leq c\, h_K^{\ell-m} |v - \pi_\ell(v)|_{W^{\ell,p}(K)}.$$

Since $\partial^\alpha(v - \pi_\ell(v))$ has zero mean-value on K for any multi-index $\alpha \in \mathbb{N}^d$ of length ℓ as well, we can apply one more time either (12.13) or (12.14) to the right-hand side. If $r \in \mathbb{N}$, we invoke the convexity of K and apply (12.13) to obtain (12.15). If $r \notin \mathbb{N}$, we apply (12.14) and invoke the regularity of the mesh sequence to obtain (12.15). □

Remark 12.14 (Comparison). The estimate (12.15) is similar in spirit to the Bramble–Hilbert lemma (Lemma 11.9), except that in Lemma 11.9 it is not known how the constant c depends on K. This difficulty was circumvented in Theorem 11.13 (bounding the local interpolation error) by invoking the fact that all the mesh cells are generated from a fixed reference cell. This assumption is not used in the proof of (12.15), which instead assumes the mesh cells to be convex sets. The estimate (12.15) can be extended to cells that are composed of a uniformly finite number of convex subsets (e.g., simplices). The key point is that the Poincaré–Steklov inequality (12.13) can be generalized to such sets; see Remark 22.11. □

12.3.2 Multiplicative trace inequality

Let $K \in \mathcal{T}_h$ and let $F \in \mathcal{F}_K$ be a face of K. Consider a function $v \in W^{1,p}(K)$. Then v has a trace in $L^p(F)$ (see Theorem 3.10). The following result (called multiplicative trace inequality) gives an estimate of $\|v\|_{L^p(F)}$ in terms of powers of $\|v\|_{L^p(K)}$ and $\|\nabla v\|_{\boldsymbol{L}^p(K)}$.

Lemma 12.15 (Multiplicative trace inequality). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape regular sequence of affine simplicial meshes in $\mathbb{R}^d$. There is c s.t. for all $p \in [1,\infty]$, all $v \in W^{1,p}(K)$, all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_K$, and all $h \in \mathcal{H}$,*

$$\|v\|_{L^p(F)} \leq c\, \|v\|_{L^p(K)}^{1-\frac{1}{p}} \left(h_K^{-\frac{1}{p}} \|v\|_{L^p(K)}^{\frac{1}{p}} + \|\nabla v\|_{\boldsymbol{L}^p(K)}^{\frac{1}{p}} \right). \tag{12.16}$$

Proof. Let $K \in \mathcal{T}_h$ and $v \in W^{1,p}(K)$. Assume first that $p \in [1, \infty)$. Let F be a face of K and let $\boldsymbol{z}_F$ be the vertex of K opposite to F. Consider the Raviart–Thomas function $\boldsymbol{\theta}_F(\boldsymbol{x}) = \frac{|F|}{d|K|}(\boldsymbol{x} - \boldsymbol{z}_F)$ (see §14.1). The normal component of $\boldsymbol{\theta}_F$ is equal to 1 on F and 0 on the other faces of K. Since $\nabla{\cdot}\boldsymbol{\theta}_F = \frac{|F|}{|K|}$, we infer using the divergence theorem that

$$\begin{aligned}
\|v\|^p_{L^p(F)} &= \int_{\partial K} |v|^p (\boldsymbol{\theta}_F{\cdot}\boldsymbol{n})\,\mathrm{d}s = \int_K \nabla{\cdot}(|v|^p \boldsymbol{\theta}_F)\,\mathrm{d}x \\
&= \int_K \left(|v|^p \nabla{\cdot}\boldsymbol{\theta}_F + p v |v|^{p-2} \boldsymbol{\theta}_F{\cdot}\nabla v\right)\,\mathrm{d}x \\
&= \frac{|F|}{|K|}\|v\|^p_{L^p(K)} + \frac{p}{d}\frac{|F|}{|K|}\int_K v|v|^{p-2}(\boldsymbol{x} - \boldsymbol{z}_F){\cdot}\nabla v\,\mathrm{d}x.
\end{aligned}$$

Using Hölder's inequality and introducing the length $\ell_F^{\perp}$ defined as the largest length of an edge of K having $\boldsymbol{z}_F$ as endpoint, we infer that

$$\|v\|^p_{L^p(F)} \le \frac{|F|}{|K|}\|v\|^p_{L^p(K)} + \frac{p}{d}\frac{|F|\ell_F^{\perp}}{|K|}\|v\|^{p-1}_{L^p(K)}\|\nabla v\|_{\boldsymbol{L}^p(K)},$$

which implies the bound (12.16) using the regularity of the mesh sequence and the fact that $p^{\frac{1}{p}} \le e^{\frac{1}{e}} < \frac{3}{2}$. Finally the bound for $p = \infty$ is obtained by passing to the limit $p \to \infty$ in (12.16) since c is uniform w.r.t. p and since $\lim_{p\to\infty} \|{\cdot}\|_{L^p(K)} = \|{\cdot}\|_{L^\infty(K)}$. □

Remark 12.16 (Literature). The idea of using a Raviart–Thomas function to prove (12.16) can be traced to Monk and Süli [139, App. B] and Carstensen and Funken [60, Thm. 4.1]. See also Ainsworth [5, Lem. 10] and Veeser and Verfürth [184, Prop. 4.2]. □

Remark 12.17 (Nonsimplicial cells). Lemma 12.15 can be extended to nonsimplicial cells s.t. one can find a vector-valued function $\boldsymbol{\theta}_F$ with normal component equal to 1 on F and 0 on the other faces, and satisfying $h_K\|\nabla{\cdot}\boldsymbol{\theta}_F\|_{L^\infty(K)} + \|\boldsymbol{\theta}_F\|_{\boldsymbol{L}^\infty(K)} \le c$ uniform w.r.t. F, K, and h. □

Remark 12.18 (Fractional trace inequality). The multiplicative trace inequality from Lemma 12.15 can be extended to functions in fractional Sobolev spaces. Let $p \in (1, \infty)$ and $s \in (\frac{1}{p}, 1)$ (we exclude the case $s = 1$ since it is already covered by Lemma 12.15). Functions in $W^{s,p}(K)$ have traces in $L^p(F)$ for every face F of K (see Theorem 3.10). Then one can show (see Exercise 12.5 or Ciarlet [68, Prop. 3.1] and Ern and Guermond [93, Lem. 7.2]) that there is c s.t. for all $v \in W^{s,p}(K)$, all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_K$, and all $h \in \mathcal{H}$,

$$\|v\|_{L^p(F)} \le c\left(h_K^{-\frac{1}{p}}\|v\|_{L^p(K)} + h_K^{s-\frac{1}{p}}|v|_{W^{s,p}(K)}\right). \tag{12.17}$$

The constant c is uniform w.r.t. s and p as long as sp is bounded from below away from 1, but can grow unboundedly as $sp \downarrow 1$. $\square$

Exercises

Exercise 12.1 (Comparison of ℓ^p and ℓ^r-norms). Let p, r be two non-negative real numbers. Let $\{a_i\}_{i\in I}$ be a finite sequence of non-negative numbers. Set $\|a\|_{\ell^p(\mathbb{R}^I)} := (\sum_{i\in I} a_i^p)^{\frac{1}{p}}$ and $\|a\|_{\ell^r(\mathbb{R}^I)} := (\sum_{i\in I} a_i^r)^{\frac{1}{r}}$. (i) Prove that $\|a\|_{\ell^p(\mathbb{R}^I)} \le \|a\|_{\ell^r(\mathbb{R}^I)}$ for $r \le p$. (*Hint*: set $\theta_i := a_i^r/\|a\|_{\ell^r(\mathbb{R}^I)}^r$.) (ii) Prove that $\|a\|_{\ell^p(\mathbb{R}^I)} \le \text{card}(I)^{\frac{r-p}{pr}} \|a\|_{\ell^r(\mathbb{R}^I)}$ for $r > p$.

Exercise 12.2 (L^p-norm of shape functions). Let $\theta_{K,i}$, $i \in \mathcal{N}$, be a local shape function. Let $p \in [1,\infty]$. Assume that the mesh sequence is shape regular. Prove that $\|\theta_{K,i}\|_{L^p(K)}$ is equivalent to $h_K^{d/p}$ uniformly w.r.t. K and h.

Exercise 12.3 (dof norm). Prove Proposition 12.5. (*Hint*: use Lemma 11.7.)

Exercise 12.4 (Markov inequality). (i) Justify that the constant $C_{2,k}$ in the Markov inequality (12.7) can be determined as the largest eigenvalue of the stiffness matrix $\mathcal{A}$. (ii) Compute numerically the constant $C_{2,k}$ for $k \in \{1, 2, 3\}$.

Exercise 12.5 (Fractional trace inequality). Prove (12.17). (*Hint*: use a trace inequality in $W^{s,p}(\widehat{K})$.)

Exercise 12.6 (Mapped polynomial approximation). Let $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ be a reference finite element such $\mathbb{P}_{k,d} \subset \widehat{P}$, $k \in \mathbb{N}$. Let $\mathcal{T}_h$ be a member of a shape regular mesh sequence. Let $\boldsymbol{T}_K(\widehat{K}) = K \in \mathcal{T}_h$ and let (K, P_K, Σ_K) be the finite element generated by the geometric mapping $\boldsymbol{T}_K$ and the functional transformation $\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K)$. Recall that $P_K = \psi_K^{-1}(\widehat{P})$. Show that there is c s.t.

$$\inf_{q\in P_K} |v - q|_{W^{m,p}(K)} \le c\, h_K^{r-m} |v|_{W^{r,p}(K)}, \tag{12.18}$$

for all $r \in [0, k+1]$, all $p \in [1,\infty)$ if $r \notin \mathbb{N}$ or all $p \in [1,\infty]$ if $r \in \mathbb{N}$, every integer $m \in \{0{:}\lfloor r \rfloor\}$, all $v \in W^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, where the mesh cells are supposed to be convex sets if $r \ge 1$. (*Hint*: use Lemma 11.7 and Corollary 12.13.)

Exercise 12.7 (Trace inequality). Let U be a Lipschitz domain in $\mathbb{R}^d$. Prove that there are $c_1(U)$ and $c_2(U)$ such that $\|v\|_{L^p(\partial U)} \le c_1(U)\|v\|_{L^p(U)} + c_2(U)\|\nabla v\|_{\boldsymbol{L}^p(U)}^{\frac{1}{p}} \|v\|_{L^p(U)}^{1-\frac{1}{p}})$ for all $p \in [1,\infty)$ and all $v \in W^{1,p}(U)$. (*Hint*: accept as a fact that there exists a smooth vector field $\boldsymbol{N} \in \boldsymbol{C}^1(\overline{U})$ and $c_0(U) > 0$ such that $(\boldsymbol{N}{\cdot}\boldsymbol{n})_{|\partial U} \ge c_0(U)$ and $\|\boldsymbol{N}(\boldsymbol{x})\|_{\ell^2(\mathbb{R}^d)} = 1$ for all $\boldsymbol{x} \in U$.)

Exercise 12.8 (Weighted inverse inequalities). Let $k \in \mathbb{N}$. (i) Prove that $\|(1-t^2)^{\frac{1}{2}} v'\|_{L^2(-1,1)} \leq \big(k(k+1)\big)^{\frac{1}{2}} \|v\|_{L^2(-1,1)}$ for all $v \in \mathbb{P}_{k,1}$. (*Hint*: let $\tilde{L}_m := \left(\frac{2m+1}{2}\right)^{1/2} L_m$, L_m being the Legendre polynomial from Definition 6.1, and prove that $\int_{-1}^{1} (1-t^2)(\tilde{L}_m)'(t)(\tilde{L}_n)'(t)\,\mathrm{d}t = \delta_{mn} m(m+1)$ for every integers $m, n \in \{0{:}k\}$.) (ii) Prove that $\|v\|_{L^2(-1,1)} \leq (k+2)\|(1-t^2)^{\frac{1}{2}} v\|_{L^2(-1,1)}$ for all $v \in \mathbb{P}_{k,1}$. (*Hint*: consider a Gauss–Legendre quadrature with $l_{\mathrm{q}} = k+2$ and use the fact that the rightmost Gauss–Legendre node satisfies $\xi_{l_{\mathrm{q}}} \leq \cos(\frac{\pi}{2l_{\mathrm{q}}})$.) Note: see also Verfürth [186].

Solution to exercises

Exercise 12.1 (Comparison of ℓ^p and ℓ^r-norms). (i) We observe that $\theta_i = a_i^r/\|a\|^r_{\ell^r(\mathbb{R}^I)} \in [0,1]$ and $\sum_{i\in I}\theta_i = 1$. Since $\frac{p}{r} \ge 1$, we infer thya $\sum_{i\in I}\theta_i^{\frac{p}{r}} \le \sum_{i\in I}\theta_i = 1$ and re-arranging the terms leads to the expected estimate.
(ii) Using Hölder's inequality, we infer that

$$\sum_{i\in I}\theta_i^{\frac{p}{r}} \le \left(\sum_{i\in I}\theta_i^{\frac{p}{r}\frac{r}{p}}\right)^{\frac{p}{r}}\left(\sum_{i\in I}1^{\frac{r}{r-p}}\right)^{1-\frac{p}{r}} \le \operatorname{card}(I)^{1-\frac{p}{r}}.$$

Exercise 12.2 (L^p-norm of shape functions). Observe that

$$\|\theta_{K,i}\|_{L^p(K)} = \left(\frac{|K|}{|\widehat{K}|}\right)^{\frac{1}{p}}\|\widehat{\theta}_i\|_{L^p(\widehat{K})},$$

and use the regularity of the mesh sequence to conclude.

Exercise 12.3 (dof norm). Owing to (12.3), it is sufficient to prove the equivalence for $p=\infty$. Let $v_h = \sum_{i\in\mathcal{N}}\sigma_{K,i}(v_h)\theta_{K,i} \in P_K$. Recalling that $\theta_{K,i} = \psi_K^{-1}(\widehat{\theta}_i)$ for all $i\in\mathcal{N}$, we infer that

$$\begin{aligned}
\|v_h\|_{L^\infty(K;\mathbb{R}^q)} &\le \sum_{i\in\mathcal{N}}|\sigma_{K,i}(v_h)|\|\theta_{K,i}\|_{L^\infty(K;\mathbb{R}^q)}\\
&\le \sum_{i\in\mathcal{N}}|\sigma_{K,i}(v_h)|\|\psi_K^{-1}\|_{\mathcal{L}(L^\infty(\widehat{K};\mathbb{R}^q);L^\infty(K;\mathbb{R}^q))}\|\widehat{\theta}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}\\
&\le c_1\|\psi_K^{-1}\|_{\mathcal{L}(L^\infty(\widehat{K};\mathbb{R}^q);L^\infty(K;\mathbb{R}^q))}\sum_{i\in\mathcal{N}}|\sigma_{K,i}(v_h)|
\end{aligned}$$

where $c_1 = \max_{i\in\mathcal{N}}\|\widehat{\theta}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}$ only depends on the reference element. Using (11.7b) with $l=0$ and $p=\infty$, we infer that

$$|v_h\|_{L^\infty(K;\mathbb{R}^q)} \le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\sum_{i\in\mathcal{N}}|\sigma_{K,i}(v_h)|.$$

Let us now prove the reverse bound. Let $\widehat{v}_h = \psi_K(v_h)$. Since $(\widehat{K},\widehat{P},\widehat{\Sigma})$ is a finite element, $\sum_{i\in\mathcal{N}}|\sigma_n(\widehat{v}_h)|$ is a norm on $\widehat{P}$. The equivalence of norms in $\widehat{P}$ implies that there is c_2, depending only on $(\widehat{K},\widehat{P},\widehat{\Sigma})$, such that

$$\begin{aligned}\sum_{i\in\mathcal{N}} |\sigma_{K,i}(v_h)| = \sum_{i\in\mathcal{N}} |\widehat{\sigma}_i(\widehat{v}_h)| &\le c_2 \|\widehat{v}_h\|_{L^\infty(\widehat{K};\mathbb{R}^q)} = c_2 \|\psi_K(v_h)\|_{L^\infty(\widehat{K};\mathbb{R}^q)} \\ &\le c_2 \|\psi_K\|_{\mathcal{L}(L^\infty(K;\mathbb{R}^q);L^\infty(\widehat{K};\mathbb{R}^q))} \|v_h\|_{L^\infty(K;\mathbb{R}^q)} \\ &\le c_2' \|\mathbb{A}_K\|_{\ell^2} \|v_h\|_{L^\infty(K;\mathbb{R}^q)},\end{aligned}$$

where the last bound follows from (11.7a) with $l = 0$ and $p = \infty$. The conclusion follows from the fact that $\|\mathbb{A}_K\|_{\ell^2} \|\mathbb{A}_K^{-1}\|_{\ell^2}$ is bounded by a constant that only depends on the regularity of the mesh sequence owing to (18.16).

Exercise 12.4 (Markov inequality). (i) Any function in $\mathbb{P}_{k,1}$ can be written $v(t) = \sum_{l=0}^{k} v_l \tilde{L}_l(t)$. Exploiting the L^2-orthonormality of the basis and the definition of the stiffness matrix $\mathcal{A}$, we infer that

$$\frac{\|v'\|_{L^2(-1,1)}}{\|v\|_{L^2(-1,1)}} = \frac{V^{\mathsf{T}}\mathcal{A}V}{V^{\mathsf{T}}V} \le \rho(\mathcal{A}).$$

(ii) A direct computation of $\mathcal{A}$ for $k \in \{1,2,3\}$ respectively yields

$$\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & \sqrt{21} \\ 0 & 0 & 15 & 0 \\ 0 & \sqrt{21} & 0 & 42 \end{pmatrix},$$

with spectral radius 3, 15, and $\frac{45+\sqrt{1605}}{2}$, respectively.

Exercise 12.5 (Fractional trace inequality). Let $v \in W^{s,p}(K)$. Let $K \in \mathcal{T}_h$ be a mesh cell and F be a face of K. Since the mapping $\boldsymbol{T}_K$ is affine, using the trace theorem (Theorem 3.10) in $W^{s,p}(\widehat{K})$, we infer that

$$\|v\|_{L^p(F)} = \frac{|F|^{\frac{1}{p}}}{|\widehat{F}|^{\frac{1}{p}}} \|\psi_K^{\mathrm{g}}(v)\|_{L^p(\widehat{F})} \le c_{s,p} |F|^{\frac{1}{p}} (\|\psi_K^{\mathrm{g}}(v)\|_{L^p(\widehat{K})} + |\psi_K^{\mathrm{g}}(v)|_{W^{s,p}(\widehat{K})}),$$

where $c_{s,p}$ can grow unboundedly as $sp \downarrow 1$ if $p > 1$. Using Lemma 11.7, this inequality is rewritten

$$\|v\|_{L^p(F)} \le c'_{s,p} |F|^{\frac{1}{p}} |K|^{-\frac{1}{p}} (\|v\|_{L^p(K)} + \|\mathbb{J}_K\|_{\ell^2}^{-s} |v|_{W^{s,p}(K)}).$$

The conclusion follows from the regularity of the mesh sequence (see (11.3)).

Exercise 12.6 (Mapped polynomial approximation). Let $k \in \mathbb{N}$. Let $r \in [0, k+1]$, let $p \in [1,\infty)$ if $r \notin \mathbb{N}$ or $p \in [1,\infty]$ if $r \in \mathbb{N}$, and let $m \in \{0{:}\lfloor r \rfloor\}$. Let $v \in W^{r,p}(K)$ and set $\widehat{v} := \psi_K(v)$. Let $\widehat{q}^* \in \widehat{P}$ be s.t. $|\widehat{v} - \widehat{q}^*|_{W^{m,p}(\widehat{K})} = \inf_{\widehat{q}\in\widehat{P}} |\widehat{v} - \widehat{q}|_{W^{m,p}(\widehat{K})}$. Then we have (the value of c changes at each occurrence)

$$\begin{aligned}
\inf_{q\in P_K} |v-q|_{W^{m,p}(K)} &\le |v-\psi_K^{-1}(\widehat{q}^*)|_{W^{m,p}(K)} = |\psi_K^{-1}(\widehat{v})-\psi_K^{-1}(\widehat{q}^*)|_{W^{m,p}(K)} \\
&\le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^m\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,|\widehat{v}-\widehat{q}^*|_{W^{m,p}(\widehat{K})} \\
&\le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^m\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,\inf_{\widehat{q}\in\mathbb{P}_{k,d}}|\widehat{v}-\widehat{q}|_{W^{m,p}(\widehat{K})} \\
&\le c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^m\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,|\widehat{v}|_{W^{r,p}(\widehat{K})} \\
&\le c\,\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K\|_{\ell^2}^r\|\mathbb{J}_K^{-1}\|_{\ell^2}^m\,|v|_{W^{r,p}(K)} \\
&\le c\,h_K^{r-m}\,|v|_{W^{r,p}(K)},
\end{aligned}$$

where we used that $\psi_K^{-1}(\widehat{q}^*)\in P_K$ in the first line, (11.7b) in the second line, the definition of $\widehat{q}^*$ and $\mathbb{P}_{k,d}\subset\widehat{P}$ in the third line, Corollary 12.13 in the fourth line, (11.7a) in the fifth line, and the -regularity of the mesh sequence in the last line. This proves (12.18).

Exercise 12.7 (Trace inequality). We first observe that

$$\begin{aligned}
c_0(U)\int_{\partial U}|v|^p\,\mathrm{d}x &\le \int_{\partial_U}(\boldsymbol{n}\cdot\boldsymbol{N})|v|^p\,\mathrm{d}x = \int_U \nabla\cdot(\boldsymbol{N}|v|^p)\,\mathrm{d}x \\
&\le \int_U((\nabla\cdot\boldsymbol{N})|v|^p + p(\boldsymbol{N}\cdot\nabla v)|v|^{p-1})\,\mathrm{d}x \\
&\le c_1(U)\|v\|_{L^p(U)}^p + p\|\nabla v\|_{\boldsymbol{L}^p(U)}\|v\|_{L^p(U)}^{p-1}
\end{aligned}$$

where $c_1(U)=\|\nabla\cdot\boldsymbol{N}\|_{L^\infty(U)}$, since $\|\boldsymbol{N}(\boldsymbol{x})\|_{\ell^2(\mathbb{R}^d)}=1$ for all $\boldsymbol{x}\in U$ and we used Hölder's inequality to bound $\int_U\|\nabla u\|_{\ell^2}|v|^{p-1}\,\mathrm{d}x$. The conclusion follows by applying the inequality $(a+b)^{\frac{1}{p}}\le a^{\frac{1}{p}}+b^{\frac{1}{p}}$ for all $a,b\ge 0$, i.e.,

$$\|v\|_{L^p(U)} \le \left(\frac{c_1(U)}{c_0(U)}\right)^{\frac{1}{p}}\|v\|_{L^p(U)} + p^{\frac{1}{p}}c_0(U)^{-\frac{1}{p}}\|\nabla v\|_{\boldsymbol{L}^p(U)}^{\frac{1}{p}}\|v\|_{L^p(U)}^{1-\frac{1}{p}}.$$

Exercise 12.8 (Weighted inverse inequalities). (i) Without loss of generality, assume $n\le m$. Integrating by parts and since $(1-t^2)$ vanishes at $t=\pm1$, we infer that

$$\int_{-1}^1(1-t^2)(\tilde{L}_m)'(t)(\tilde{L}_n)'(t)\,\mathrm{d}t = -\int_{-1}^1\tilde{L}_m(t)\Big((1-t^2)(\tilde{L}_n)'(t)\Big)'\,\mathrm{d}t.$$

Since $\Big((1-t^2)(\tilde{L}_n)'(t)\Big)'$ is a polynomial of degree n whose leading coefficient is equal to that of $\tilde{L}_n$ multiplied by $-n(n+1)$, the orthonormality of the (normalized) Legendre polynomials implies that $\int_{-1}^1(1-t^2)(\tilde{L}_m)'(t)(\tilde{L}_n)'(t)\,\mathrm{d}t = \delta_{mn}m(m+1)$. As a result, writing any $v\in\mathbb{P}_{k,1}$ in the form $v(t)=\sum_{l=0}^k v_l\tilde{L}_l(t)$, we infer that

$$\int_{-1}^{1}(1-t^2)|v'(t)|^2\,\mathrm{d}t = \sum_{l=0}^{k} v_l^2 l(l+1) \le k(k+1)\sum_{l=0}^{k} v_l^2 = k(k+1)\|v\|^2_{L^2(-1,1)}.$$

(ii) Since $(1-t^2)v^2$ is of degree $2k+2$ and the quadrature is of order $2l_{\mathrm{q}}-1 = 2k+3$, we infer that

$$\int_{-1}^{1}(1-t^2)v(t)^2\,\mathrm{d}t = \sum_{l=1}^{l_{\mathrm{q}}} \omega_l(1-\xi_l^2)v(\xi_l)^2$$
$$\ge (1-\xi_{l_{\mathrm{q}}}^2)\sum_{l=1}^{l_{\mathrm{q}}} \omega_l v(\xi_l)^2 = (1-\xi_{l_{\mathrm{q}}}^2)\int_{-1}^{1} v(t)^2\,\mathrm{d}t.$$

The conclusion follows from

$$\frac{1}{1-\xi_{l_{\mathrm{q}}}^2} \le \frac{1}{\sin^2(\frac{\pi}{2l_{\mathrm{q}}})} \le (l_{\mathrm{q}})^2,$$

since $\sin(x) \ge \frac{2}{\pi}x$ for all $x \in [0, \frac{\pi}{2}]$.