In Part VII, composed of Chapters 31 to 35, we study the approximation of scalar second-order elliptic PDEs by $H^1$-conforming finite elements. The prototypical example is the Laplacian with homogeneous Dirichlet conditions, but we also consider more general settings including lower-order terms in the PDE and other boundary conditions. Among the topics we address in this part are the a priori and a posteriori error analysis, the discrete maximum principle, and the impact of quadratures. We also study the $H^1$-conforming approximation of the Helmholtz problem as an example of elliptic PDE without coercivity.

The present chapter addresses fundamental properties of scalar-valued second-order elliptic PDEs. We focus on weak formulations endowed with a coercivity property so that well-posedness hinges on the Lax–Milgram lemma. The key example is a diffusion-advection-reaction PDE where the lower-order terms are small enough so as not to pollute the coercivity provided by the diffusion operator. We study in some detail how various boundary conditions (Dirichlet, Neumann, Robin) can be enforced in the weak formulation. Moreover important smoothness properties of the exact solution are listed at the end of the chapter. These results will be useful later to establish error estimates for the finite element approximation.

### 31.1 Model problem

Let $D$ be a domain in $\mathbb{R}^d$, i.e., $D$ is a nonempty, open, bounded, connected subset of $\mathbb{R}^d$ (see Definition 3.1). Let $a$, $\beta$, and $\mu$ be functions defined on $D$ that take values in $\mathbb{R}^{d\times d}$, $\mathbb{R}^d$, and $\mathbb{R}$, respectively. Given a function $f : D \rightarrow \mathbb{R}$, we look for a function $u : D \rightarrow \mathbb{R}$ that solves the following linear PDE:

$$-\nabla \cdot (a \nabla u) + \beta \nabla u + \mu u = f \quad \text{in } D. \quad (31.1)$$
Boundary conditions are discussed later. Using Cartesian coordinates (31.1) amounts to
\[ \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} \beta_i \frac{\partial u}{\partial x_i} + \mu u = f. \]
The PDE reduces to the Poisson equation \(-\Delta u = f\) studied in §24.1 if \(a\) is the identity tensor in \(\mathbb{R}^d\) and \(\beta\) and \(\mu\) vanish identically. More generally (31.1) is a diffusion-advection-reaction equation modeling for instance heat or mass transfer or flows in porous media. The first term on the left-hand side of (31.1) accounts for diffusion processes, the second one for advection processes, and the third one for reaction processes (depletion occurs when \(\mu\) is positive).

### 31.1.1 Ellipticity and assumptions on the data

We assume that \(a \in L^\infty(D) := L^\infty(D; \mathbb{R}^{d \times d})\) and that \(a\) takes symmetric values. We also assume that \(\beta \in W^{1,\infty}(D) := W^{1,\infty}(D; \mathbb{R}^d), \mu \in L^\infty(D),\) and \(f \in L^2(D)\). For dimensional consistency we equip the space \(H^1(D)\) with the norm \(\|v\|_{H^1(D)} := (\|\nabla v\|^2_{L^2(D)} + \ell_D^2 \|v\|^2_{L^2(D)})^{\frac{1}{2}}\), where \(\ell_D\) is a length scale associated with \(D\), e.g., \(\ell_D := \text{diam}(D)\). A key notion for the second-order PDE (31.1) is that of ellipticity.

**Definition 31.1 (Ellipticity).** For a.e. \(x \in D\), let \([\lambda_{\min}(x), \lambda_{\max}(x)]\) be the smallest interval containing the eigenvalues of \(a(x)\). We say that the PDE (31.1) is elliptic if
\[
0 < \lambda := \text{ess inf}_{x \in D} \lambda_{\min}(x) \leq \text{ess sup}_{x \in D} \lambda_{\max}(x) =: \lambda_D < \infty. \tag{31.2}
\]

**Example 31.2 (Anisotropic diffusion).** We say that the diffusion process is anisotropic if the diffusion matrix is not proportional to the identity, as in the PDE \(-\frac{\partial^2 u}{\partial x_1^2} + 2\kappa \frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_2^2} = f\) which is is elliptic if \(\kappa \in (-1, 1)\).

The following important result, which is similar to the unique continuation principle for real analytic functions, hinges on the ellipticity property.

**Theorem 31.3 (Unique continuation principle).** Let \(D\) be a connected subset of \(\mathbb{R}^d\) with \(0 \in D\). Assume that \(a\) satisfies the ellipticity condition (31.2), \(a_{ij} \in C^0(D; \mathbb{R})\), \(a_{ij}\) is Lipschitz continuous in \(D\setminus\{0\}\), and there are \(c > 0\) and \(\delta > 0\) such that \(\|\nabla a_{ij}(x)\|_{L^\infty} \leq c \|x\|^{-\delta}_{L^\infty}\). Let \(u \in H^1_{\text{loc}}(D)\) and assume that
\[
|a|D^2 u \leq c \sum_{|\alpha| \leq 1} \|x\|^{\delta + |\alpha| - 2} |\partial^\alpha u|, \tag{31.3a}
\]
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^d} \int_{\|x\|_2 < \epsilon} u^2(x) \, dx = 0. \tag{31.3b}
\]
Then \(u \equiv 0\) in \(D\).

**Proof.** See Hörmander [192, Thm. 17.2.6]. We also refer the reader to Reed and Simon [255, Thm. XIII.57] and [255, Thm. XIII.63] for variations on the unique continuation principle that are somewhat easier to grasp.
The above result, known in the literature as the Aronszajn–Cordes uniqueness theorem, can be used to establish the uniqueness of the solution to the PDE (31.1). Assume that \( u_1, u_2 \) are two solutions of (31.1), and assume that one can show that \( u_1 \in H^1_\text{loc}(D), u_2 \in H^1_\text{loc}(D) \), and there is a open set \( S \subset D \) s.t. \( (u_1 - u_2)|_S = 0 \). One can always assume that \( \theta \in S \). Setting \( u := u_1 - u_2 \), one has \(-d:D^2 u = (\nabla d - \beta)\nabla u - \mu u \). Let us assume that \( d \) satisfies the assumptions of Theorem 31.3. Then one immediately deduces that (31.3a) holds true with some appropriate constant \( c \). Using that \( u|_S = 0 \), the second condition (31.3b) is trivially satisfied, and uniqueness follows.

Remark 31.4 (Divergence form, Cordes condition). The PDE (31.1) is said to be in divergence form because of the way the second-order term is written. One can also consider the PDE in non-divergence form \(-d:D^2 u + \beta \nabla u + \mu u = f\), where \( d:D^2 u = \sum_{i,j \in \{1:d\}} d_{ij} \partial^2 u / \partial x_i \partial x_j \). In this case one usually adds the Cordes condition [108] to the ellipticity assumption: There is \( \epsilon \in (0,1) \) s.t. \( \frac{|d|}{(tr(d))^2} \leq \frac{1}{\epsilon} \) uniformly in \( D \), where \( ||d||_F = (d:d)^{\frac{1}{2}} \) is the Frobenius norm of \( d \) and \( tr(d) \) its trace (note that \( tr(d) > 0 \) owing to the ellipticity condition). We refer the reader to Smears and Süli [269] for further insight in the context of Hamilton–Jacobi–Bellman equations. \( \square \)

31.1.2 Towards a weak formulation

Proceeding informally as in §24.1, e.g., assuming \( u \in H^2(D) \), we multiply (31.1) by a test function \( w \in H^1(D) \) and integrate over \( D \) to obtain

\[
\int_D (-\nabla \cdot (d \nabla u) w + (\beta \cdot \nabla u) w + \mu uw) \, dx = \int_D f w \, dx. \tag{31.4}
\]

Integrating by parts the first term on the left-hand side leads to

\[
\int_D -\nabla \cdot (d \nabla u) w \, dx = \int_D (d \nabla u) \cdot \nabla w \, dx - \int_{\partial D} (n \cdot (d \nabla u)) w \, ds, \tag{31.5}
\]

where \( n \) denotes the outward unit normal to \( D \). We then arrive at

\[
a(u, w) - \int_{\partial D} (n \cdot (d \nabla u)) w \, ds = \int_D f w \, dx, \quad \forall w \in H^1(D), \tag{31.6}
\]

where \( a \) is defined for all \((v, w) \in H^1(D) \times H^1(D)\) as follows:

\[
a(v, w) := \int_D ((d \nabla v) \cdot \nabla w + (\beta \cdot \nabla v) w + \mu vw) \, dx. \tag{31.7}
\]

Notice in passing that using Cartesian coordinates, the symmetry of \( d \) implies

\[
(d \nabla v) \cdot \nabla w = \sum_{i,j \in \{1:d\}} d_{ij} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i} = \sum_{i,j \in \{1:d\}} d_{ij} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} = \nabla v \cdot (d \nabla w),
\]

that is, \((d \nabla v) \cdot \nabla w = \nabla v \cdot (d \nabla w)\). Moreover, using the Cauchy–Schwarz inequality for the three integrals leads to
\[ |a(v, w)| \leq (\lambda_2 \ell_D^2 + \beta_2 \ell_D^{-1} + \mu_2) \|v\|_{H^1(D)} \|w\|_{H^1(D)}, \quad (31.8) \]

for all \( v, w \in H^1(D) \), with \( \beta_2 := \|\beta\|_{L^\infty(D)} \) and \( \mu_2 := \|\mu\|_{L^\infty(D)} \), which proves that the bilinear form \( a \) is bounded on \( H^1(D) \times H^1(D) \). Equation (31.6) is the starting point to derive weak formulations for the PDE (31.1) with various types of boundary conditions.

**31.2 Dirichlet boundary condition**

Our goal is now to prove the well-posedness of the weak formulation when a Dirichlet boundary condition is enforced. In what follows, we identify \( L^2(D) \) with its dual space \( L^2(D)^\prime \) so that we are in the situation where \( H^1_0(D) \hookrightarrow L^2(D) \equiv L^2(D)^\prime \hookrightarrow H^{-1}(D) = H^1_0(D)^\prime \), \( (31.9) \)

with bounded and densely defined embeddings (recall that the notation \( V \hookrightarrow W \) means that the embedding of \( V \) into \( W \) is bounded).

**31.2.1 Homogeneous Dirichlet condition**

We consider the homogeneous Dirichlet condition

\[ u = 0 \quad \text{on } \partial D, \quad (31.10) \]

which we are going to enforce strongly by using the space \( H^1_0(D) \) for both the trial and the test spaces. Recall from the Trace theorem (Theorem 3.10) that \( u \in H^1_0(D) \) implies that \( \gamma^g(u) = 0 \), where \( \gamma^g : H^1(D) \rightarrow H^{1/2}(\partial D) \) is the trace map such that \( \gamma^g(v) = v|_{\partial D} \) if the function \( v \) is smooth. Since the test functions vanish at the boundary, we can drop the boundary term on the left-hand side of (31.6), leading to the following weak formulation:

\[
\begin{cases}
\text{Find } u \in V := H^1_0(D) \text{ such that} \\
[a(u, w)] = \int_D fdw \, dx, \quad \forall w \in V.
\end{cases} \quad (31.11)
\]

**Proposition 31.5 (Weak solution).** Let \( u \) solve (31.11) with \( f \in L^2(D) \). Then the PDE (31.1) is satisfied a.e. in \( D \), and the boundary condition (31.10) a.e. on \( \partial D \).

**Proof.** Let \( u \) be a weak solution. Testing the weak formulation (31.11) against an arbitrary function \( \varphi \in C_0^\infty(D) \subset H^1_0(D) \) and using the notion of weak derivatives leads to \( \langle -\nabla \cdot (\beta \nabla u), \varphi \rangle = \int_D (f - \beta \cdot \nabla u - \mu u) \varphi \, dx \) since \( f \in L^2(D) \) and \( \beta \cdot \nabla u + \mu u \in L^2(D) \) owing to the assumptions on the data. Hence \( -\nabla \cdot (\beta \nabla u) \) defines a bounded linear form on \( L^2(D) \) with Riesz–Fréchet representative equal to \( f - \beta \cdot \nabla u - \mu u \). This means that \( u \) solves the PDE (31.1) a.e.
in $D$. Moreover $u \in H^1_0(D)$ implies that $\gamma^s(u) = 0$ in $H^1(\partial D) \hookrightarrow L^2(\partial D)$, i.e., the boundary condition (31.10) holds a.e. on $\partial D$.

**Remark 31.6** ($f \in H^{-1}(D)$). When $f \in H^{-1}(D)$, the term $\int_D fw \, dx$ in (31.11) must be understood as $\langle f, w \rangle_{H^{-1}(D), H^1_0(D)}$. More specifically, recalling from Theorem 4.12 that the assumption $f \in H^{-1}(D)$ is equivalent to assuming that there are $g_0 \in L^2(D)$ and $g_1 \in L^2(D)$ such that $\langle f, w \rangle_{H^{-1}(D), H^1_0(D)} = \int_D (g_0 w + g_1 \nabla w) \, dx$, each time we write $a(u, w) = \langle f, w \rangle_{H^{-1}(D), H^1_0(D)}$, we actually mean $a(u, w) = \int_D (g_0 w + g_1 \nabla w) \, dx$, and the PDE we actually solve is $-\nabla \cdot (\ell u) + \beta \nabla u + \mu u = g_0 - \nabla g_1$ in $H^{-1}(D)$. □

We now make assumptions on the PDE coefficients that are sufficient to prove well-posedness of (31.11) by invoking coercivity. Recall the Poincaré–Steklov inequality (3.11) (with $p = 2$), i.e., there is $C_{\text{ps}} > 0$ such that

$$C_{\text{ps}} \|v\|_{L^2(D)} \leq \ell_D \|\nabla v\|_{L^2(D)}, \quad \forall v \in H^1_0(D).$$

(31.12)

Owing to (31.12), we can equip the space $V := H^1_0(D)$ with the norm

$$\|v\|_V := \|\nabla v\|_{L^2(D)} = |v|_{H^1(D)}.$$  

(31.13)

The space $V$ equipped with this norm is a Hilbert space since $\|v\|_V \leq \ell_D^{-1} \|v\|_{H^1(D)} \leq (1 + C_{\text{ps}}^2)^{\frac{1}{2}} \|v\|_V$ for all $v \in V$.

**Proposition 31.7** (Well-posedness). Assume the ellipticity condition (31.2). Assume that there exists $\theta > 0$ such that

$$\mu_s := \text{ess inf}_{x \in D} (\mu - \frac{1}{2} \nabla \cdot \beta)(x) \geq - (1 - \theta) C_{\text{ps}}^2 \ell_D^{-2} \lambda_s.$$  

(31.14)

Then the bilinear form $a$ is $V$-coercive:

$$a(v, v) \geq \lambda_s \min(1, \theta) \|v\|_V^2, \quad \forall v \in V,$$  

(31.15)

and the problem (31.11) is well-posed.

**Proof.** The boundedness property (31.8) of $a$ can be rewritten as

$$|a(v, w)| \leq (\lambda_s + C_{\text{ps}}^{-1} \ell_D \beta_s + C_{\text{ps}}^{-2} \ell_D^2 \mu_s) \|v\|_V \|w\|_V,$$

for all $v, w \in V$. Moreover the linear form $\ell(w) := \int_D fw \, dx$ is bounded on $V$ since $|\ell(w)| \leq \|f\|_{L^2(D)} \|w\|_{L^2(D)} \leq \|f\|_{L^2(D)} C_{\text{ps}}^{-1} \ell_D \|w\|_V$ for all $w \in V$. Let us now prove the coercivity property (31.15). Using the divergence formula for the field $(\frac{1}{2} v^2) \beta$, we infer that

$$\int_D v(\beta \cdot \nabla v) \, dx = -\frac{1}{2} \int_D (\nabla \cdot \beta) v^2 \, dx + \frac{1}{2} \int_{\partial D} (\beta \cdot \mathbf{n}) v^2 \, ds,$$

(31.16)

for all smooth functions $v \in C^\infty(D)$. A density argument then shows that the formula (31.16) remains valid for all $v \in H^1(D)$. Using the definition
of $\mu_y$, the identity (31.16), and that $v$ vanishes at the boundary, we obtain
\[ a(v, v) \geq \int_D \left( \lambda_0 \| \nabla v \|^2_{L^2(D)} + \mu_y |v|^2 \right) \, dx \] for all $v \in V$. The assumptions on $\lambda_0$ and $\mu_y$ imply that
\[ a(v, v) \geq \lambda_0 \left( \| \nabla v \|^2_{L^2(D)} - (1 - \theta) C_{\text{PS}}^2 \| v \|^2_{L^2(D)} \right). \]
If $\theta > 1$, the last term is positive, whereas if $\theta \in (0, 1]$, we have
\[ \| \nabla v \|^2_{L^2(D)} - (1 - \theta) C_{\text{PS}}^2 \| v \|^2_{L^2(D)} = \theta \| \nabla v \|^2_{L^2(D)} + (1 - \theta)(\| \nabla v \|^2_{L^2(D)} - C_{\text{PS}}^2 \| v \|^2_{L^2(D)}) \geq \theta \| \nabla v \|^2_{L^2(D)}, \]
where the last bound follows from the Poincaré–Steklov inequality. The coercivity property (31.15) then results from the Poincaré–Steklov inequality. The coercivity property (31.15) then results from Proposition 25.7, proceeding as in the homogeneous case, one can prove the following result.

**Remark 31.9 (Variational formulation).** Assume that $\beta = 0$ in $D$. Owing to Proposition 25.7, $u$ solves (31.11) if $u$ minimizes in $H^1_0(D)$ the energy functional $E_D(v) := \frac{1}{2} \int_D (\nabla v \cdot \nabla v + \mu |v|^2 - 2f v) \, dx$.

**Remark 31.10 (Helmholtz).** The condition (31.14) is only sufficient to ensure the well-posedness of (31.11) by means of a coercivity argument. We will see in Chapter 35, which deals with the Helmholtz problem, that well-posedness can also hold without invoking (31.14). In this case we will establish well-posedness by means of an inf-sup argument.

### 31.2.2 Non-homogeneous Dirichlet condition

Let $g \in H^{\frac{1}{2}}(\partial D)$. We consider the non-homogeneous Dirichlet condition
\[ u = g \quad \text{on } \partial D. \tag{31.17} \]
Since the map $\gamma^* : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is surjective, there is a uniform constant $C_{\gamma^*}$ and $u_g \in H^1(D)$ such $\gamma^*(u_g) = g$ and $\| u_g \|_{H^1(D)} \leq C_{\gamma^*} \| g \|_{H^{\frac{1}{2}}(\partial D)}$; see Theorem 3.10(iii). Setting $u_0 := u - u_g$, we obtain $\gamma^*(u - u_g) = g - g = 0$, i.e., $u_0 \in H^1_0(D)$. This leads to the following weak formulation:

\[
\begin{cases}
\text{Find } u \in H^1(D) \text{ such that } u_0 := u - u_g \in V := H^1_0(D) \text{ satisfies } \\
a(u_0, w) = \int_D f w \, dx - a(u_g, w), \quad \forall w \in V. \tag{31.18}
\end{cases}
\]

The right-hand side in (31.18) defines a bounded linear form on $V$ owing to the boundedness of $a$ on $H^1(D) \times H^1(D)$ and the above bound on $u_g$. Proceeding as in the homogeneous case, one can prove the following result.
Proposition 31.11 (Weak solution, well-posedness). Let \( u \) solve (31.18) with \( f \in L^2(D) \) and \( g \in H^{\frac{1}{2}}(\partial D) \). Then the PDE (31.1) is satisfied a.e. in \( D \) and the boundary condition (31.17) a.e. on \( \partial D \). Moreover, under the assumptions of Proposition 31.7, (31.18) is well-posed.

31.3 Robin/Neumann conditions

The Dirichlet conditions are called essential boundary conditions since they are imposed explicitly in the solution space. The Robin and the Neumann conditions belong to the class of natural boundary conditions. These conditions are not explicitly enforced in the solution space, but they are enforced in the weak formulations by using test functions that are not zero at the boundary.

31.3.1 Robin condition

Let \( \rho \in L^\infty(\partial D) \) and \( g \in L^2(\partial D) \). We consider the Robin boundary condition

\[
\rho u + \mathbf{n} \cdot (d\nabla u) = g \quad \text{on } \partial D.
\]

(31.19)

Starting from (31.6) and still proceeding informally, we consider test functions in \( H^1(D) \) (i.e., they are no longer in \( H^1_0(D) \) as for the Dirichlet conditions), and we use the Robin condition in the boundary integral on the left-hand side of (31.6), thereby replacing \( \mathbf{n} \cdot (d\nabla u) \) by \( g - \rho u \). This leads to \( a(u, w) + \int_{\partial D} (g - \rho u) w \, ds = \int_D f w \, dx \).

Introducing the trace map \( \gamma g \) in the boundary term and rearranging the expression, we obtain the following weak formulation:

\[
\begin{cases}
\text{Find } u \in V := H^1(D) \text{ such that } \\
a_\rho(u, w) = \int_D f w \, dx + \int_{\partial D} \gamma g (w) \, ds, \quad \forall w \in V,
\end{cases}
\]

(31.20)

with the bilinear form \( a_\rho \) on \( H^1(D) \times H^1(D) \) s.t.

\[
a_\rho(v, w) := a(v, w) + \int_{\partial D} \rho \gamma g (v) \gamma g (w) \, ds.
\]

(31.21)

The boundedness of the trace map (see Theorem 3.10) implies that there is \( M_{\gamma g} \) s.t. \( \| \gamma g(v) \|_{L^2(\partial D)} \leq M_{\gamma g} \ell^\frac{1}{2}(D) \| v \|_{H^1(D)} \). Using the Cauchy–Schwarz inequality yields \( \int_{\partial D} \rho \gamma g (v) \gamma g (w) \, ds \leq \rho_2 M_{\gamma g}^2 \ell^1(D) \| v \|_{H^1(D)} \| w \|_{H^1(D)} \), with \( \rho_2 := \| \rho \|_{L^\infty(\partial D)} \). Since \( a \) is bounded on \( H^1(D) \times H^1(D) \), so is \( a_\rho \). Similarly, the right-hand side in (31.20) defines a bounded linear form in \( H^1(D) \).

We identify \( L^2(\partial D) \) with its dual space \( L^2(\partial D)' \) in order to interpret the boundary condition satisfied by weak solutions to (31.20). Hence we have \( H^{\frac{1}{2}}(\partial D) \hookrightarrow L^2(\partial D) \equiv L^2(\partial D)' \hookrightarrow H^{-\frac{1}{2}}(\partial D) \) with dense embeddings, where
\( H^{-\frac{1}{2}}(\partial D) \) is the dual space of \( H^{\frac{1}{2}}(\partial D) \). Recall from Theorem 4.15 the normal trace map \( \gamma^d : H(\text{div}; D) \to H^{-\frac{1}{2}}(\partial D) \) defined such that the following identity holds true for all \( \phi \in H(\text{div}; D) \) and all \( w \in H^1(D) \):

\[
\langle \gamma^d(\phi), \gamma^s(w) \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = \int_D (\phi \cdot \nabla w + (\nabla \cdot \phi)w) \, dx. \tag{31.22}
\]

We have \( \gamma^d(\phi) = n \cdot \phi \) whenever \( \phi \) is smooth, e.g., if \( \phi \in H^s(D) \), \( s > \frac{1}{2} \).

**Proposition 31.12 (Weak solution).** Let \( f \in L^2(D) \), \( \rho \in L^\infty(\partial D) \) and \( g \in L^2(\partial D) \). Let \( u \) solve (31.20). Then the PDE (31.1) is satisfied a.e. in \( D \). The boundary condition (31.19) is satisfied a.e. in \( \partial D \) in the sense that \( \rho \gamma^s(u) + \gamma^d(d \nabla u) = g \) in \( L^2(\partial D) \).

**Proof.** As in the proof of Proposition 31.5, one can show that the PDE (31.1) is satisfied a.e. in \( D \). In particular, introducing the diffusive flux \( \sigma := -d \nabla u \), we obtain \( \sigma \in L^2(D) \) and \( \nabla \sigma = f - \beta \nabla u - \mu u \in L^2(D) \), i.e., \( \sigma \in H(\text{div}; D) \).

Using the weak formulation, we infer that

\[-\langle \gamma^d(\sigma), \gamma^s(w) \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \int_{\partial D} (\rho \gamma^s(u) - g) \gamma^s(w) \, ds = 0,
\]

for all \( w \in H^1(D) \). Since the trace operator \( \gamma^s : H^1(D) \to H^{\frac{1}{2}}(\partial D) \) is surjective and the above equality is valid for all \( w \in H^1(D) \), we infer that \( \gamma^d(\sigma) \) defines a bounded linear form on \( L^2(\partial D) \) with Riesz–Fréchet representative equal to \( \rho \gamma^s(u) - g \). Hence the boundary condition is satisfied a.e. on \( \partial D \).

**Remark 31.13 (Data smoothness).** Notice that \( f \in L^2(D) \) is needed to establish that \( \nabla \sigma \in L^2(D) \). It is possible to assume that \( g \) is only in \( H^{-\frac{1}{2}}(\partial D) \). Then the boundary term in (31.20) becomes \( \langle g, \gamma^s(w) \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \), and the Robin boundary condition is satisfied only in \( H^{-\frac{1}{2}}(\partial D) \).

We now address the well-posedness of (31.20). One can show (see Exercise 31.2 and (3.15)) that there is \( C_{\text{ps}} > 0 \) such that for all \( v \in H^1(D) \),

\[
C_{\text{ps}} \| v \|_{L^2(D)} \leq \ell_D \| v \|_V, \quad \| v \|_V := \left\{ \| \nabla v \|_{L^2(D)}^2 + \ell_D^{-1} \| \gamma^s(v) \|_{L^2(\partial D)}^2 \right\}^{\frac{1}{2}}.
\tag{31.23}
\]

Thus \( (1 + C_{\text{ps}}^{-2})^{-\frac{1}{2}} \| v \|_{H^1(D)} \leq \ell_D \| v \|_V \leq (1 + M_{\gamma^s}^2)^{\frac{1}{2}} \| v \|_{H^1(D)} \), so that the space \( V := H^1(D) \) equipped with the norm \( \| v \|_V \) is a Hilbert space. Let \( \mu_\circ := \text{ess inf}_{x \in D}(\mu - \frac{1}{2} \nabla \cdot \beta)(x) \) and \( \nu_\circ := \text{ess inf}_{x \in \partial D}(\rho + \frac{1}{2} \beta \cdot n)(x) \).

**Proposition 31.14 (Well-posedness).** Assume that the ellipticity assumption (31.2) holds. Assume that either \( \mu_\circ > 0 \), \( \nu_\circ \geq 0 \) or \( \mu_\circ \geq 0 \), \( \nu_\circ > 0 \). Then the bilinear form \( a_\circ \) is \( V \)-coercive, and the problem (31.20) is well-posed.

**Proof.** The boundedness of \( a_\circ \) follows from
Remark 31.16 (Variational formulation). Assume that \( \gamma \) follows from \( \mathcal{F}_\text{neumann} \) and \( \mathcal{F}_\text{coercivity} \) holds if
\[
\ell(D) := \int_D f w \, dx + \int_{\partial D} g \gamma_k(w) \, ds
\]
is bounded on \( V \) since \( |\ell(w)| \leq (C_{\text{neumann}} + \ell(D)^2) ||w||_V \) for all \( w \in V \). Let us now prove the V-coercivity of \( a_\rho \). Let \( v \in V \). Using (31.16), we infer that
\[
a_\rho(v, v) \geq \lambda \| \nabla v \|^2_{L^2(D)} + \rho \| \gamma_k(v) \|^2_{L^2(\partial D)}. \tag{31.24}
\]
If \( \mu > 0 \), we can drop the term multiplied by \( \nu_\rho \) in (31.24), and coercivity follows from
\[
a_\rho(v, v) \geq \min(\lambda, \mu \ell(D)^2) \| v \|^2_{H^1(D)} \geq \min(\lambda, \mu \ell(D)^2)(1 + M_\gamma^2)^{-1} \| v \|^2_V.
\]
If \( \nu > 0 \), we can drop the term multiplied by \( \nu_\rho \) in (31.24), and coercivity follows from
\[
a_\rho(v, v) \geq \min(\lambda, \nu \ell(D)^2) \| v \|^2_V.
\]
That (31.20) is well-posed follows from the Lax–Milgram lemma. \( \square \)

Example 31.15 (Pure diffusion). For a purely diffusive problem, coercivity holds if \( \rho \) is uniformly bounded from below away from zero. \( \square \)

Remark 31.16 (Variational formulation). Assume that \( \beta \) is identically zero in \( D \). Owing to Proposition 25.7, \( u \) solves (31.20) iff \( u \) minimizes in \( H^1(D) \) the energy functional \( \mathcal{E}_R(v) := \frac{1}{2} \int_D (\nabla v \cdot (d \nabla v) + \mu v^2 - 2f v) \, dx + \frac{1}{2} \int_{\partial D} (\rho \gamma_k(v)^2 - 2g \gamma_k(v)) \, ds. \) \( \square \)

31.3.2 Neumann condition

The Neumann condition is a particular case of the Robin condition in which \( \rho \) vanishes identically on \( \partial D \), i.e., we want to enforce
\[
\mathbf{n} \cdot \nabla u = g \quad \text{on} \quad \partial D. \tag{31.25}
\]
The following weak formulation is obtained by setting \( \rho \) to zero in (31.20):
\[
\begin{cases}
\text{Find } u \in V := H^1(D) \text{ such that } \\
a(u, w) = \int_D f w \, dx + \int_{\partial D} g \gamma_k(w) \, ds, \quad \forall w \in V.
\end{cases} \tag{31.26}
\]

Proposition 31.17 (Weak solution, well-posedness). Let \( f \in L^2(D) \) and \( g \in L^2(\partial D) \). Let \( u \) solve (31.26). Then the PDE (31.1) is satisfied a.e. in \( D \). The boundary condition (31.25) is satisfied a.e. in \( \partial D \) in the sense that \( \gamma_k(\mathbf{n} \cdot \nabla u) = g \) in \( L^2(\partial D) \). Moreover, if the ellipticity assumption (31.2) holds true and if \( \mu_\rho > 0 \) and \( \text{ess inf}_{x \in \partial D} (\mathbf{\beta} \cdot \mathbf{n})(x) \geq 0, \) the bilinear form \( a \) is \( V \)-coercive, and the problem (31.26) is well-posed.

Proof. Set \( \rho \equiv 0 \) in Propositions 31.12 and 31.14. \( \square \)
The coercivity assumption invoked in Proposition 31.17 fails when \( \mu \) and \( \beta \) vanish identically in \( D \), i.e., for the purely diffusive problem
\[
- \nabla \cdot (\frac{1}{\sqrt{\mu}} \nabla u) = f \quad \text{in} \quad D, \quad n \cdot (\frac{1}{\sqrt{\mu}} \nabla u) = g \quad \text{on} \quad \partial D. \tag{31.27}
\]
Indeed we observe that if \( u \) is a solution, then \( u + c \) is also a solution for all \( c \in \mathbb{R} \). A simple way to deal with this arbitrariness is to restrict the solution space to functions whose mean-value over \( D \) is zero, i.e., we consider the space \( H^1_0(D) := \{ v \in H^1(D) \mid \underline{v}_D = 0 \} \) where \( \underline{v}_D := |D|^{-1} \int_D v \, dx \). Note that a necessary condition for a solution to exist is the following compatibility condition on \( f \) and \( g \):
\[
\int_D f \, dx + \int_{\partial D} g \, ds = 0. \tag{31.28}
\]
Indeed (31.22) implies that if (31.27) has a solution \( u \), then
\[
\int_D f \, dx + \int_{\partial D} g \, ds = - \int_D \nabla \cdot (\frac{1}{\sqrt{\mu}} \nabla u) \, dx + (\gamma^d(\frac{1}{\sqrt{\mu}} \nabla u), 1)_{H^{-1/2}_0(D), H^{1/2}_0(D)} = 0.
\]
We now consider the following weak formulation:
\[
\begin{cases}
\text{Find} \ u \in V := H^1_0(D) \text{ such that } \\
a_d(u, w) = \int_D f w \, dx + \int_{\partial D} g \gamma_d(w) \, ds, \quad \forall w \in V.
\end{cases} \tag{31.29}
\]
with the bilinear form \( a_d(v, w) := \int_D (\frac{1}{\sqrt{\mu}} \nabla v) \cdot \nabla w \, dx \). Note that the test functions in (31.29) have also zero mean-value over \( D \).

**Proposition 31.18 (Well-posedness).** Let \( f \in L^2(D) \) and \( g \in L^2(\partial D) \) satisfy (31.28). Let \( u \) solve (31.29). Then the PDE (31.27) is satisfied a.e. in \( D \). The boundary condition is satisfied a.e. in \( \partial D \) in the sense that \( \gamma^d(\frac{1}{\sqrt{\mu}} \nabla u) = g \) in \( L^2(\partial D) \). Moreover, under the ellipticity condition (31.2), \( a_d \) is \( V \)-coercive, and the problem (31.29) is well-posed.

**Proof.** See Exercise 31.4. \( \square \)

### 31.3.3 Mixed Dirichlet–Neumann conditions
It is possible to combine the Dirichlet and the Neumann conditions. Let \( \partial D_d \) be a closed subset of \( \partial D \) and set \( \partial D_n := \partial D \setminus \partial D_d \). We assume that both subsets \( \partial D_d \) and \( \partial D_n \) have positive (surface) measures, and we enforce a Dirichlet and a Neumann condition on \( \partial D_d \) and \( \partial D_n \), respectively:
\[
u = g_d \quad \text{on} \quad \partial D_d, \quad n \cdot (\frac{1}{\sqrt{\mu}} \nabla u) = g_n \quad \text{on} \quad \partial D_n, \tag{33.30}
\]
with \( g_d \) and \( g_n \) defined on \( \partial D_d \) and \( \partial D_n \), respectively. We assume that there exists a bounded extension operator \( \mathcal{H}^{\frac{1}{2}}(\partial D_d) \rightarrow \mathcal{H}^{\frac{1}{2}}(\partial D) \), i.e., there exists \( C_{\partial D_d} > 0 \) s.t. for all \( \alpha \in \mathcal{H}^{\frac{1}{2}}(\partial D_d) \), there is \( \tilde{\alpha} \in \mathcal{H}^{\frac{1}{2}}(\partial D) \) s.t. \( \tilde{\alpha}|_{\partial D_d} := \alpha \) and \( C_{\partial D_d} \| \tilde{\alpha} \|_{\mathcal{H}^{\frac{1}{2}}(\partial D)} \leq \| \alpha \|_{\mathcal{H}^{\frac{1}{2}}(\partial D_d)} \). Owing to Theorem 2.29 this assumption holds true if the interface between \( \partial D_d \) and \( \partial D_n \) is Lipschitz. Then
let \( \tilde{u}_d \in H^1(D) \) be s.t. \( \gamma(\tilde{u}_d) = \tilde{g}_d \) and let \( V := \{ v \in H^1(D) \mid \gamma(v) = 0 \text{ a.e. on } \partial D_d \} \). Consider the weak formulation:

\[
\begin{align*}
\text{Find } u_0 & \in V \text{ such that } \\
& a(u_0, w) = \int_D f w \, dx + \int_{\partial D_a} g_n \gamma(w) \, ds - a(\tilde{u}_d, w), \quad \forall w \in V.
\end{align*}
\] (31.31)

Let \( \tilde{H}^\perp(\partial D_a) := \{ v \in H^\perp(\partial D_a) \mid \tilde{v} \in H^\perp(\partial D) \} \), where \( \tilde{v} \) is the zero-extension of \( v \) to \( \partial D \).

**Proposition 31.19 (Well-posedness).** Let \( f \in L^2(D), \ g_d \in H^\perp(\partial D_d) \), and \( g_n \in L^2(\partial D_n) \). Under the above assumptions, if \( u_0 \) solves (31.31), then \( u = u_0 + \tilde{u}_d \) solves the PDE (31.1) a.e. in \( D \), the Dirichlet condition is satisfied a.e. on \( \partial D_d \), and the Neumann condition a.e. on \( \partial D_n \) in the sense that \( \langle \gamma(d\nabla u), \tilde{v} \rangle_{H^\perp, H^\perp(\partial D)} = \int_{\partial D_n} g v \, ds \) for all \( v \in \tilde{H}^\perp(\partial D_n) \). Finally the problem (31.31) is well-posed under the assumptions of Proposition 31.7.

**Proof.** We only sketch the proof. That \( u = u_0 + \tilde{u}_d \) satisfies the PDE in \( D \) is shown as above. The Dirichlet condition results from \( \gamma(\tilde{u}_d|_{\partial D_d}) = \gamma(\tilde{u}_d|_{\partial D_d}) = \gamma(\tilde{u}_d|_{\partial D_d}) = \tilde{g}_d|_{\partial D_d} = g_d \). To obtain the Neumann condition we observe that for all \( v \in \tilde{H}^\perp(\partial D_n) \), there is \( w \in V \) s.t. \( \gamma(w) = \tilde{v} \). Using \( w \) as a test function in (31.31) we infer that \( \langle \gamma(d\nabla u), \tilde{v} \rangle_{H^\perp, H^\perp(\partial D)} = \int_{\partial D_n} g v \, ds \). To prove the well-posedness of (31.31) we first notice that \( V \) is a closed subspace of \( H^1(D) \). Indeed if \( (v_n)_{n \in \mathbb{N}} \) is a Cauchy sequence in \( V \), then \( v_n \to v \) in \( H^1(D) \) as \( n \to \infty \). This implies that \( \gamma(v) = 0 \text{ a.e. on } \partial D_d \) since \( \gamma(v_n) \to \gamma(v) \) in \( H^\perp(\partial D) \). To conclude the proof we use the following Poincaré–Steklov inequality in \( V \): There is \( \tilde{C}_\gamma > 0 \) such that \( \tilde{C}_\gamma \| v \|_{L^2(D)} \leq \ell_D \| \nabla v \|_{L^2(D)} \) for all \( v \in V \). This inequality is a consequence of Lemma 3.30 applied with the linear form \( f(v) = \int_{\partial D_n} \gamma(v) \, ds \) and \( p = 2 \) (note that \( V \ni v \mapsto \int_{\partial D_n} \gamma(v) \, ds \) restricted to constant functions is nonzero since \( \partial D_d \) has positive measure).

**Remark 31.20 (Data in \( \tilde{H}^\perp(\partial D_n)' \)).** The weak formulation (31.31) still makes sense if \( \int_{\partial D_n} g_n \gamma(w) \, ds \) is replaced by \( g_n(\gamma(w)|_{\partial D_n}) \) where \( g_n \in \tilde{H}^\perp(\partial D_n)' \), since the map \( V \ni v \mapsto \gamma(w)|_{\partial D_n} \in \tilde{H}^\perp(\partial D_n) \) is bounded.

**Remark 31.21 (\( \tilde{H}^\perp(\partial D_n) \) vs. \( H^\perp_{00}(\partial D_n) \)).** In the literature the interpolation space \( H^\perp_{00}(\partial D_n) \) introduced in Lions and Magenes [218, Thm. 11.7] is sometimes invoked instead of \( \tilde{H}^\perp(\partial D_n) \). More precisely, if \( U \) is a Lipschitz domain in \( \mathbb{R}^d \) (think of \( d := d-1 \)), we define \( H^\perp_{00}(U) := [L^2(U), H^\perp(\partial U)]_{1/2,2} \) (see Definition A.22). Then \( H^\perp_{00}(U) \to \tilde{H}^\perp(U) \) follows from Theorem A.27, since the zero-extension operator maps boundedly \( L^2(U) \) to \( L^2(\mathbb{R}^d) \) and \( H^\perp(U) \) to \( H^1(\mathbb{R}^d) \) (since \( 1 - \frac{1}{2} \notin \mathbb{N} \)). Moreover, as observed in [218, Thm. 11.7] and Tartar [280, p. 160], “\( H^\perp_{00}(U) \) is characterized as the space of functions
u in $H^\frac{1}{2}(U)$ such that $\frac{u}{\sqrt{d(x)}} \in L^2(U)$, where $d(x)$ is the distance to the boundary $\partial U$". Hence $H^\frac{1}{2}(U) = H^\frac{1}{2}_{00}(U)$. We also refer the reader to Chandler-Wilde et al. [85, Cor. 4.10], where it is shown that \{\tilde{H}^s(U) \mid s \in \mathbb{R}\} is an interpolation scale (i.e., for all $s_1 < s_2$ and all $s \in (s_1, s_2)$, we have $H^s(U) = [H^{s_1}(U), H^{s_2}(U)]_{\theta,2}$ with $\theta := (s-s_1)/(s_2-s_1)$). The above argument leads us to conjecture that the spaces $\tilde{H}^\frac{1}{2}(\partial D_n)$ and $H^\frac{1}{2}_{00}(\partial D_n)$ are identical provided the interface between $\partial D_n$ and $\partial D_d$ is smooth enough. Since we do not know any precise result from the literature establishing this equality, we prefer to work with the space $\tilde{H}^\frac{1}{2}(\partial D_n)$.

\section{31.4 Elliptic regularity}

The solution space $V$ for scalar second-order elliptic PDEs is such that $H^1_0(D) \subseteq V \subseteq H^1(D)$ depending on the type of boundary condition that is enforced. Since functions in $V$ may not have weak second-order derivatives, a natural question is whether it is possible to prove that the weak solution enjoys higher regularity. The elliptic regularity theory provides theoretical results allowing one to assert that under suitable assumptions on the smoothness of the domain and the data, the weak solution sits indeed in a Sobolev space with higher regularity, e.g., in $H^{1+r}(D)$ with $r > 0$. These results are important for the finite element error analysis since convergence rates depend on the smoothness of the weak solution. In this section we consider elliptic regularity results in the interior of the domain and then up to the boundary, with a particular attention paid to the practical case of a Lipschitz domain, e.g., a polygon or a polyhedron. Most of the results are just stated and we provide pointers to the literature for proofs.

Besides the hypotheses on the PDE coefficients from §31.1.1, we implicitly assume that the lower-order terms $\beta$ and $\mu$ are s.t. the advection-reaction term $\beta \nabla v + \mu v$ has the same smoothness as that requested for the source $f$ for all $v \in H^1(D)$. For instance, when we assume $f \in L^2(D)$, we also implicitly assume that $\beta \in L^\infty(D)$ and $\mu \in L^r(D)$, with $r > 2$ and $r \geq d$, so that $\beta \nabla v + \mu v \in L^2(D)$ for all $v \in H^1(D)$.

\subsection{31.4.1 Interior regularity}

We first present a general result concerning interior regularity, i.e., regularity in any subset $S \subset D$ (meaning that $\overline{S} \subseteq D$). Notice that we do not make any assumption on the boundary condition satisfied by the weak solution or on the smoothness of $D$.

\textbf{Theorem 31.22 (Interior regularity).} Let $D$ be a bounded open set. Assume that $a \in C^1(\overline{D})$ and $f \in L^2(D)$. Let $u \in H^1(D)$ be any of the above
weak solutions. Then for every open subset $S \subset D$, there are $C_1, C_2$ (depending on $S$, $D$, and the PDE coefficients) such that

$$\|u\|_{H^2(S)} \leq C_1\|f\|_{L^2(D)} + C_2\|u\|_{L^2(D)}.$$  \hspace{1cm} (31.32)

**Proof.** See Evans [150, §6.3.1]. The main tool for the proof is the technique of difference quotients by Nirenberg [239], Agmon et al. [5]. \hfill \Box

**Remark 31.23 (Sharper bound).** If the weak formulation is well-posed, the bound (31.32) takes the form $\|u\|_{H^2(S)} \leq C\|f\|_{L^2(D)}$ owing to the a priori estimate $\|u\|_{H^1(D)} \leq C\|f\|_{L^2(D)}$. \hfill \Box

**Remark 31.24 (Higher-order interior regularity).** Let $m$ be a nonnegative integer. Assume that $d \in C^{m+1}(\overline{D})$, that the coefficients $\{\beta_i\}_{i \in \{1:d\}}$ and $\mu$ are in $C^m(\overline{D})$, and that $f \in H^m(D)$. Let $u \in H^1(D)$ be any of the above weak solutions. Then for every open subset $S \subset D$, there are $C_1, C_2$ (depending on $S$, $D$, $m$, and the PDE coefficients) s.t. $\|u\|_{H^{m+2}(S)} \leq C_1\|f\|_{H^m(D)} + C_2\|u\|_{L^2(D)}$; see [150, §6.3.1]. \hfill \Box

### 31.4.2 Regularity up to the boundary

We are now concerned with the smoothness of the weak solution up to the boundary. In this context the smoothness of $\partial D$ and the nature of the boundary condition enforced on $\partial D$ play a role. The following theorems gather results established over the years by many authors. We refer the reader to the textbooks by Grisvard [170, 171], Dauge [117] for more detailed presentations. We consider three situations: domains having a smooth boundary, convex domains, and Lipschitz domains. In what follows we assume that the weak formulations are well-posed.

**Theorem 31.25 (Smooth domain).** Let $D$ be a domain in $\mathbb{R}^d$ with a $C^{1,1}$ boundary. Assume that $d$ is Lipschitz in $D$, i.e., there is $L$ s.t. $\|d(x) - d(y)\|_{L^\infty(\mathbb{R}^d)} \leq L\|x - y\|_{L^2(\mathbb{R}^d)}$ for all $x, y \in D$. Let $p \in (1, \infty)$ and assume that $f \in L^p(D)$. (i) The weak solution to the Dirichlet problem with boundary data $g \in W^{2-\frac{1}{p},p}(\partial D)$ is in $W^{2,p}(D)$. (ii) The weak solution to the Neumann problem with boundary data $g \in W^{1-\frac{1}{p},p}(\partial D)$ is in $W^{2,p}(D)$. The same conclusion holds true for the Robin problem if $\rho$ is Lipschitz on $\partial D$.

**Proof.** See [170, Thm. 2.4.2.5-2.4.2.7] (see also [150, §6.3.2] for the Dirichlet problem and $p = 2$). \hfill \Box

**Remark 31.26 (Neumann problem).** In the literature elliptic regularity for the Neumann problem is often considered with the assumptions of Proposition 31.17, that is to say, the coefficient $\mu$ is uniformly bounded away from zero from below. The Neumann problem (31.27) with the compatibility condition (31.28) can be treated by observing that if $u$ is the weak solution to this problem, then $u$ is also the weak solution to the Neumann problem
set in $H^1(D)$ with the coefficient $\mu \equiv \mu_0 > 0$ and the source term $f$ replaced by $f + \mu_0 u$, where $\mu_0$ is a nonzero constant with the appropriate units. □

**Theorem 31.27 (Higher-order regularity).** Let $m$ be a positive integer. Assume that $\partial D$ is of class $\mathcal{C}^{m+1,1}$, $d \in \mathcal{C}^{m,1}(D)$, and $f \in W^{m,p}(D)$. Assume that the coefficients $\{\beta_i\}_{i \in \{1:d\}}$ and $\mu$ are in $\mathcal{C}^m(\overline{D})$. Then the weak solution of the Dirichlet problem with $g \in W^{m+2-\frac{1}{p},p}(\partial D)$ is in $W^{m+2,p}(D)$. The same conclusion holds true for the Robin and Neumann problems if $g \in W^{m+1-\frac{1}{p},p}(\partial D)$ and $\rho \in \mathcal{C}^{m,1}(\partial D)$.

**Proof.** See [170, Thm. 2.5.1.1]. □

The smoothness assumption on $\partial D$ can be relaxed if the domain $D$ is convex. Notice that a convex domain is Lipschitz; see [170, Cor. 1.2.2.3].

**Theorem 31.28 (Convex domain).** Let $D$ be a convex domain. Assume that $d$ is Lipschitz in $D$. Let $f \in L^2(D)$. (i) The weak solution to the Dirichlet problem with $g = 0$ is in $H^2(D)$. (ii) The weak solution to the Robin or Neumann problem with $g = 0$ is in $H^2(D)$.

**Proof.** See [170, Thm. 3.2.1.2, 3.2.1.3, 3.2.3.1]. □

Elliptic regularity in Lipschitz domains is widely studied in the literature; see, e.g., Kondrat’ev [209], Maz’Ja and Plamenevskii [224], Jerison and Kenig [198, 199]. We first consider polygons in $\mathbb{R}^2$ and quote results from [170, Chap. 4].

**Theorem 31.29 (Polygon, $d = 1$).** Let $D \subset \mathbb{R}^2$ be a polygon with boundary vertices $\{S_j\}_{1 \leq j \leq J}$ where the segment joining $S_j$ to $S_{j+1}$ corresponds to the boundary face denoted by $F_j$ (setting conventionally $J + 1 := 1$). Let $\theta_j \in (0,2\pi)$ be the interior angle formed by the faces $F_j$ and $F_{j+1}$. Assume that $d$ is the identity matrix and that $\theta_j \neq \pi$ for all $j \in \{1:J\}$. Let $f \in L^2(D)$. (i) There is $s_0 \in (\frac{1}{2},1]$ such that the weak solution to the Dirichlet problem enforcing $u|_{F_j} = g_j$, with $g_j \in H^{-\frac{1}{2}}(F_j)$ and $g_j(S_j) = g_{j+1}(S_{j+1})$ for all $j \in \{1:J\}$, is in $H^{1+s}(D)$ for all $s \in [0,s_0]$ and $s_0 = 1$ if $D$ is convex. (ii) The same conclusion holds true for Neumann problem enforcing $\frac{\partial u}{\partial n}|_{F_j} = g_j$ with $g_j \in H^{\frac{1}{2}}(F_j)$ for all $j \in \{1:J\}$.

**Proof.** See [170, Cor. 4.4.4.14] (which treats mixed Dirichlet–Neumann conditions and $L^p$-Sobolev spaces). The weak solution is in $H^2(D)$ up to singular perturbations that behave in radial coordinates as $r^{\frac{1}{p} - s} \sin(\theta \frac{\partial}{\partial \theta} + \varphi_j)$ in the vicinity of $S_j$ with $\varphi_j \in \mathbb{R}$; see Exercise 31.5. □

**Remark 31.30 (Variable coefficients).** This case can be treated by freezing the diffusion tensor at each polygon vertex and applying locally a coordinate transformation to recover the Laplace operator; see [170, §5.2]. □
The analysis of elliptic regularity in a polyhedron is more intricate since vertex, edge, and edge-vertex singularities can occur; see Grisvard [170, §8.2], Dauge [117, §5], Lubuma and Nicaise [219, 220], Nicaise [238], Guo and Babuška [176, 177], Costabel et al. [113, 114]. For \( s \in (0, 1) \), let us define the space \( H^{-1+s}(D) \) either by interpolation between \( L^2(D) \) and \( H^{-1}(D) \) or as the dual of \( H^{1-s}_0(D) \) (the subspace of \( H^{1-s}(D) \) spanned by functions with zero trace on \( \partial D \) for \( s \in (0, \frac{1}{2}) \)). These two definitions give the same space with equivalent norms.

**Theorem 31.31 (Polyhedron, \( dI = 1 \), Dirichlet).** Let \( D \subset \mathbb{R}^3 \) be a polyhedron. There exists \( s_0 > \frac{1}{2} \), depending on \( D \), such that the Laplace operator is an isomorphism from \( H^{1+s}(D) \cap H^1_0(D) \) to \( H^{-1+s}(D) \) for all \( s \in [0, s_0] \).

**Proof.** This is a consequence of Theorem 18.13 in Dauge [117, p. 158]. \( \square \)

We finally consider the case of Lipschitz domains.

**Theorem 31.32 (Lipschitz diffusion).** Let \( D \) be a Lipschitz domain in \( \mathbb{R}^d \). Assume that \( dI \) is Lipschitz in \( D \). There is \( s_0 \in (0, \frac{1}{2}) \) such that the following holds true for all \( s \in [0, s_0] \): (i) The weak solution to the Dirichlet problem with \( f \in L^2(D) \) and \( g \in H^{2+s}(\partial D) \) is in \( H^{1+s}(D) \). (ii) The weak solution to the Neumann problem with \( f \in L^2(D) \) and \( g \in H^{-2+s}(\partial D) \) is in \( H^{1+s}(D) \).

**Proof.** See Theorems 3 and 4 in Savaré [263]. Notice also that the lowest-order terms in the PDE are in \( L^2(D) \) and that \( L^2(D) \subset H^{-1+s}(D) \) for all \( s \leq 1 \), so that \( f \) can be replaced by \( f - \beta \cdot \nabla u - \mu u \). \( \square \)

**Remark 31.33 (Very weak solution).** It is possible to extend the notion of elliptic regularity to the very weak solutions. Such solutions do not necessarily belong to the space \( H^1(D) \). For instance, using the transposition technique from Lions and Magenes [218, Chap. 2], it is shown in Savaré [263] that the statement of Theorem 31.32 also holds true for all \( s \in (-\frac{1}{2}, 0) \). \( \square \)

The Lipschitz property of \( dI \) is rather restrictive since it excludes domains composed of different materials. Following Jochmann [201], it is possible to replace this hypothesis by a (usually called) multiplier assumption, which consists of assuming that there is \( s_0 \in (0, \frac{1}{2}) \) such that

\[
\text{the map } H^{s_0}(D) \ni \xi \mapsto dI \xi \in H^{s_0}(D) \text{ is bounded.} \quad (31.33)
\]

It is shown in Jochmann [201, Lem. 2] (see also Bonito et al. [46, Prop. 2.1]) that this property holds true if \( D \) is partitioned into \( M \) disjoint Lipschitz subdomains \( \{D_m\}_{m \in \{1:M\}} \) and if there is a real number \( \alpha > s_0 \) and there are diffusion tensors \( dI_m \in C^{0,\alpha}(D_m) \), for all \( m \in \{1:M\} \), s.t. \( dI := \sum_{m \in \{1:M\}} \mathbb{1}_{D_m} dI_m \), where \( \mathbb{1}_{D_m} \) is the indicator function of \( D_m \).

**Theorem 31.34 (Piecewise smooth diffusion).** Assume that there is \( s_0 \in (0, \frac{1}{2}) \) such that the multiplier assumption (31.33) holds true. Then
there is \( s \in (0, s_0) \), depending on \( D \) and \( d \), s.t. the weak solution to the homogeneous Dirichlet or Neumann problem with \( f \in L^2(D) \) (and \( g = 0 \)) is in \( H^{1+s}(D) \).

**Proof.** See Theorem 3 in [201] or Lemma 3.2 in [46]. The statement also holds true for \( f \) in the dual space of \( H_0^{1-s}(D) \) for the Dirichlet problem and for \( f \) in the dual space of \( H^{1-s}(D) \) for the Neumann problem. See also Bernardi and Verfürth [38] for Dirichlet conditions and piecewise constant (or pcw. twice continuously differentiable) isotropic diffusion. \( \square \)

Theorem 31.34 also holds true for the mixed Dirichlet-Neumann problem. We refer the reader to Jochmann [201] for more details on this question.

### Exercises

**Exercise 31.1 (Cordes).** Prove that ellipticity implies the Cordes condition if \( d = 2 \). (Hint: use that \( \|d\|_F^2 = (\tr(d))^2 - 2 \det(d) \).)

**Exercise 31.2 (Poincaré–Steklov).** Prove (31.23). (Hint: use (3.12).)

**Exercise 31.3 (Potential flow).** Consider the PDE \( \nabla(-\kappa \nabla u + \beta u) = f \) in \( D \) with homogeneous Dirichlet conditions and assume that \( \kappa \) is a positive real number. Assume that \( \beta := \nabla \psi \) for some smooth function \( \psi \) (we say that \( \beta \) is a potential flow). Find a functional \( \mathcal{E} : H^1_0(D) \to \mathbb{R} \) of which the weak solution \( u \) is a minimizer on \( H^1_0(D) \). (Hint: consider the function \( e^{-\psi/\kappa} u \).)

**Exercise 31.4 (Purely diffusive Neumann).** Prove Proposition 31.18. (Hint: for all \( w \in H^1(D) \), the function \( \tilde{w} := w - \underline{u}_D \) is in \( H^1(D) \), use also the Poincaré–Steklov inequality from Lemma 3.24.)

**Exercise 31.5 (Mixed Dirichlet–Neumann).** The goal is to show by a counterexample that one cannot assert that the weak solution is in \( H^2(D) \) for the mixed Dirichlet–Neumann problem even if the domain and the boundary data are smooth. Using polar coordinates, set \( D := \{(r, \theta) \in (0, 1) \times (0, \pi)\} \), \( \partial D_n := \{(r \in (0, 1), \theta = \pi)\} \), and \( \partial D_a := \partial D \setminus \partial D_n \). Verify that the function \( u(r, \theta) := r^4 \sin^2(\frac{\theta}{2}) \) satisfies \( -\Delta u = 0 \) in \( D \), \( \frac{\partial u}{\partial n}|_{\partial D_n} = 0 \), and \( u|_{\partial D_a} = r^4 \sin^2(\frac{\theta}{2}) \). (Hint: in polar coordinates, \( \Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \). Verify that \( u \notin H^2(D) \).)

**Exercise 31.6 (\( H^2(\mathbb{R}^d) \)-seminorm).** Prove that \( \|\phi\|_{H^2(\mathbb{R}^d)} = \|\Delta \phi\|_{L^2(\mathbb{R}^d)} \) for all \( \phi \in C^0(\mathbb{R}^d) \). (Hint: use Lemma B.3.)

**Exercise 31.7 (Counterexample to elliptic regularity in \( W^{2,\infty}(D) \)).** Let \( D \) be the unit disk in \( \mathbb{R}^2 \). Consider the function \( u(x_1, x_2) := x_1 x_2 \ln(r) \) with \( r^2 := x_1^2 + x_2^2 \) (note that \( u|_{\partial D} = 0 \)). Verify that \( \Delta u \in L^\infty(D) \), but that \( u \notin W^{2,\infty}(D) \). (Hint: consider the cross-derivative.)
Exercise 31.8 (Domain with slit). Let \( D := \{ r \in (0,1), \theta \in (0,2\pi) \} \), where \( (r,\theta) \) are the polar coordinates, i.e., \( \overline{D} \) is the closed ball of radius 1 centered at 0. Let \( u(r,\theta) := r \cos(\frac{1}{2}\theta) \) for all \( r > 0 \) and \( \theta \in [0, 2\pi) \). (i) Let \( p \in [1, \infty) \). Is \( u|_D \in W^{1,p}(D) \)? Is \( u|_{\text{int}(D)} \in W^{1,p}(\text{int}(D)) \)? (Hint: recall Example 4.3.) (ii) Is the restriction to \( D \) of the functions in \( C^1(D) \) dense in \( W^{1,p}(D) \)? (Hint: argue by contradiction and use that \( \| u|_D \|_{W^{1,p}(D)} = \| u|_{\text{int}(D)} \|_{W^{1,p}(\text{int}(D))} \) for all \( v \in C^1(D) \).

Exercise 31.9 (A priori estimate). Consider the PDE \(-\kappa_0 \Delta u + \beta \cdot \nabla u + \mu_0 u = f\) with homogeneous Dirichlet conditions. Assume that \( \kappa_0, \mu_0 \in \mathbb{R} \), \( \kappa_0 > 0 \), \( \nabla \cdot \beta = 0 \), \( \beta|_{\partial D} = 0 \), and \( f \in H^1_0(D) \). Let \( \nabla_s \beta := \frac{1}{2}(\nabla \beta + (\nabla \beta)^T) \) denote the symmetric part of the gradient of \( \beta \), and assume that there is \( \mu'_0 > 0 \) s.t. \( \nabla_s \beta + \mu_0 I_d \geq \mu'_0 I_d \) in the sense of quadratic forms. Prove that \( \| u \|_{H^1(D)} \leq (\mu'_0)^{-\frac{1}{2}} \| f \|_{H^1(D)} \) and \( \| \Delta u \|_{L^2(D)} \leq (4\mu_0^2 \kappa_0^{-\frac{1}{2}}) \| f \|_{H^1(D)} \). (Hint: use \(-\Delta u\) as a test function.) Note: these results are established in Beirão da Veiga [36], Burman [68].

Exercise 31.10 (Complex-valued diffusion). Assume that the domain \( D \) is partitioned into two disjoint subdomains \( D_1 \) and \( D_2 \). Let \( \kappa_1, \kappa_2 \) be two complex numbers, both with positive modulus and such that \( \frac{\kappa_1}{\kappa_2} \not\in \mathbb{R}^- \). Set \( \kappa(x) := \kappa_1 1_{D_1}(x) + \kappa_2 1_{D_2}(x) \) for all \( x \in D \). Let \( f \in L^2(D) \). Show that the problem of seeking \( u \in V := H^1_0(D; \mathbb{C}) \) such that \( a(u, w) := \int_D \kappa \nabla u \cdot \nabla \overline{w} \, dx = \int_D f \overline{w} \, dx \) for all \( w \in V \) is well-posed. (Hint: use (25.7).)
Solution to exercises

Exercise 31.1 (Cordes). Using the symmetry of $d$, we have $\|d\|_F^2 = (\text{tr}(d))^2 - 2 \det(d)$, where $\det(d)$ is the determinant of $d$, so that $\|d\|_F^2 = \frac{2 \det(d)}{1 + \epsilon}$. Since $\det(d) > 0$ by the ellipticity condition, we have $\epsilon > 0$. Since $(\text{tr}(d))^2 \geq 4 \det(d)$ if $d = 2$, we have $\|d\|_F^2 \geq 2 \det(d)$, so that $\epsilon \leq 1$, the case $\epsilon = 1$ being reached when both eigenvalues of $d$ are equal, i.e., $d = \lambda I$, with $\lambda > 0$.

Exercise 31.2 (Poincaré–Steklov). Let us define the linear form $\ell_D : H^1(D) \to \mathbb{R}$ by

$$\ell_D(v) = \int_{\partial D} \gamma^g(v) \, ds,$$

for all $v \in H^1(D)$. Using the Cauchy–Schwarz inequality for the rightmost term yields $|\int_{\partial D} \gamma^g(v) \, ds| \leq \|\partial D\|^2 \|\gamma^g(v)\|_{L^2(\partial D)}$. Hence we have

$$\sqrt{2} C_{\text{ps}} \|v\|_{L^2(D)} \leq \ell_D(\nabla v) + \|\partial D\|^2 \|\gamma^g(v)\|_{L^2(\partial D)},$$

and we conclude using Young's inequality: $(a + b) \leq (2(a^2 + b^2))^{\frac{1}{2}}$.

Exercise 31.3 (Potential flow). We observe that

$$\nabla (e^{-\psi/\kappa} u) = \frac{1}{\kappa} e^{-\psi/\kappa} (\kappa \nabla u - \beta u),$$

since $\nabla \psi = \beta$. Consider the functional $\mathcal{E} : H^1_0(D) \to \mathbb{R}$ such that

$$\mathcal{E}(v) = \frac{1}{2} \int_D e^{\psi/\kappa} \left| \nabla (e^{-\psi/\kappa} v) \right|^2 \, dx - \int_D e^{-\psi/\kappa} f v \, dx,$$

for all $v \in H^1_0(D)$. Proceeding as in the proof of Proposition 25.7, we see that $u \in H^1_0(D)$ is a global minimizer of $\mathcal{E}$ if and only if

$$\int_D e^{\psi/\kappa} \kappa \nabla (e^{-\psi/\kappa} u) \cdot \nabla (e^{-\psi/\kappa} w) \, dx = \int_D e^{-\psi/\kappa} f w \, dx,$$

for all $w \in H^1_0(D)$. Taking $w$ to be arbitrary in $C_c^\infty(D)$, we infer that

$$\int_D e^{-\psi/\kappa} \nabla (-\kappa \nabla u + \beta u) w \, dx = \int_D e^{-\psi/\kappa} f w \, dx$$

which shows that $u$ satisfies the PDE $\nabla (-\kappa \nabla u + \beta u) = f$ a.e. in $D$.

Exercise 31.4 (Purely diffusive Neumann). For all $w \in H^1(D)$, writing $w = \tilde{w} + \underline{w}_D$ with $\tilde{w} \in H^1_0(D)$ and testing the weak formulation against $\tilde{w}$,
we infer that the weak solution satisfies
\[ a_d(u, w) = a_d(u, \tilde{w}) = \int_D f \tilde{w} \, dx + \int_{\partial D} g \gamma^s(\tilde{w}) \, ds = \int_D f w \, dx + \int_{\partial D} g \gamma^s(w) \, ds, \]
where we used the compatibility condition (31.28) in the last equality. Since the equality \( a_d(u, w) = \int_D f w \, dx + \int_{\partial D} g \gamma^s(w) \, ds \) is valid for every function \( w \in H^1(D) \), we infer as in the case of Robin conditions that the PDE and the boundary condition in (31.27) are satisfied a.e. in \( D \) and a.e. on \( \partial D \), respectively. To prove well-posedness, we use the Poincaré–Steklov Lemma 3.24 with \( p = 2 \), i.e., \( C_{\text{ps}} \| v \|_{L^2(D)} \leq \ell_D \| \nabla v \|_{L^2(\partial D)} \) for all \( v \in H^1_0(D) \), so that \( V = H^1_0(D) \) equipped with the norm \( \| v \|_V := \| \nabla v \|_{L^2(\partial D)} \) is a Hilbert space. Since \( a_d(v, v) \geq \lambda_0 \| v \|_V^2 \), this proves coercivity, and thus well-posedness follows from the Lax–Milgram lemma.

**Exercise 31.5 (Mixed Dirichlet–Neumann).** A direct computation yields \( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{1}{r} r^{-\frac{3}{2}} \sin \left( \frac{\theta}{2} \right) \) and \( \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{2} r^{-\frac{3}{2}} \sin \left( \frac{\theta}{2} \right) \) so that \( \Delta u = 0 \). The Dirichlet condition is clearly satisfied on \( \partial D_A \). Concerning the Neumann condition on \( \partial D_n \), we observe that \( \frac{\partial u}{\partial n} = \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{3} r^{-\frac{3}{2}} \cos \left( \frac{\theta}{2} \right) \) which vanishes for \( \theta = \pi \). Finally we observe that \( \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{2} r^{-\frac{3}{2}} \sin \left( \frac{\theta}{2} \right) \) and that \( \int_0^r r^{-3} \, dr \) is not bounded.

**Exercise 31.6 (\( H^2(\mathbb{R}^d) \)-seminorm).** Let \( \phi \in C^\infty_0(\mathbb{R}^d) \). Integrating by parts, we infer that
\[
\| \phi \|_{H^2(\mathbb{R}^d)}^2 = \sum_{i,j \in \{1:d\}} \int_{\mathbb{R}^d} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \, dx
\]
\[= - \sum_{i,j \in \{1:d\}} \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial x_i} \frac{\partial}{\partial x_j} \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) \, dx
\]
\[= - \sum_{i,j \in \{1:d\}} \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial x_i} \frac{\partial^3 \phi}{\partial x_i \partial x_j^2} \, dx
\]
\[= \sum_{i,j \in \{1:d\}} \int_{\mathbb{R}^d} \frac{\partial^3 \phi}{\partial x_i \partial x_j^2} \, dx = \| \Delta \phi \|_{L^2(\mathbb{R}^d)}^2,
\]
where we used Lemma B.3 to exchange the order of the partial derivatives.

**Exercise 31.7 (Counterexample to elliptic regularity in \( W^{2,\infty}(D) \)).** We observe that
\[
\Delta u = x_1 x_2 \Delta (\ln(r)) + 2 \nabla (x_1 x_2) \cdot \nabla (\ln(r)) + \Delta (x_1 x_2) \ln(r)
\]
\[= 2 \nabla (x_1 x_2) \cdot \nabla (\ln(r)) + 4 \frac{x_1 x_2}{r^2},
\]
so that \( \Delta u \in L^\infty(D) \). Moreover we have
\[ \frac{\partial^2 u}{\partial x_1 \partial x_2} = \ln(r) + 1 - \frac{2x_1^2 x_2^2}{r^4}, \]

which is unbounded at the origin.

**Exercise 31.8 (Domain with slit).** (i) Since \( \partial_{\theta} u = -\frac{1}{r} \sin\left(\frac{\theta}{2}\right) + 2r \delta_{\theta=0} \), where \( \delta_{\theta=0} \) is the Dirac measure whose support is the segment \{ \( r \in (0, 1), \ \theta = 0 \} = \{ x_1 \in (0, 1), \ x_2 = 0 \} \), we infer that \( u_{|D} \in W^{1,p}(D) \), but \( u_{|\text{int}(D)} \notin W^{1,p}(\text{int}(\overline{D})) \) since \( \delta_{\theta=0} \) cannot be identified with any function in \( L^p(\text{int}(\overline{D})) \); see Example 4.3.

(ii) Assume that the restriction to \( D \) of the functions in \( C^1(\overline{D}) \) is dense in \( W^{1,p}(D) \). Since \( u_{|D} \in W^{1,p}(D) \), there is a sequence of functions in \( C^1(\overline{D}) \), say \( (v_n)_{n \in \mathbb{N}} \), such that \( v_n|_{D} \to u_{|D} \) in \( W^{1,p}(D) \). But, since \( v_n \in C^1(\overline{D}) \subset W^{1,p}(\overline{D}) \) and \( |\text{int}(\overline{D})| \setminus D = 0 \), we have

\[
\| v_n|_{D} \|_{W^{1,p}(D)} = \| v_n|_{\text{int}(\overline{D})} \|_{W^{1,p}(\text{int}(\overline{D}))}.
\]

This means that \( (v_n|_{\text{int}(\overline{D}})_{n \in \mathbb{N}} \) is a Cauchy sequence in \( W^{1,p}(\text{int}(\overline{D})) \). Let \( w \) be the limit in question. Then

\[
w_{|D} = u_{|D}, \quad \text{a.e.}
\]

This proves that \( u_{|\text{int}(\overline{D})} = u_{|\text{int}(\overline{D})} \) since \( |\text{int}(\overline{D})| \setminus D = 0 \). This in turn establishes that \( u_{|\text{int}(\overline{D})} \in W^{1,p}(\text{int}(\overline{D})) \), which is a contradiction. Hence the restriction to \( D \) of the functions in \( C^1(\overline{D}) \) is not dense in \( W^{1,p}(D) \).

**Exercise 31.9 (A priori estimate).** Following the hint and integrating by parts, we infer that

\[
\kappa_0 \| \Delta u \|_{L^2(D)}^2 - (\beta \cdot \nabla u, \Delta u)_{L^2(D)} + \mu_0 |u|_{H^1(D)}^2 = -(f, \Delta u)_{L^2(D)} = (\nabla f, \nabla u)_{L^2(D)},
\]

where we used that \( u \in H^1_0(D) \) in the third term on the left-hand side and \( f \in H^1_0(D) \) on the right-hand side. Using that \( \beta_{|\partial D} = 0 \), we infer that

\[
-(\beta \cdot \nabla u, \Delta u)_{L^2(D)} = \sum_{i,j \in \{1:d\}} (\beta_{i} \partial_i u, \partial_j \partial_j u)_{L^2(D)} = \sum_{i,j \in \{1:d\}} (\partial_j \beta_i \partial_i u, \partial_j \partial_j u)_{L^2(D)} + (\beta_i \partial_i (\partial_j u), \partial_j u)_{L^2(D)} =: \mathcal{L}_1 + \mathcal{L}_2.
\]

We have \( \mathcal{L}_1 = (\nabla \beta) \nabla u, \nabla u)_{L^2(D)} \). Using that \( \nabla \beta = 0 \) and using again that \( \beta \) vanishes at the boundary, we obtain that
\[ T_2 = \sum_{i,j \in \{1:d\}} (\beta \nabla \partial_j u, \partial_j u)_{L^2(D)} = \int_D \frac{1}{2} \nabla (\beta \| \nabla u \|^2) \, dx = 0. \]

In summary, we have shown that

\[ \kappa_0 \| \Delta u \|^2_{L^2(D)} + ((\nabla \beta) \nabla u, \nabla u)_{L^2(D)} + \mu_0 |u|^2_{H^1(D)} = (\nabla f, \nabla u)_{L^2(D)}. \]

Our assumption on \( \nabla s \beta \) implies that

\[ \kappa_0 \| \Delta u \|^2_{L^2(D)} + \mu'_0 |u|^2_{H^1(D)} \leq (\nabla f, \nabla u)_{L^2(D)}. \]

The estimate on \( |u|_{H^1(D)} \) follows by applying the Cauchy–Schwarz inequality to the right-hand side. The estimate on \( \| \Delta u \|_{L^2(D)} \) follows by bounding the right-hand side as \( \mu'_0 |u|^2_{H^1(D)} + (4\mu'_0)^{-1} |f|^2_{H^1(D)} \).

**Exercise 31.10 (Complex-valued diffusion).** Let us write \( \kappa_m := |\kappa_m| e^{i \varphi_m} \) for all \( m \in \{1,2\} \). Set \( \xi := e^{-i \frac{\varphi_1 + \varphi_2}{2}} \). Then the real part of \( \xi \kappa_1 \) is \( |\kappa_1| \cos \left( \frac{\varphi_1 - \varphi_2}{2} \right) \) and that of \( \xi \kappa_2 \) is \( |\kappa_2| \cos \left( \frac{\varphi_2 - \varphi_1}{2} \right) \). It is readily seen that these two real numbers have the same sign and that they are both nonzero since \( \frac{\varphi_2 - \varphi_1}{2} \neq \pm \frac{\pi}{2} \) (since otherwise \( \kappa_1 \kappa_2 \) would be a negative real number). Hence up to a possible sign change in \( \xi \), the bilinear form \( a \) satisfies the coercivity property (25.7). We conclude using the Lax–Milgram lemma.