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Splittings and calculational techniques for higher THH

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Tensoring finite pointed simplicial sets X with commutative ring spectra R yields important homology theories such as (higher) topological Hochschild homology and torus homology. We prove several structural properties of these constructions relating $X \otimes (-)$ to $\Sigma X \otimes (-)$ and we establish splitting results. This allows us, among other important examples, to determine $\mathsf{THH}_*^{[n]}(\mathbb{Z}/p^m; \mathbb{Z}/p)$ for all $n \geqslant 1$ and for all $m \geqslant 2$.

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Introduction

For any (finite) simplicial set X one can define the tensor product of X with a commutative ring spectrum A, $X \otimes A$, where the case $X = S^1$ gives the topological Hochschild homology of A. More generally, for any sequence of maps of commutative ring spectra $R \to A \to C$ and any (finite) pointed simplicial set X, we can define $\mathcal{L}_X^R(A; C)$, the Loday construction with respect to X of A over R with coefficients in C. Important examples are $X = S^n$ or X a torus. The construction specializes to $X \otimes A$ in the case $\mathcal{L}_X^S(A; A)$, where S denotes the sphere spectrum. For details see Definition 1.1.

An important question about the Loday construction concerns the dependence on X: given two pointed simplicial sets X and Y, with $\Sigma X \simeq \Sigma Y$, does that imply that $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$? If it does, the Loday construction would be a "stable invariant". Positive cases arise from the work of Berest, Ramadoss and Yeung [5, Theorem 5.2]: they identify the homotopy groups of the Loday construction with respect to a simplicial set X of a commutative Hopf algebra over a field with representation homology of the Hopf algebra with respect to $\Sigma(X_+)$, where X_+ denotes X with an added disjoint

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basepoint. In [16], Dundas and Tenti prove that stable invariance holds if A is a smooth algebra over a commutative ring k. However, in [16] they also provide a counterexample: $\mathcal{L}^{H\mathbb{Q}}_{S^2\vee S^1\vee S^1}(H\mathbb{Q}[t]/t^2)$ is *not* equivalent to $\mathcal{L}^{H\mathbb{Q}}_{S^1\times S^1}(H\mathbb{Q}[t]/t^2)$ even though $\Sigma(S^2\vee S^1\vee S^1)\simeq\Sigma(S^1\times S^1)$. Our juggling formula (Theorem 3.3) and our generalized Brun splitting (Theorem 4.1) relate the Loday construction on ΣX to that of X. One application among others of these results is to establish stable invariance in certain examples.

For commutative $\mathbb{F}_p$–algebras A one often observes a splitting of the spectrum $\mathsf{THH}(A)$ as $\mathsf{THH}(\mathbb{F}_p)\wedge_{H\mathbb{F}_p}\mathsf{THH}^{H\mathbb{F}_p}(HA)$, so $\mathsf{THH}(A)$ splits as the topological Hochschild homology of $\mathbb{F}_p$ tensored with the Hochschild homology of A; see Larsen and Lindenstrauss [25]. It is natural to ask in what generality such splittings occur. If one replaces $\mathbb{F}_p$ by $\mathbb{Z}$, then there are many counterexamples. For instance, if $A=\mathcal{O}_K$ is a number ring then $\mathsf{THH}_*(\mathcal{O}_K)$ is known by Lindenstrauss and Madsen [26, Theorem 1.1] and is far from being equivalent to $\pi_*(\mathsf{THH}(\mathbb{Z})\wedge^L_{H\mathbb{Z}}\mathsf{THH}^{H\mathbb{Z}}(H\mathcal{O}_K))$ in general (see Halliwell, Höning, Lindenstrauss, Richter and Zakharevich [19, Remark 4.12.] for a concrete example). We prove several splitting results for higher THH and use one of them to determine higher THH of $\mathbb{Z}/p^m$ with $\mathbb{Z}/p$–coefficients for all $m\geqslant 2$.

Content

We start with a brief recollection of the Loday construction in Section 1.

In [6] we determined the higher Hochschild homology of $R=\mathbb{F}_p[x]$ and of $R=\mathbb{F}_p[x]/x^{p^\ell}$, both Hopf algebras. In Section 2 we generalize our results to the cases $R=\mathbb{F}_p[x]/x^m$ if p divides m, which is not a Hopf algebra unless $m=p^\ell$ for some ℓ.

Building on work in [19] we prove a juggling formula (see Theorem 3.3): for every sequence of cofibrations of commutative ring spectra $S\to R\to A\to B\to C$ there is an equivalence

$$\mathcal{L}^R_{\Sigma X}(B;C)\simeq\mathcal{L}^R_{\Sigma X}(A;C)\wedge^L_{\mathcal{L}^A_X(C)}\mathcal{L}^B_X(C).$$

This result is crucial for the applications that we present in this paper. It has to be handled with care: we explain in Section 3.1 what happens if one oversimplifies this formula.

Using a geometric argument, Brun [9] constructs a spectral sequence for calculating THH_*–groups. As a consequence of the above juggling formula we obtain a generalization of his splitting: For any sequence of cofibrations of commutative ring spectra

$S \to R \to A \to B$ (see Theorem 4.1) there is an equivalence of commutative B–algebra spectra

$$\mathcal{L}^R_{\Sigma X}(A;B) \simeq B \wedge^L_{\mathcal{L}^R_X(B)} \mathcal{L}^A_X(B).$$

Note that B, which only appears at the basepoint on the left, now appears almost everywhere on the right. This splitting also gives rise to associated spectral sequences for calculating higher THH_*–groups.

We apply our results to prove a generalization of Greenlees' splitting formula [18, Remark 7.2]: For an augmented commutative k–algebra A we obtain in Corollary 3.6 that

$$\mathcal{L}_{\Sigma X}(HA;Hk) \simeq \mathcal{L}_{\Sigma X}(Hk) \wedge^L_{Hk} \mathcal{L}^{HA}_X(Hk)$$

and this can be written as

$$\mathcal{L}_{\Sigma X}(HA;Hk) \simeq \mathcal{L}_{\Sigma X}(Hk) \wedge^L_{Hk} \mathcal{L}^{Hk}_{\Sigma X}(HA;Hk),$$

where all the Loday constructions are over the same simplicial set. For $X = S^n$, for example, this yields Theorem 5.9,

$$\mathsf{THH}^{[n]}(A;k) \simeq \mathsf{THH}^{[n]}(k) \wedge^L_{Hk} \mathsf{THH}^{[n],k}(A;k),$$

where $\mathsf{THH}^{[n]} = \mathcal{L}_{S^n}$.

Shukla homology is a derived version of Hochschild homology. We define higher-order Shukla homology in Section 5 and calculate some examples that will be used in subsequent results. We prove that the Shukla homology of order n of a ground ring k over a flat augmented k–algebra is isomorphic to the reduced Hochschild homology of order $n+1$ of the flat augmented algebra (Proposition 5.8).

Tate [39] shows how to control Tor groups for certain quotients of regular local rings. We use this to develop a splitting on the level of homotopy groups for

$$\mathsf{THH}(R/(a_1,\dots,a_r);R/\mathfrak{m})$$

as

$$\mathsf{THH}_*(R/(a_1,\dots,a_r);R/\mathfrak{m}) \cong \mathsf{THH}_*(R;R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1,\dots,S_r)$$

if R is regular local, $\mathfrak{m}$ is the maximal ideal and the a_i are in $\mathfrak{m}^2$. Here, Γ denotes a divided power algebra and $|S_i| = 2$ for all $1 \leqslant i \leqslant r$.

We prove a splitting result for $\mathsf{THH}^{[n]}(R/a, R/p)$ in Section 7, where R is a commutative ring and $p, a \in R$ are not zero divisors, (p) is a maximal ideal and $a \in (p)^2$. In this situation,

$$\textbf{(0.1)} \qquad \mathsf{THH}^{[n]}(R/a, R/p) \simeq \mathsf{THH}^{[n]}(R, R/p) \wedge^L_{HR/p} \mathsf{THH}^{[n];R}(R/a, R/p).$$

The juggling formula is an important ingredient in the proof, but it doesn't suffice to deduce this splitting result. We need a careful analysis of the maps involved; this is the content of Section 7.

In many cases the homotopy groups of the factors on the right-hand side of (0.1) can be completely determined. Among other important examples we get explicit formulas for $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m, \mathbb{Z}/p)$ for all $n \geqslant 1$ and all $m \geqslant 2$ (compare Theorem 9.1):

$$\mathsf{THH}^{[n]}_*(\mathbb{Z}/p^m, \mathbb{Z}/p) \cong \mathsf{THH}^{[n]}_*(\mathbb{Z}, \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} \mathsf{THH}^{[n],\mathbb{Z}}_*(\mathbb{Z}/p^m, \mathbb{Z}/p).$$

We know $\mathsf{THH}^{[n]}_*(\mathbb{Z}, \mathbb{Z}/p)$ from Dundas, Lindenstrauss and Richter [13] and we determine $\mathsf{THH}^{[n],\mathbb{Z}}_*(\mathbb{Z}/p^m, \mathbb{Z}/p)$ explicitly for all n.

This generalizes previous results by Pirashvili [32], Brun [9] and Angeltveit [1] from $n = 1$ to all n.

In Section 8 we provide a splitting result for commutative ring spectra of the form $A \times B$: we show in Proposition 8.4 that for any finite connected simplicial set X, we have

$$\mathcal{L}_X(A \times B) \xrightarrow{\simeq} \mathcal{L}_X(A) \times \mathcal{L}_X(B).$$

We present some sample applications of our splitting results in Section 9: a splitting of higher THH of ramified number rings with reduced coefficients (Section 9.2), a version of Galois descent for higher THH (Section 9.3) and a calculation of higher THH of function fields over $\mathbb{F}_p$ (Section 9.4). We close with a discussion of the case of higher THH of $\mathbb{Z}/p^m$ (with unreduced coefficients) in Section 9.5.

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1 The Loday construction: basic features

We recall some definitions concerning the Loday construction and we fix notation.

For our work we can use any good symmetric monoidal category of spectra whose category of commutative monoids is Quillen equivalent to the category of E_∞–ring spectra, such as symmetric spectra [23], orthogonal spectra [27] or S–modules [17]. As parts of the paper require working with a specific model category, we work with the category of S–modules unless otherwise specified.

Let X be a finite pointed simplicial set and let $R \to A \to C$ be a sequence of maps of commutative ring spectra. We assume that R is a cofibrant commutative S–algebra and that A and C are cofibrant commutative R–algebras.

Definition 1.1 The Loday construction with respect to X of A over R with coefficients in C is the simplicial commutative augmented C–algebra spectrum $\mathcal{L}^R_X(A;C)$ whose p–simplices are

$$C \wedge \bigwedge_{x \in X_p \setminus *} A,$$

where the smash products are taken over R. Here, $*$ denotes the basepoint of X and we place a copy of C at the basepoint. As the smash product over R is the coproduct in the category of commutative R–algebra spectra, the simplicial structure is straightforward: Face maps d_i on X induce multiplication in A or the A–action on C if the basepoint is involved. Degeneracies s_i on X correspond to the insertion of the unit maps $\eta_A\colon R \to A$ over all n–simplices which are not hit by $s_i\colon X_{n-1} \to X_n$.

The cofibrancy assumptions on R, A and C ensure that the homotopy type of $\mathcal{L}^R_X(A;C)$ is well defined.

As defined above, $\mathcal{L}^R_X(A;C)$ is a simplicial commutative augmented C–algebra spectrum. If M is an A–module spectrum (which is cofibrant as an R–module spectrum), then $\mathcal{L}^R_X(A;M)$ is defined. By slight abuse of notation we won't distinguish $\mathcal{L}^R_X(A;C)$ or $\mathcal{L}^R_X(A;M)$ from their geometric realization. For $C = A$ we abbreviate $\mathcal{L}^R_X(A;A)$ by $\mathcal{L}^R_X(A)$. If $R = S$, then we omit it from the notation. Note that $\mathcal{L}^R_X(A)$ is by definition [17, Chapter VII, Sections 2–3] equal to $X \otimes A$, where $X \otimes A$ is formed in the category of commutative R–algebras.

If X is an arbitrary pointed simplicial set, then we can write it as the colimit of its finite pointed subcomplexes and the Loday construction with respect to X can then also be expressed as the colimit of the Loday construction for the finite subcomplexes.

Let A be a fixed cofibrant commutative R–algebra. As the model structure on commutative R–algebras is a topological model category [17, Corollary VII.4.8], the functor $X \mapsto X \otimes A$ sends homotopy pushouts to homotopy pushouts; in particular,

$$\Sigma X \otimes A \cong CX \otimes A \wedge_{(X \otimes A)} CX \otimes A \simeq A \wedge^L_{(X \otimes A)} A,$$

where $CX \simeq *$ denotes the cone on X. Similarly, for a cofibrant commutative R–algebra C with a morphism of commutative R–algebras $A \to C$ one has

$$\mathcal{L}^R_{\Sigma X}(A; C) \simeq C \wedge^L_{\mathcal{L}^R_X(A;C)} C.$$

An important case of the Loday construction is $X = S^n$. In this case we write $\mathsf{THH}^{[n],R}(A; C)$ for $\mathcal{L}^R_{S^n}(A; C)$; this is the *higher-order topological Hochschild homology of order n of A over R with coefficients in C*.

Let k be a commutative ring, A be a commutative k–algebra and M be an A–module. Then we abbreviate the spectrum $\mathsf{THH}^{[n],Hk}(HA; HM)$ by $\mathsf{THH}^{[n],k}(A; M)$ and $\mathsf{THH}^{[n],S}(HA; HM)$ by $\mathsf{THH}^{[n]}(A; M)$.

If A is flat over k, then $\pi_* \mathsf{THH}^k(A; M) \cong \mathsf{HH}^k_*(A; M)$ [17, Theorem IX.1.7] and this also holds for higher-order Hochschild homology in the sense of Pirashvili [33]: $\pi_* \mathsf{THH}^{[n],k}(A; M) \cong HH^{[n],k}_*(A; M)$ if A is k–flat [6, Proposition 7.2].

To avoid visual clutter, given a commutative ring A and an element $a \in A$, we write A/a instead of $A/(a)$.

2 Higher THH of truncated polynomial algebras

In [40, Section 3] Veen uses the fact that the Loday construction sends homotopy pushouts of pointed simplicial sets to homotopy pushouts of commutative ring spectra in order to express higher THH as a "topological Tor" of a lower THH. For any cofibrant commutative S–algebra A,

$$\mathsf{THH}^{[n]}(A) \simeq A \wedge^L_{\mathsf{THH}^{[n-1]}(A)} A$$

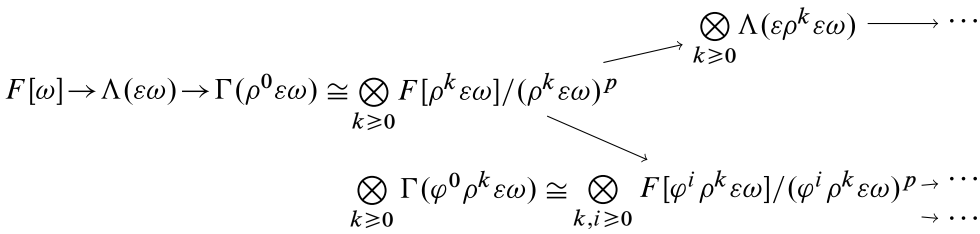

Figure 1: Iterated Tor flow chart.

and this yields a spectral sequence

$$E^2_{s,*} = \mathrm{Tor}_s^{\mathsf{THH}_*^{[n-1]}(A)}(A_*, A_*) \Rightarrow \mathsf{THH}_*^{[n]}(A).$$

In particular cases, this spectral sequence collapses for all $n \geqslant 1$, making it possible to calculate $\mathsf{THH}_*^{[n]}(A)$ as iterated Tor's of A_*. In [6, Figures 1 and 2] we had a flow chart showing the results of iterated Tor of $\mathbb{F}_p$ over some $\mathbb{F}_p$–algebras with a particularly convenient form. We can do similar calculations over any field:

Proposition 2.1 *If F is a field of characteristic p and $|\omega|$ is even, there is a flow chart as in Figure 1 showing the calculation of iterated* Tor*'s of F: if $\mathcal{A}$ is a term in the n^{th} generation in the flow chart, then $\mathrm{Tor}^{\mathcal{A}}(F, F)$ is the tensor product of all the terms in the $(n+1)^{st}$ generation that arrows from $\mathcal{A}$ point to. Here $|\rho^0 y| = |y| + 1$, $|\varepsilon z| = |z| + 1$ and $|\varphi^0 z| = 2 + p|z|$.*

If F a field of characteristic zero, the analogous flow chart for $|x|$ even is

$$F[x] \to \Lambda(\varepsilon x) \to F[\rho^0\varepsilon x] \to \Lambda(\varepsilon\rho^0\varepsilon x) \to \cdots .$$

So, for instance, in prime characteristic p in the flow chart $F[\omega]$ is the first and $\Lambda(\varepsilon\omega)$ is the second generation whereas for the fourth generation we obtain

$$\left(\bigotimes_{k\geqslant 0} \Lambda(\varepsilon\rho^k\varepsilon\omega)\right) \otimes \left(\bigotimes_{k,i\geqslant 0} F[\varphi^i\rho^k\varepsilon\omega]/(\varphi^i\rho^k\varepsilon\omega)^p\right).$$

Proof In characteristic p, all divided power algebras split as tensor products of truncated polynomial algebras. This allows us to use the resolutions of [6, Section 2], where the tensor products in the respective bar constructions are all taken to be over F, to pass from each stage to the next.

In characteristic 0, the Tor dual of an exterior algebra is a divided power algebra, but this is isomorphic to a polynomial algebra. Thus the resolutions of [6, Section 2], with the tensor products in the bar constructions again taken to be over F, can be used analogously to get the alternation between exterior and polynomial algebras. □

Let x be a generator of even nonnegative degree. In [6, Theorem 8.8] we calculated the higher HH of truncated polynomial rings of the form $\mathbb{F}_p[x]/x^{p^\ell}$ for any prime p. The decomposition due to Bökstedt which is described before the statement of the theorem there does not work for $\mathbb{F}_p[x]/x^m$ when m is not a power of p, but we can nevertheless use a similar kind of argument to determine higher HH of $\mathbb{F}_p[x]/x^m$ as long as p divides m. This generalization of [6, Theorem 8.8] is interesting because if m is not a power of p, $\mathbb{F}_p[x]/x^m$ is no longer a Hopf algebra, which the cases we discussed in [6] were.

In the following, $\mathsf{HH}^{[n]}_*$ will denote Hochschild homology groups of order n whereas $\mathsf{HH}^{[n]}$ denotes the corresponding simplicial object whose homotopy groups are $\mathsf{HH}^{[n]}_*$.

Theorem 2.2 *Let x be of even degree and let m be a positive integer divisible by p. Then, for all $n \geqslant 1$,*

$$\mathsf{HH}^{[n],\mathbb{F}_p}_*(\mathbb{F}_p[x]/x^m) \cong \mathbb{F}_p[x]/x^m \otimes B''_n(\mathbb{F}_p[x]/x^m),$$

where $B''_1(\mathbb{F}_p[x]/x^m) \cong \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$, with $|\varepsilon x| = |x|+1$, $|\varphi^0 x| = 2+m|x|$ and

$$B''_n(\mathbb{F}_p[x]/x^m) \cong \mathrm{Tor}_{*,*}^{B''_{n-1}(\mathbb{F}_p[x]/x^m)}(\mathbb{F}_p, \mathbb{F}_p).$$

Since the ring $\mathbb{F}_p[x]/x^m$ (with x of degree zero) is monoidal over $\mathbb{F}_p$, this gives a higher THH calculation,

$$\begin{aligned} \text{(2.3)} \qquad \mathsf{THH}^{[n]}_*(\mathbb{F}_p[x]/x^m) &\cong \mathsf{THH}^{[n]}_*(\mathbb{F}_p) \otimes \mathsf{HH}^{[n],\mathbb{F}_p}_*(\mathbb{F}_p[x]/x^m) \\ &\cong B^n_{\mathbb{F}_p}(\mu) \otimes \mathbb{F}_p[x]/x^m \otimes B''_n(\mathbb{F}_p[x]/x^m), \end{aligned}$$

where $B^1_{\mathbb{F}_p}(\mu) \cong \mathbb{F}_p[\mu]$ with $|\mu| = 2$ and $B^n_{\mathbb{F}_p}(\mu) = \mathrm{Tor}_{*,*}^{B^{n-1}_{\mathbb{F}_p}(\mu)}(\mathbb{F}_p, \mathbb{F}_p)$ for $n > 1$ (see [13, Remark 3.6]).

Proof We use the standard resolution [24, (1.6.1)] and get that the Hochschild homology of $\mathbb{F}_p[x]/x^m$ is the homology of the complex

$$\cdots \xrightarrow{0} \Sigma^{m|x|}\mathbb{F}_p[x]/x^m \xrightarrow{\Delta(x,x)} \Sigma^{|x|}\mathbb{F}_p[x]/x^m \xrightarrow{0} \mathbb{F}_p[x]/x^m.$$

Since p divides m, we have $\Delta(x,x) = mx^{m-1} \equiv 0$, so the above differentials are all trivial and

$$\mathsf{HH}_*^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) \cong \mathbb{F}_p[x]/x^m \otimes \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$$

at least as an $\mathbb{F}_p[x]/x^m$–module, with $|\varepsilon x| = |x|+1$, $|\varphi^0 x| = 2+m|x|$. The map $G_\bullet$ from [24, (1.8.6)] embeds this small complex quasi-isomorphically with its stated multiplicative structure into the standard Hochschild complex for $\mathbb{F}_p[x]/x^m$. It sends $\mathbb{F}_p[x]/x^m$ to itself in degree zero of the Hochschild complex and

$$\varepsilon x \mapsto 1 \otimes x - x \otimes 1, \quad \varphi^0 x \mapsto \sum_{i_0=1}^{m} \sum_{j_0=0}^{1} (-1)^{1+j_0} x^{i_0-j_0} \otimes x^{m-i_0} \otimes x^{j_0},$$

which generate an exterior and divided power subalgebra, respectively, inside the standard Hochschild complex equipped with the shuffle product. The map $G_\bullet$ is shown in [24] to be half of a chain homotopy equivalence between the small complex and the standard Hochschild complex. So we get that $\mathsf{HH}_*^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m) = \mathsf{HH}_*^{[1],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)$ has the desired form as an algebra and sits as a deformation retract inside the standard complex calculating it.

For the higher HH_* computation we use that the E^2–term of the spectral sequence for $\mathsf{HH}_*^{[2]}$ is

$$E^2_{*,*} = \mathrm{Tor}^{\mathbb{F}_p[x]/x^m \otimes \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)}(\mathbb{F}_p[x]/x^m, \mathbb{F}_p[x]/x^m)$$

and, as the generators εx and $\varphi^0 x$ come from homological degree one and two, the module structure of $\mathbb{F}_p[x]/x^m$ over $\Lambda_{\mathbb{F}_p}(\varepsilon x)$ and $\Gamma_{\mathbb{F}_p}(\varphi^0 x)$ factors over the augmentation to $\mathbb{F}_p$. Therefore the above Tor term splits as

$$\mathbb{F}_p[x]/x^m \otimes \mathrm{Tor}_{*,*}^{\Lambda_{\mathbb{F}_p}(\varepsilon x)}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathrm{Tor}_{*,*}^{\Gamma_{\mathbb{F}_p}(\varphi^0 x)}(\mathbb{F}_p, \mathbb{F}_p). \tag{2.4}$$

Now we can argue as in [6] to show that there cannot be any differentials or extensions in this spectral sequence: although we are calculating the homology of the total complex of the bisimplicial $\mathbb{F}_p$–vector space of the bar construction

$$B(\mathbb{F}_p[x]/x^m, \mathsf{HH}^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m), \mathbb{F}_p[x]/x^m),$$

which involves both vertical and horizontal boundary maps, we can map the bar construction

$$B(\mathbb{F}_p[x]/x^m, \mathbb{F}_p[x]/x^m \otimes \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x), \mathbb{F}_p[x]/x^m)$$

quasi-isomorphically into it, and the latter complex involves only nontrivial horizontal maps (all vertical differentials vanish) and has homology exactly equal to the algebra in (2.4). When all the vertical differentials in the original double complex are zero, there can be no nontrivial spectral sequence differentials d^r for $r \geqslant 2$. Also, the trivial vertical differentials mean that there can be no nontrivial extensions involving anything but the i^{th} and $(i+1)^{\text{st}}$ filtration, but since we can produce explicit generators whose p^{th} powers (in the even-dimensional case) or squares (in the odd-dimensional case) actually vanish, we do not need to worry about extensions at all. Thus we obtain the claim about $\mathsf{HH}_*^{[2],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)$.

An iteration of this argument yields the result for higher Hochschild homology, since now we only have exterior algebras on odd-dimensional classes and truncated algebras, truncated at the p^{th} power, on even-dimensional ones, where the powers that vanish do so for combinatorial reasons not relating to the power of x that was truncated at in the original algebra. At each stage, the tensor factor $\mathbb{F}_p[x]/x^m$ will split off the E^2–term for degree reasons. What remains will be the Tor of $\mathbb{F}_p$ with itself over a differential graded algebra that can be chosen up to chain homotopy equivalence to be a graded algebra A with a zero differential which is moreover guaranteed by the flow chart to have the property that $B(\mathbb{F}_p, A, \mathbb{F}_p)$ is chain homotopy equivalent to its homology embedded as a subcomplex with trivial differential inside it.

The splitting for higher THH follows from the splitting for higher HH by arguing as in [6, Proposition 6.1] (following [20, Theorem 7.1]) for the abelian pointed monoid $\{1, x, \dots, x^{m-1}, x^m = 0\}$. □

Reducing the coefficients via the augmentation simplifies things even further. Here, the result does not depend on the p–valuation of m, because x augments to zero and therefore Hochschild homology of $\mathbb{F}_p[x]/x^m$ with coefficients in $\mathbb{F}_p$ is the homology of the complex

$$\cdots \xrightarrow{0} \Sigma^{m|x|}\mathbb{F}_p \xrightarrow{\Delta(x,x)=0} \Sigma^{|x|}\mathbb{F}_p \xrightarrow{0} \mathbb{F}_p.$$

Thus we obtain the following result:

Proposition 2.5 *For all primes p and for all $m > 1$, for x of even degree we get*

$$\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m; \mathbb{F}_p) \cong B_n''(\mathbb{F}_p[x]/x^m),$$

where $B_1''(\mathbb{F}_p[x]/x^m) \cong \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$ and

$$B_n''(\mathbb{F}_p[x]/x^m) = \operatorname{Tor}_{*,*}^{B_{n-1}''(\mathbb{F}_p[x]/x^m)}(\mathbb{F}_p, \mathbb{F}_p)$$

for $n>1$. *Therefore we obtain* (*when* x *has degree zero*)

$$\mathsf{THH}_*^{[n]}(\mathbb{F}_p[x]/x^m;\mathbb{F}_p)\cong \mathsf{THH}_*^{[n]}(\mathbb{F}_p)\otimes B_n''(\mathbb{F}_p[x]/x^m). \tag{2.6}$$

This is shown as in the proof of the previous theorem using the method of [6], embedding

$$\varepsilon x\mapsto 1\otimes x\otimes 1,\quad \varphi^0 x\mapsto 1\otimes x^{m-1}\otimes x\otimes 1$$

inside the bar complex $B(\mathbb{F}_p,\mathbb{F}_p[x]/x^m,\mathbb{F}_p)$, where they generate exterior and divided power algebras, respectively, regardless of the divisibility of m.

Remark 2.7 The calculation becomes drastically different if $(p,m)=1$ and we look at $\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)$ rather than reducing coefficients to get $\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]/x^m;\mathbb{F}_p)$. Then multiplication by m is an isomorphism on $\mathbb{F}_p[x]/x^m$–modules and hence

$$\mathsf{HH}_*^{\mathbb{F}_p}(\mathbb{F}_p[x]/x^m)\cong\begin{cases}\mathbb{F}_p[x]/x^m & \text{for } *=0,\\ (\Sigma^{|x|(km+1)}\mathbb{F}_p[x]/x^m)/x^{m-1} & \text{for } *=2k+1,\\ \Sigma^{km|x|}\ker(\cdot x^{m-1}) & \text{for } *=2k \text{ with } k>0.\end{cases}$$

3 A juggling formula

In this section we generalize juggling formulas from [19, Sections 2–3]: we allow working relative to a ring spectrum R that can be different from the sphere spectrum and we relate the Loday construction on a suspension on a pointed simplicial set X to the Loday construction on X. In [19] we mainly considered the cases where X is a sphere.

Lemma 3.1 *Let* X *be a pointed simplicial set and let* R *be a cofibrant commutative* S*–algebra. For a sequence of commutative* S*–algebras*

$$R\xrightarrow{\eta_A}A\xrightarrow{\alpha}B\xrightarrow{\beta}C$$

such that A *is a cofibrant commutative* R*–algebra and* B *and* C *are cofibrant commutative* A*–algebras, there is an equivalence of augmented commutative* C*–algebras,*

$$\mathcal{L}_X^A(B;C)\simeq C\wedge_{\mathcal{L}_X^R(A;C)}\mathcal{L}_X^R(B;C).$$

The following proof replaces a less transparent one that we gave in an earlier version of the paper. The new proof was suggested by one of the referees of our paper and we thank her or him for it.

Proof The Loday construction $\mathcal{L}_X^A(B;C)$ in simplicial degree n can be written as the colimit of the diagram

$$C \xleftarrow{\beta\circ\alpha} A \xrightarrow{\alpha} B \xleftarrow{\alpha} A \xrightarrow{\alpha} \cdots \xleftarrow{\alpha} A \xrightarrow{\alpha} B,$$

where C is placed at the basepoint of X_n and the copies of B sit at the points $x \in X_n \setminus *$. This colimit is equivalent to the iterated colimit of the expanded diagram

$$\begin{array}{ccccccccccccc}
C & \xleftarrow{\quad\beta\circ\alpha\quad} & A & = & A & = & A & = & \cdots & = & A & = & A \\
\| & & \uparrow{\scriptstyle\eta_A} & & \| & & \uparrow{\scriptstyle\eta_A} & & & & \uparrow{\scriptstyle\eta_A} & & \| \\
C & \xleftarrow{\beta\circ\alpha\circ\eta_A} & R & \xrightarrow{\eta_A} & A & \xleftarrow{\eta_A} & R & \xrightarrow{\eta_A} & \cdots & \xleftarrow{\eta_A} & R & \xrightarrow{\eta_A} & A \\
\| & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & & & \| & & \downarrow{\scriptstyle\alpha} \\
C & \xleftarrow{\beta\circ\alpha\circ\eta_A} & R & \xrightarrow{\alpha\circ\eta_A} & B & \xleftarrow{\alpha\circ\eta_A} & R & \xrightarrow{\alpha\circ\eta_A} & \cdots & \xleftarrow{\alpha\circ\eta_A} & R & \xrightarrow{\alpha\circ\eta_A} & B
\end{array}$$

where we first form the colimit of the columns and then take the colimit of the rows.

Forming the colimit first with respect to the rows yields

$$\begin{array}{c}
C \\
\uparrow \\
(\mathcal{L}_X^R(A;C))_n \\
\downarrow \\
(\mathcal{L}_X^R(B;C))_n
\end{array}$$

and then the colimit of the columns just gives the pushout

$$C \wedge_{(\mathcal{L}_X^R(A;C))_n} (\mathcal{L}_X^R(B;C))_n.$$

This yields an isomorphism for any fixed simplicial degree n. As the simplicial structure maps cause the insertion of the unit maps of A and B in the case of degeneracies or the multiplication in A or B, or the action of A or B on C in the case of face maps, these equivalences are compatible with the simplicial structure maps. Thus we obtain an isomorphism of simplicial spectra which is compatible with the structure of simplicial augmented commutative C–algebras.

A colimit argument over finite pointed subcomplexes then proves the claim for general, not necessarily finite, X. □

Remark 3.2 Under the assumptions of Lemma 3.1 the map $\mathcal{L}_X^R(A;C) \to \mathcal{L}_X^R(B;C)$ is a cofibration of commutative augmented C–algebras (see [19, Section 3] for a proof), thus $C \wedge_{\mathcal{L}_X^R(A;C)} \mathcal{L}_X^R(B;C)$ is actually a homotopy pushout and models the derived smash product $C \wedge^L_{\mathcal{L}_X^R(A;C)} \mathcal{L}_X^R(B;C)$.

Theorem 3.3 (juggling formula) *Let X be a pointed simplicial set. Then, for any sequence of cofibrations of commutative S–algebras $S \to R \to A \to B \to C$, we get an equivalence of augmented commutative C–algebras*

$$\mathcal{L}^R_{\Sigma X}(B;C) \simeq \mathcal{L}^R_{\Sigma X}(A;C) \wedge^L_{\mathcal{L}_X^A(C)} \mathcal{L}_X^B(C).$$

Proof Consider the diagram

$$\begin{array}{ccccc} C & \longleftarrow & \mathcal{L}_X^R(A;C) & \longrightarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{L}_X^R(C) & \longleftarrow & \mathcal{L}_X^R(A;C) & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}_X^R(C) & \longleftarrow & \mathcal{L}_X^R(B;C) & \longrightarrow & C \end{array}$$

By Lemma 3.1, taking the homotopy pushouts of the rows produces the diagram

$$\begin{array}{c} \mathcal{L}^R_{\Sigma X}(A;C) \\ \uparrow \\ \mathcal{L}_X^A(C) \\ \downarrow \\ \mathcal{L}_X^B(C) \end{array}$$

whose homotopy pushout is

$$\mathcal{L}^R_{\Sigma X}(A;C) \wedge^L_{\mathcal{L}_X^A(C)} \mathcal{L}_X^B(C).$$

We get an equivalent result by first taking the homotopy pushouts of the columns and then of the rows. Homotopy pushouts on the columns produces

$$C \wedge^L_{\mathcal{L}_X^R(C)} \mathcal{L}_X^R(C) \leftarrow \mathcal{L}_X^R(A;C) \wedge^L_{\mathcal{L}_X^R(A;C)} \mathcal{L}_X^R(B;C) \to C \wedge^L_C C,$$

which simplifies to

$$C \leftarrow \mathcal{L}_X^R(B;C) \to C,$$

whose homotopy pushout is equivalent to $\mathcal{L}^R_{\Sigma X}(B;C)$. □

Restricting our attention to spheres, we obtain the following result. This is a relative variant of [19, Theorem 3.6].

Corollary 3.4 *Let $S \to R \to A \to B \to C$ be a sequence of cofibrations of commutative S–algebras. Then, for all $n \geqslant 0$, there is an equivalence of augmented commutative C–algebras,*

$$\mathsf{THH}^{[n+1],R}(B;C) \simeq \mathsf{THH}^{[n+1],R}(A;C) \wedge^L_{\mathsf{THH}^{[n],A}(C)} \mathsf{THH}^{[n],B}(C).$$

Remark 3.5 This corollary gives a splitting of the same form as [19, Theorem 3.6]. However, as the proof is different it is not obvious that the maps in the smash product are the same. Thus (unlikely as it may be) it may turn out to be the case that this gives two different but similar-looking splittings.

Let B be an augmented commutative A–algebra spectrum. We apply Theorem 3.3 to the sequence $S \xrightarrow{=} S \to A \to B \to A$, so we take a cofibrant model of A as a commutative S–algebra, a cofibrant model of B as a commutative A–algebra and we factor the augmentation map $B \to A$ into a cofibration followed by an acyclic fibration of B–algebras, $B \rightarrowtail A' \xrightarrow{\sim} A$. Note that

$$A \simeq \mathcal{L}^A_X(A) \simeq \mathcal{L}^A_X(A')$$

because A is cofibrant over A and the map $A \to A'$ is a weak equivalence by 2-out-of-3.

Corollary 3.6 *There is a splitting*

$$\mathcal{L}_{\Sigma X}(B;A') \simeq \mathcal{L}_{\Sigma X}(A;A') \wedge^L_A \mathcal{L}^B_X(A').$$

In particular, if k is a commutative ring, $A = Hk$ and $B = HQ$ for an augmented commutative k–algebra Q, then, factoring the augmentation $HQ \to Hk$ through a cofibration $HQ \rightarrowtail (Hk)' \xrightarrow{\sim} Hk$ as above,

$$\mathcal{L}_{\Sigma X}(HQ;(Hk)') \simeq \mathcal{L}_{\Sigma X}(Hk;(Hk)') \wedge^L_{Hk} \mathcal{L}^{HQ}_X((Hk)')$$

and, if k is a field, then we obtain on the level of homotopy groups

$$\pi_*\mathcal{L}_{\Sigma X}(HQ;Hk) \cong \pi_*(\Sigma X \otimes Hk) \otimes_k \pi_*(\mathcal{L}^{HQ}_X((Hk)')).$$

Remark 3.7 We stress that in Corollary 3.6 there is a spectrum-level splitting of $\mathcal{L}_{\Sigma X}(HQ;Hk)$ into $\mathcal{L}_{\Sigma X}(Hk)$ smashed with an additional factor. In particular, for $X = S^n$, the higher THH of an augmented commutative k–algebra splits as

$$\mathsf{THH}^{[n+1]}(Q;k) \simeq \mathsf{THH}^{[n+1]}(k) \wedge^L_{Hk} \mathsf{THH}^{[n],Q}(k).$$

Greenlees proposed a splitting result in [18, Remark 7.2]: if k is a field and Q is an augmented commutative k–algebra, then his results yield a splitting

$$\mathsf{THH}_*(Q;k) \simeq \mathsf{THH}_*(k) \otimes_k \mathrm{Tor}_*^Q(k,k).$$

Our result generalizes his because for $X = S^0$ the term $\mathcal{L}_{S^0}^{HQ}(Hk)$ is nothing but $Hk \wedge_{HQ}^L Hk$, whose homotopy groups are isomorphic to $\mathrm{Tor}_*^Q(k,k)$. We will revisit this splitting result later in Theorem 5.9, relating it to higher-order Hochschild homology.

3.1 Beware the phony right-module structure!

In some cases it is tempting to use the (valid) splitting of $\mathsf{THH}^{[n+1],R}(A;C)$ as $\mathsf{THH}^{[n+1],R}(A) \wedge_A C$ and oversimplify the juggling formula we got in Corollary 3.4 to the *invalid* identification of $\mathsf{THH}^{[n+1],R}(B;C)$ with

$$\mathsf{THH}^{[n+1],R}(A) \wedge_A (C \wedge^L_{\mathsf{THH}^{[n],A}(C)} \mathsf{THH}^{[n],B}(C)),$$

which in the case $B = C$ becomes

(3.8) $$\mathsf{THH}^{[n+1],R}(A) \wedge_A \mathsf{THH}^{[n+1],A}(C).$$

This transformation is incorrect because it disregards the module structures, without which the maps of pushouts are not well defined. The spectrum $\mathsf{THH}^{[n+1],R}(A;C)$ is *not* equivalent to $\mathsf{THH}^{[n+1],R}(A) \wedge_A C$ as a right-module spectrum over $\mathsf{THH}^{[n],A}(C)$. Assuming that the rearrangement that leads to (3.8) were valid, any cofibration of commutative S–algebras $S \to A \to B$ would produce an equivalence between $\mathsf{THH}^{[n]}(B)$ and $\mathsf{THH}^{[n]}(A) \wedge_A^L \mathsf{THH}^{[n],A}(B)$. But this equivalence does *not* hold in many examples, eg for $A = H\mathbb{Z}$ and B equal to the Eilenberg–Mac Lane spectrum of $\mathbb{F}_p$ or of the ring of integers in a number field.

4 A generalization of Brun's spectral sequence

Morten Brun [9] uses the geometry of the circle to identify $\mathsf{THH}(HQ; HQ \wedge^L_{Hk} HQ)$ with $\mathsf{THH}(Hk; HQ)$, where k is a commutative ring and Q is a commutative k–algebra: $\mathsf{THH}(Hk; HQ)$ is a circle with HQ at the basepoint and Hk sitting at every nonbasepoint of S^1. Homotopy invariance says that we can let the point take over half the circle, so that it covers an interval. This idea identifies $\mathsf{THH}(Hk; HQ)$ with

$\mathsf{THH}(Hk; B(HQ, HQ, HQ))$, where B denotes the two-sided bar construction. Brun [9, Lemma 6.2.3] then shows that the latter is equivalent to

$$\mathsf{THH}(HQ; B(HQ, Hk, HQ))$$

by a shift of perspective. This idea inspired our juggling formula of Theorem 3.3 and also the following result, which can be obtained as a corollary of the juggling formula applied to the sequence of maps $S \to R = R \to A \to B$:

Theorem 4.1 (Brun juggling) *Let X be a pointed simplicial set. For any sequence of cofibrations of commutative S–algebras $S \to R \to A \to B$ we get an equivalence of commutative B–algebras*

$$\mathcal{L}^R_{\Sigma X}(A; B) \simeq B \wedge^L_{\mathcal{L}^R_X(B)} \mathcal{L}^A_X(B).$$

Note that B, which only appears at the basepoint on the left, now appears almost everywhere on the right. Thus we can think of the basepoint as having "eaten" most of ΣX.

In the following we will use the notation from [19, Section 2]. If Y is a pointed simplicial subset of X, then we denote by $\mathcal{L}^R_{(X,Y)}(A, B; B)$ the relative Loday construction where we attach B to every point in Y including the basepoint, A to every point in the complement and we use the structure maps to turn this into an augmented commutative B–algebra spectrum. Recall that when the simplicial set X is not necessarily finite, the construction is defined as the direct limit of the construction on finite subsimplicial sets. Note that if $Y = *$, then $\mathcal{L}^R_{(X,*)}(A, B; B) = \mathcal{L}^R_X(A; B)$, so in this case we omit the $*$ from the notation as in Definition 1.1.

We offer a direct proof of the Brun juggling theorem that uses the geometric features of suspensions to make it easier to see what is going on in this application of Theorem 3.3; this generalizes the geometric intuition behind Brun's original $X = S^0$ case.

Alternative proof of Theorem 4.1 We use the homotopy equivalence between the pair $(\Sigma X, *)$ and the pair $(CX \cup_X CX, CX)$, with the cone sitting as the upper half of the suspension. Then, since the Loday construction is homotopy invariant,

$$\begin{aligned}\mathcal{L}^R_{\Sigma X}(A; B) = \mathcal{L}^R_{(\Sigma X,*)}(A, B; B) &= \mathcal{L}^R_{(CX\cup_X CX,*)}(A, B; B)\\ &\simeq \mathcal{L}^R_{(CX\cup_X CX,CX)}(A, B; B)\\ &= \mathcal{L}^R_{(CX\cup_X CX,CX\cup_X X)}(A, B; B).\end{aligned}$$

By [19, Proposition 2.10(b)],

$$\mathcal{L}^R_{(CX\cup_X CX, CX\cup_X X)}(A,B;B) \simeq \mathcal{L}^R_{(CX,CX)}(A,B;B)\wedge_{\mathcal{L}^R_{(X,X)}(A,B;B)} \mathcal{L}^R_{(CX,X)}(A,B;B).$$

By definition $\mathcal{L}^R_{(CX,CX)}(A,B;B)=\mathcal{L}^R_{CX}(B)$ and $\mathcal{L}^R_{(X,X)}(A,B;B)=\mathcal{L}^R_X(B)$ and by homotopy invariance $\mathcal{L}^R_{CX}(B)\simeq B$, hence

$$\mathcal{L}^R_{(CX\cup_X CX, CX\cup_X X)}(A,B;B)\simeq B\wedge_{\mathcal{L}^R_X(B)}\mathcal{L}^R_{(CX,X)}(A,B;B).$$

Using [19, (3.0.1)] we can identify $\mathcal{L}^R_{(CX,X)}(A,B;B)$ with

$$\mathcal{L}^R_{CX}(A;B)\wedge^L_{\mathcal{L}^R_X(A;B)}\mathcal{L}^R_X(B;B) \tag{4.2}$$

and as CX is contractible we obtain $B\simeq\mathcal{L}^R_{CX}(A;B)$. Then Lemma 3.1 yields an equivalence of (4.2) with $\mathcal{L}^A_X(B)$. □

Example 4.3 Consider the case when $X=S^0$:

$$\mathsf{THH}(A;B)\simeq B\wedge_{B\wedge B}(B\wedge_A B)=\mathsf{THH}(B;B\wedge_A B). \tag{4.4}$$

Let R be connective. There is an Atiyah–Hirzebruch spectral sequence (see [17, Theorem IV.3.7])

$$E^2_{p,q}=\pi_p(E\wedge_R H\pi_q M)\Rightarrow\pi_{p+q}(E\wedge_R M).$$

Let B be a connective A–algebra. Setting $R=B\wedge B$, $E=B$ and $M=B\wedge_A B$, we get

$$E^2_{p,q}=\pi_p(B\wedge_{B\wedge B}H\pi_q(B\wedge_A B))\Rightarrow\pi_{p+q}(B\wedge_{B\wedge B}(B\wedge_A B)).$$

Setting $B=HQ$ and $A=Hk$ gives us

$$E^2_{p,q}=\mathsf{THH}_p(Q;\operatorname{Tor}^k_q(Q,Q))\Rightarrow\pi_{p+q}(HQ\wedge_{HQ\wedge HQ}(HQ\wedge_{Hk}HQ)) \cong\mathsf{THH}_{p+q}(k;Q)$$

by (4.4). This recovers a spectral sequence with the same E^2 page and limit as Brun's [9, Theorem 6.2.10]. A substantial generalization of Brun's spectral sequence for THH can be found in [22, Theorem 1.1].

Example 4.5 We can generalize Example 4.3 to any X. In particular, consider a commutative ring k and a commutative k–algebra Q. If we apply the Atiyah–Hirzebruch spectral sequence in the case

$$E = HQ, \quad R = \mathcal{L}_{S^n}(HQ), \quad M = \mathcal{L}_{S^n}^{Hk}(HQ)$$

then the Brun juggling formula of Theorem 4.1 gives us a spectral sequence

$$E^2_{p,q} = \pi_p(HQ \wedge_{\mathsf{THH}^{[n]}(Q)} H\mathsf{THH}^{[n],k}_q(Q)) \Rightarrow \mathsf{THH}^{[n+1]}_{p+q}(k;Q).$$

In the next section, we will see that we can identify $\mathsf{THH}^{[n],k}(Q)$ with higher-order Shukla homology, $\mathsf{Sh}^{[n],k}(Q)$, so we get the simpler description

$$E^2_{p,q} = \pi_p\big(HQ \wedge_{\mathsf{THH}^{[n]}(Q)} H(\mathsf{Sh}^{[n],k}_q(Q))\big) \Rightarrow \mathsf{THH}^{[n+1]}_{p+q}(k;Q).$$

5 Higher Shukla homology

Let k be a commutative ring. Ordinary Shukla homology [38] of a k–algebra A with coefficients in an A–bimodule M is a derived version of Hochschild homology and it can be identified with $\mathsf{THH}^k(A;M)$. We will define higher-order Shukla homology in the context of commutative algebras as an iterated bar construction and identify it with higher-order topological Hochschild homology.

Definition 5.1 Let A be a commutative k–algebra and B be a commutative A–algebra. We define

$$\mathsf{Sh}^{[0],k}(A;B) = HA \wedge^L_{Hk} HB.$$

For $n \geqslant 1$ we define *n^{th} order Shukla homology of A over k with coefficients in B* as

$$\mathsf{Sh}^{[n],k}(A;B) = B^S(HB, \mathsf{Sh}^{[n-1],k}(A;B), HB),$$

where the latter is the two-sided bar construction with respect to HB over the sphere spectrum.

It is consistent to set $\mathsf{Sh}^{[-1],k}(A) = Hk$. Again, we abbreviate $\mathsf{Sh}^{[n],k}(A;A)$ by $\mathsf{Sh}^{[n],k}(A)$.

For $n=1$ we have $\mathsf{Sh}^{[1],k}(A;B) = B^S(HB, HA \wedge^L_{Hk} HB, HB) \simeq \mathsf{THH}^k(A;B)$. For example, when $k=\mathbb{Z}$ and p is a prime,

$$\mathsf{Sh}^{\mathbb{Z}}_*(\mathbb{Z}/p^m;\mathbb{Z}/p) \cong \Gamma_{\mathbb{Z}/p}(x(m))$$

with $|x(m)| = 2$.

Remark 5.2 Both the geometric realization of $B^S(HB, \mathsf{Sh}^{[n-1],k}(A;B), HB)$ and that of $B^{Hk}(HB, \mathsf{Sh}^{[n-1],k}(A;B), HB)$ are equivalent to $HB \wedge^L_{\mathsf{Sh}^{[n-1],k}(A;B)} HB$ and the derived smash product is the homotopy pushout in the category of commutative S–algebras and is also the homotopy pushout in the category of commutative S–algebras under Hk which is equivalent to the category of commutative Hk–algebras. A priori, $\mathsf{THH}^k(A;B)$ is a simplicial spectrum and $\mathsf{Sh}^{[n],k}(A;B)$ is therefore an n–simplicial spectrum, but we take iterated diagonals to get a simplicial spectrum and can then use geometric realization to get an honest spectrum.

In particular, we can identify $\mathsf{Sh}^{[n],k}(A;B)$ with $\mathsf{THH}^{[n],k}(A;B)$.

Note that $\mathsf{Sh}^{[n],k}(A;B)$ is a commutative HB–algebra spectrum for all $n \geqslant 1$, so in particular all Shukla homology spectra are generalized Eilenberg–Mac Lane spectra.

Proposition 5.3 *Let R be a commutative ring and let $a, p \in R$ be elements which are not zero divisors such that (p) is maximal and $a \in (p)^2$. Then*

$$\mathsf{Sh}^{[0],R}_*(R/p) \cong \Lambda_{R/p}(\tau_1) \cong \mathsf{Sh}^{[0],R}_*(R/a; R/p), \quad |\tau_1| = 1$$

and for $n \geqslant 1$,

$$\mathsf{Sh}^{[n],R}_*(R/p) \cong \mathrm{Tor}_*^{\mathsf{Sh}^{[n-1],R}_*(R/p)}(R/p; R/p)$$

and

$$\mathsf{Sh}^{[n],R}_*(R/a; R/p) \cong \mathrm{Tor}_*^{\mathsf{Sh}^{[n-1],R}_*(R/a,R/p)}(R/p; R/p).$$

Warning The reduction $R/a \to R/p$ does *not* induce an isomorphism

$$\mathsf{Sh}^{[n],R}_*(R/a; R/p) \to \mathsf{Sh}^{[n],R}_*(R/p).$$

By considering resolutions we can see that at $n = 0$ the induced map is the map taking τ_1 to 0. In fact, in Corollary 7.4 we show that the map induced by $R/a \to R/p$ is zero on all generators other than the R/p in dimension 0.

Proof We prove this by induction on n. At $n = 0$,

$$\mathsf{Sh}^{[0],R}(R/p; R/p) = HR/p \wedge^L_{HR} HR/p.$$

There is a Künneth spectral sequence,

$$E^2_{s,t} = \mathrm{Tor}^R_{s,t}(R/p, R/p) \Rightarrow \pi_{s+t}(HR/p \wedge^L_{HR} HR/p),$$

which in this case is concentrated in internal degree $t=0$. The short resolution

$$R \xrightarrow{\cdot p} R \to R/p$$

gives

$$\mathrm{Tor}^R_{s,t}(R/p, R/p) \cong H_s(R/p \xrightarrow{0} R/p)_t \cong \begin{cases} R/p & \text{if } s=0=t,\\ R/p & \text{if } s=1 \text{ and } t=0,\\ 0 & \text{otherwise.}\end{cases}$$

For degree reasons, there cannot be any differentials or extensions in this spectral sequence, and the product of τ_1 with itself has to vanish. Thus $\mathsf{Sh}^{[0]}_*(R/p) \cong \Lambda_{R/p}(\tau_1)$, as desired. Note that this proof works (almost) verbatim for $\mathsf{Sh}^{[0],R}_*(R/a; R/p)$.

By [13, Proposition 2.1], as an augmented commutative HR/p–algebra,

$$\mathsf{Sh}^{[0],R}(R/p) \simeq HR/p \vee \Sigma HR/p \simeq \mathsf{Sh}^{[0],R}(R/a; R/p).$$

Thus

$$\mathsf{Sh}^{[1],R}(R/p) = B(HR/p, \mathsf{Sh}^{[0],R}(R/p), HR/p) \simeq HB(R/p, \Lambda_{R/p}(\tau_1), R/p).$$

In the following let $\mathbb{F}$ be R/p. By [6, Section 2], if we start with $B^{\mathbb{F}}(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F})$ for $\mathbb{F}$ a field of positive characteristic, we know that the spectral sequence

$$\mathrm{Tor}^{\Lambda_{\mathbb{F}}(\tau_1)}_{*,*}(\mathbb{F}, \mathbb{F}) \Rightarrow H_*(B^{\mathbb{F}}(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F}))$$

collapses at E^2, which concludes the proof of the $n=1$ case. We have that

$$B^{\mathbb{F}}(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F}) \simeq B(\mathbb{F}, \Lambda_{\mathbb{F}}(\tau_1), \mathbb{F})$$

because both calculate the homology of $\mathbb{F} \otimes^L_{\Lambda_{\mathbb{F}}(\tau_1)} \mathbb{F}$. Moreover, in [6, Section 8] we show that if we keep applying $B^{\mathbb{F}}(\mathbb{F}, -, \mathbb{F})$ to the result, having started with $\Lambda_{\mathbb{F}}(\tau_1)$, the spectral sequences $\mathrm{Tor}^{H_*(-)}_{*,*}(\mathbb{F}, \mathbb{F}) \Rightarrow H_*(B^{\mathbb{F}}(\mathbb{F}, -, \mathbb{F}))$ will keep collapsing. Since in the case of a characteristic zero field a divided power algebra is isomorphic to a polynomial one, we can use the method of [6], adjusted as in the proof of Proposition 2.1, to get the same result.

Finally, in [13, Section 3] we show that once we can exhibit a commutative $H\mathbb{F}$–algebra as the image of the Eilenberg–Mac Lane functor on some simplicial algebra, we can continue doing that when we apply $B(\mathbb{F}, -, \mathbb{F})$ to that algebra — once we get to the algebraic setting we can stay there. This concludes the proof for the collapsing of the spectral sequences both for R/p and for R/a. □

The following argument allows us to identify higher-order Shukla homology of order n with higher-order Hochschild homology of order $n+1$ for augmented algebras in good cases.

Let A be an augmented commutative k–algebra. We apply the juggling formula of Theorem 3.3 to the sequence $Hk \to Hk \to HA \to (Hk)'$, where $HA \to (Hk)'$ is a cofibration of commutative HA–algebras with $(Hk)' \simeq Hk$, and obtain

$$\mathcal{L}^{Hk}_{\Sigma X}(HA;(Hk)') \simeq \mathcal{L}^{Hk}_{\Sigma X}(Hk;(Hk)') \wedge^{L}_{\mathcal{L}^{Hk}_{X}((Hk)')} \mathcal{L}^{HA}_{X}((Hk)').$$

But $\mathcal{L}^{Hk}_{X}((Hk)') \simeq Hk \simeq \mathcal{L}^{Hk}_{\Sigma X}(Hk;(Hk)')$ as before, so we have the following consequence of Theorem 3.3:

Corollary 5.4 *In the special case of the sequence of cofibrations of commutative S–algebras $R = Hk \to Hk \to HA \to (Hk)'$ we obtain*

$$\mathcal{L}^{Hk}_{\Sigma X}(HA;(Hk)') \simeq \mathcal{L}^{HA}_{X}((Hk)') \tag{5.5}$$

for any X.

Remark 5.6 Corollary 5.4 implies that $\mathcal{L}^{HA}_{X}((Hk)')$ depends only on the homotopy type of ΣX, so $\mathcal{L}^{HA}_{X}((Hk)')$ is a stable invariant of X. Note that $\mathcal{L}^{Hk}_{\Sigma X}(HA;(Hk)') \simeq \mathcal{L}^{Hk}_{\Sigma X}(HA;Hk)$.

Consider the case $X = S^n$. Then (5.5) gives

$$\mathcal{L}^{Hk}_{S^{n+1}}(HA;Hk) \simeq \mathcal{L}^{HA}_{S^n}((Hk)'). \tag{5.7}$$

The term on the left-hand side of (5.7) has as homotopy groups the Hochschild homology of order $n+1$ of A with coefficients in k if A is flat as a k–module. The right-hand side simplifies to $\mathsf{THH}^{[n],A}(k)$ and this is the Shukla homology of order n of k over A. Therefore we obtain:

Proposition 5.8 *Let k be a commutative ring and let A be an augmented commutative k–algebra which is flat as a k–module. Then, for all $n \geqslant 0$,*

$$\mathsf{HH}^{[n+1],k}_{*}(A;k) \cong \mathsf{Sh}^{[n],A}_{*}(k).$$

Note that for $n = 0$ we obtain the classical formula [10, Theorem X.2.1],

$$\mathsf{HH}^{k}_{*}(A;k) \cong \operatorname{Tor}^{A}_{*}(k,k).$$

For higher Hochschild homology with reduced coefficients of augmented commutative k–algebras, this last proposition allows us to deduce the kind of splitting result we are looking for:

Theorem 5.9 *Let k be a commutative ring and let A be an augmented commutative k–algebra. Then, for all $n \geqslant 1$,*

$$\mathsf{THH}^{[n]}(A;k) \simeq \mathsf{THH}^{[n]}(k) \wedge^L_{Hk} \mathsf{THH}^{[n],k}(A;k).$$

If k is a field then we obtain the following isomorphism on the level of homotopy groups:

$$\mathsf{THH}^{[n]}_*(A;k) \cong \mathsf{THH}^{[n]}_*(k) \otimes_k \mathsf{HH}^{[n],k}_*(A;k).$$

Proof By Corollary 3.6,

$$\mathsf{THH}^{[n]}(A;k) \simeq \mathsf{THH}^{[n]}(k) \wedge^L_{Hk} \mathsf{THH}^{[n-1],A}(k),$$

but (5.7) says that

$$\mathsf{THH}^{[n-1],A}(k) \simeq \mathsf{THH}^{[n],k}(A;k).$$ □

6 A weak splitting for $\mathsf{THH}(R/(a_1,\dots,a_r);R/\mathfrak{m})$

Using a Tor calculation by Tate from the 1950s we obtain a splitting on the level of homotopy groups of $\mathsf{THH}_*(R/(a_1,\dots,a_r);R/\mathfrak{m})$ in good cases. This yields an easy way of calculating $\mathsf{THH}_*(\mathbb{Z}/p^m;\mathbb{Z}/p)$ for $m \geqslant 2$. Compare [32; 9; 1] for other approaches.

Theorem 6.1 *Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and let $(a_1,\dots,a_r)$ be a regular sequence in R with $a_i \in \mathfrak{m}^2$ for $1 \leqslant i \leqslant r$. Then*

$$\mathsf{THH}_*(R/(a_1,\dots,a_r);R/\mathfrak{m}) \cong \mathsf{THH}_*(R;R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1,\dots,S_r)$$

with $|S_i| = 2$.

Proof Let $I = (a_1,\dots,a_r)$. Applying the juggling formula of Theorem 3.3 to $X = S^0$ and to the sequence $S \to HR \to HR/I \to HR/\mathfrak{m}$ gives

$$\mathsf{THH}(R/I;R/\mathfrak{m}) \simeq \mathsf{THH}(R;R/\mathfrak{m}) \wedge^L_{HR/\mathfrak{m}\wedge^L_{HR}HR/\mathfrak{m}} (HR/\mathfrak{m} \wedge^L_{HR/I} HR/\mathfrak{m}).$$

In [39] Tate determines the algebra structure on the homotopy groups of the last term,

$$\mathrm{Tor}_*^{R/I}(R/\mathfrak{m}, R/\mathfrak{m}) \cong \Lambda_{R/\mathfrak{m}}(T_1, \dots, T_d) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1, \dots, S_r).$$

Here, d is the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as an $R/\mathfrak{m}$–vector space. We can choose a regular system of generators $(t_1, \dots, t_d)$ for $\mathfrak{m}$ such that the module structure of $\mathrm{Tor}_*^{R/I}(R/\mathfrak{m}, R/\mathfrak{m})$ over $\mathrm{Tor}_*^{R}(R/\mathfrak{m}, R/\mathfrak{m}) \cong \Lambda_{R/\mathfrak{m}}(T_1, \dots, T_d)$ is the canonical one (see [39, page 22]). Hence the Künneth spectral sequence for $\mathsf{THH}(R/I; R/\mathfrak{m})$ has an E^2–term isomorphic to

$$\mathsf{THH}_*(R; R/\mathfrak{m}) \otimes_{\mathrm{Tor}_*^R(R/\mathfrak{m}, R/\mathfrak{m})} \mathrm{Tor}_*^{R/I}(R/\mathfrak{m}, R/\mathfrak{m}),$$

which is concentrated in the zeroth column, and this consists of

$$\mathsf{THH}_*(R; R/\mathfrak{m}) \otimes_{R/\mathfrak{m}} \Gamma_{R/\mathfrak{m}}(S_1, \dots, S_r).$$

Therefore, there are no nontrivial differentials and extensions in this spectral sequence. □

We call the splitting of Theorem 6.1 a *weak splitting* because it is only a splitting on the level of homotopy groups. In Section 7 we develop a stronger spectrum-level splitting of a similar form.

We apply the above result in the special case where R is a principal ideal domain. Let $p \neq 0$ be an element of R such that (p) is a maximal ideal in R and let m be bigger than or equal to 2. Then we are in the situation of Theorem 6.1 because $R_{(p)}/p^m \cong R/p^m$ so we can drop the assumption that R is local. The above result immediately gives an explicit formula for $\mathsf{THH}(R/p^n; R/p)$:

Corollary 6.2 *For all $m > 1$,*

$$\mathsf{THH}_*(R/p^m; R/p) \cong \mathsf{THH}_*(R; R/p) \otimes_{R/p} \Gamma_{R/p}(S_1).$$

Remark 6.3 One may try to use the same method for $\mathsf{THH}^{[n]}$. The juggling formula from Theorem 3.3 for $S \xrightarrow{=} S \to H\mathbb{Z} \to H\mathbb{Z}/p^m \to H\mathbb{Z}/p$ gives us

$$\mathsf{THH}^{[n]}(\mathbb{Z}/p^m; \mathbb{Z}/p) \simeq \mathsf{THH}^{[n]}(\mathbb{Z}; \mathbb{Z}/p) \wedge^L_{\mathsf{Sh}^{[n-1],\mathbb{Z}}(\mathbb{Z}/p)} \mathsf{Sh}^{[n-1],\mathbb{Z}/p^m}(\mathbb{Z}/p).$$

Thus we must understand the structure of $\mathsf{Sh}^{[n-1],\mathbb{Z}/p^m}(\mathbb{Z}/p)$ as a $\mathsf{Sh}^{[n-1],\mathbb{Z}}(\mathbb{Z}/p)$–algebra. It is not possible to do this through direct Tor computations for all n, as the computations rapidly become intractable; even $\mathsf{Sh}^{[1],\mathbb{Z}/p^2}(\mathbb{Z}/p)$ is rather involved [4, Section 5.2], but see Proposition 7.5 for a general formula.

In order to obtain calculations in this example and in related cases, we need to develop the more delicate splitting of Section 7.

7 A splitting for $\mathsf{THH}^{[n]}(R/a; R/p)$

Throughout this section, we assume that R is a commutative ring and $a, p \in R$ are elements which are not zero divisors for which (p) is a maximal ideal and $a \in (p)^2$.

Lemma 7.1 *Let R, p and a be as above, and let $\pi\colon R/a \to R/p$ be the obvious reduction. Then the map induced by π,*

$$\pi_*\colon \mathsf{Sh}_*^{[0],R}(R/a; R/p) \to \mathsf{Sh}_*^{[0],R}(R/p),$$

factors as

$$\mathsf{Sh}_*^{[0],R}(R/a; R/p) \xrightarrow{\epsilon} R/p \xrightarrow{\eta} \mathsf{Sh}_*^{[0],R}(R/p).$$

Proof The assumptions on a and p ensure that there exists a $b \in R$ such that $a = bp^2$. We have the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\cdot a} & R & \xrightarrow{\epsilon} & R/a \\ {\scriptstyle \cdot bp}\downarrow & & \| & & \\ R & \xrightarrow{\cdot p} & R & \xrightarrow{\epsilon} & R/p \end{array}$$

Thus we have a map of resolutions. When we tensor up with R/p we get the diagram

$$\begin{array}{ccc} R \otimes_R R/p & \xrightarrow{a\otimes 1} & R \otimes_R R/p \\ {\scriptstyle bp\otimes 1=0}\downarrow & & \| \\ R \otimes_R R/p & \xrightarrow{p\otimes 1} & R \otimes_R R/p \end{array}$$

We take the homology of the top and bottom row. Note that since $a, p \in (p)$, the horizontal maps are 0; thus the top and bottom row produce Tor's which are of the form $\Lambda_{R/p}(\tau_1)$. However, when we look at where τ_1 goes from the top to the bottom, it maps by multiplication by bp — which is 0 in R/p. Thus this map is 0. □

Surprisingly enough, this special case allows us to prove a spectrum-level splitting for all $n \geqslant 0$.

Definition 7.2 Let $\mathcal{A}_{HR/p}$ be the category of augmented commutative HR/p–algebras and $h\mathcal{A}_{HR/p}$ its homotopy category. Let $\mathrm{Mod}_{HR/p}$ be the category of HR/p–modules.

Lemma 7.3 *For R, p and a as above, the map*

$$\varphi_n\colon \mathsf{THH}^{[n],R}(R/a; R/p) \to \mathsf{THH}^{[n],R}(R/p)$$

induced by $R/a \to R/p$ factors through HR/p in $h\mathcal{A}_{HR/p}$.

Proof The key step is the $n = 0$ case.

From [13, Proposition 2.1] we know that $\mathsf{THH}^{[0],R}(R/a; R/p) \simeq HR/p \vee \Sigma HR/p$ and also $\mathsf{THH}^{[0],R}(R/p) \simeq HR/p \vee \Sigma HR/p$. So we need to understand

$$h\mathcal{A}_{HR/p}(HR/p \vee \Sigma HR/p, HR/p \vee \Sigma HR/p).$$

By [2, Proposition 3.2], we can identify this as

$$h\,\mathrm{Mod}_{HR/p}(\mathbb{L}Q\mathbb{R}I(HR/p \vee \Sigma HR/p), \Sigma HR/p).$$

Given an $A \in \mathcal{A}_{HR/p}$, we have a pullback

$$\begin{array}{ccc} IA & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \epsilon} \\ * & \longrightarrow & HR/p \end{array}$$

Let B be a nonunital HR/p–algebra. Basterra defines $Q(B)$ to be the pushout

$$\begin{array}{ccc} B \wedge_{HR/p} B & \longrightarrow & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q(B) \end{array}$$

We want to take the left- and right-derived versions of these functors for Basterra's result.

Let X be a fibrant replacement for $HR/p \vee \Sigma HR/p$ in $\mathcal{A}_{HR/p}$, so that $X \xrightarrow{\epsilon} HR/p$ is a fibration. Thus the square

$$\begin{array}{ccc} \mathbb{R}I(HR/p \vee \Sigma HR/p) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \epsilon} \\ * & \longrightarrow & HR/p \end{array}$$

is a homotopy pullback square (since every spectrum is fibrant [21, Proposition 13.1.2]). For conciseness we write $Y = \mathbb{R}I(HR/p \vee \Sigma HR/p)$. We have a long exact sequence of homotopy groups

$$0 \to \pi_1 Y \to \pi_1 X \to \pi_1 HR/p \to \pi_0 Y \to \pi_0 X \to \pi_0 HR/p \to \pi_{-1} Y \to 0,$$

where we have used that $X \simeq HR/p \vee \Sigma HR/p$, so that its homotopy groups are concentrated in degrees 0 and 1. Note that the map $\pi_0 X \to \pi_0 HR/p$ is the identity. We thus see that $\pi_i Y \cong 0$ for $i \neq 1$ and $\pi_1 Y \cong R/p$.

We need to identify

$$h\,\mathrm{Mod}_{HR/p}(\mathbb{L}Q(\Sigma HR/p), \Sigma HR/p) \cong \pi_0 F_{HR/p}(\mathbb{L}Q(\Sigma HR/p), \Sigma HR/p),$$

where $F_{HR/p}(\cdot,\cdot)$ is the function spectrum. We use the universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{R/p}(\pi_*\mathbb{L}Q(\Sigma HR/p), \pi_*\Sigma HR/p) \Rightarrow \pi_{t-s} F_{HR/p}(\mathbb{L}Q(\Sigma HR/p), \Sigma HR/p).$$

Note that we're working over a field, so $E_2^{s,t} = 0$ for $s \neq 0$, and $\pi_*(\Sigma HR/p)$ is zero everywhere except at π_1, so in fact the spectral sequence collapses at E_2. Since we are only interested in π_0, the only term relevant to us is

$$E_\infty^{0,0} \cong E_2^{0,0} \cong \mathrm{Hom}_{R/p}(\pi_1\mathbb{L}Q(\Sigma HR/p), R/p).$$

Note that this group cannot be 0, since our hom set contains at least two elements: the identity map and the 0 map. Thus it remains to compute $\pi_1\mathbb{L}Q(\Sigma HR/p)$.

Consider the diagram

$$\begin{array}{ccc} (\Sigma HR/p)^{\mathrm{cof}} & \xrightarrow{f} & \mathbb{L}Q(\Sigma HR/p) \\ \sim \downarrow & & \\ \Sigma HR/p & & \end{array}$$

By [3, Proposition 2.1], since $(\Sigma HR/p)^{\mathrm{cof}}$ is 0–connected, f is 1–connected; thus $\pi_1 f$ is surjective. Since R/p is a field, we must have

$$\pi_1\mathbb{L}Q(\Sigma HR/p) \cong 0 \text{ or } R/p.$$

Since it can't be 0, it must be R/p, with the induced map being the identity.

Let $\tau_1^{(a)}$ be the generator of the $\Lambda(\tau_1)$ obtained as $\mathsf{Sh}^{[0],R}(R/a; R/p)$ and let $\tau_1^{(p)}$ be the generator of the $\Lambda(\tau_1)$ obtained as $\mathsf{Sh}^{[0],R}(R/p)$ in the calculation of Lemma 7.1.

The above calculation shows that

$$h\,\mathrm{Mod}_{HR/p}(\mathbb{L}Q\mathbb{R}I(HR/p \vee \Sigma HR/p), \Sigma HR/p) \cong \mathrm{Hom}_{R/p}(R/p, R/p),$$

where the first copy of R/p is generated by $\tau_1^{(a)}$ and the second copy is generated by $\tau_1^{(p)}$. But the induced map on $\mathsf{Sh}^{[0],R}$ takes $\tau_1^{(a)}$ to 0. Thus the corresponding map in $h\mathcal{A}_{R/p}$ is also 0. This proves the $n=0$ case.

We now turn to the induction step. We have the composition

$$\begin{array}{ccc} B(HR/p, \mathsf{THH}^{[n-1],R}(R/a; R/p), HR/p) & \xrightarrow{\text{hypothesis}} & \\ \Big\downarrow {\scriptstyle B(1,\varphi_{n-1},1)} & & B(HR/p, HR/p, HR/p) \\ B(HR/p, \mathsf{THH}^{[n-1],R}(R/p; R/p), HR/p) & \longleftarrow & \end{array}$$

of maps of simplicial spectra. Taking the realization gives us the composition

$$\varphi_n\colon \mathsf{THH}^{[n]}(R/a; R/p) \to HR/p \to \mathsf{THH}^{[n]}(R/p),$$

as desired. □

By applying π_* to the result of Lemma 7.3, we get the following generalization of Lemma 7.1:

Corollary 7.4 *For R, p and a as above, for all $n \geqslant 0$ the map*

$$\mathsf{Sh}_*^{[n],R}(R/a; R/p) \to \mathsf{Sh}_*^{[n],R}(R/p)$$

induced by $R/a \to R/p$ factors as

$$\mathsf{Sh}_*^{[n],R}(R/a; R/p) \xrightarrow{\epsilon} R/p \xrightarrow{\eta} \mathsf{Sh}_*^{[n],R}(R/p).$$

Here, R/p is considered as a graded ring concentrated in degree 0; the first map in the factorization is the augmentation and the second is the unit map induced by the inclusion of the basepoint.

By Lemma 3.1,

$$\mathsf{THH}^{[n],R/a}(R/p) \simeq HR/p \wedge_{\mathsf{THH}^{[n],R}(R/a;R/p)} \mathsf{THH}^{[n],R}(R/p).$$

However, by Lemma 7.3 the map $\mathsf{THH}^{[n],R}(R/a; R/p) \to \mathsf{THH}^{[n],R}(R/p)$ factors through HR/p. This proves the following result about higher-order Shukla homology:

Proposition 7.5 *For R, p, and a as above,*

$$\begin{aligned}\mathsf{THH}^{[n],R/a}(R/p) &\simeq (HR/p \wedge_{\mathsf{THH}^{[n],R}(R/a;R/p)} HR/p) \wedge_{HR/p} \mathsf{THH}^{[n],R}(R/p)\\ &\simeq \mathsf{THH}^{[n+1],R}(R/a;R/p) \wedge_{HR/p} \mathsf{THH}^{[n],R}(R/p).\end{aligned}$$

This recovers the calculation of $\mathsf{Sh}_*^{\mathbb{Z}/p^2}(\mathbb{Z}/p)$ from [4, Section 5.2]. It also explains why these Shukla calculations are more involved than Shukla homology calculations of the form $\mathsf{Sh}_*^R(R/x)$ where x is a regular element. In the latter case we just obtain a divided power algebra over R/x on a generator of degree two, whereas for all $m \geqslant 2$,

$$\begin{aligned}\mathsf{Sh}_*^{\mathbb{Z}/p^m}(\mathbb{Z}/p) &\cong \mathsf{Sh}_*^{[2],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p) \otimes_{\mathbb{Z}/p} \mathsf{Sh}_*^{\mathbb{Z}}(\mathbb{Z}/p)\\ &\cong \bigotimes_{i\geqslant 0} \Lambda\big(\varepsilon(\varrho^i(\tau_1^{(p^m)}))\big) \otimes \Gamma_{\mathbb{Z}/p}(\varphi^0\varrho^i(\tau_1^{(p^m)})) \otimes_{\mathbb{Z}/p} \Gamma_{\mathbb{Z}/p}(\varrho^0(\tau_1^{(p)})).\end{aligned}$$

We are now ready to prove the main splitting result:

Theorem 7.6 *If R is a commutative ring and if $p, a \in R$ are elements which are not zero divisors for which (p) is a maximal ideal and $a \in (p)^2$, then, for all $n \geqslant 1$,*

$$\mathsf{THH}^{[n]}(R/a;R/p) \simeq \mathsf{THH}^{[n]}(R;R/p) \wedge^L_{HR/p} \mathsf{THH}^{[n],R}(R/a;R/p).$$

Proof Recall that in the category of commutative algebras, the smash product is the same as the pushout. Consider the diagram

$$\begin{array}{ccccc} HR/p & \longrightarrow & \mathsf{THH}^{[n-1],R}(R/p;R/p) & \longrightarrow & \mathsf{THH}^{[n]}(R;R/p)\\ \downarrow & & \downarrow & & \downarrow\\ \mathsf{THH}^{[n],R}(R/a;R/p) & \longrightarrow & \mathsf{THH}^{[n-1],R/a}(R/p;R/p) & \longrightarrow & \mathsf{THH}^{[n]}(R/a;R/p)\end{array}$$

By Proposition 7.5 the left square is a homotopy pushout square and the right square is a homotopy pushout square by the juggling formula of Theorem 3.3, with the maps of those formulas. Thus the outside of the diagram also gives a homotopy pushout, producing the formula

$$\mathsf{THH}^{[n]}(R/a;R/p) \simeq \mathsf{THH}^{[n]}(R;R/p) \wedge^L_{HR/p} \mathsf{THH}^{[n],R}(R/a;R/p),$$

as desired. □

8 The Loday construction of products

We establish a splitting formula for Loday constructions of products of ring spectra. This result is probably well known, but as we will need it later, we provide a proof. In the case of $X = S^1$ such a splitting is proved for connective ring spectra in [15] in the context of "ring functors". See also [12, Proposition 4.2.4.4]. The impatient reader is invited to proceed directly to Section 9 to see the applications of our results.

For two ring spectra A and B we consider their product $A \times B$ with the multiplication

$$(A \times B) \wedge (A \times B) \to A \times B$$

that is induced by the maps $(A \times B) \wedge (A \times B) \to A$ and $(A \times B) \wedge (A \times B) \to B$ that are given by the projection maps to A and B and the multiplication on A and B:

$$\begin{array}{ccccc} (A \times B) \wedge (A \times B) & \xrightarrow{\mathrm{pr}_A \wedge \mathrm{pr}_A} & A \wedge A & \xrightarrow{\mu_A} & A \\ \big\downarrow{\scriptstyle \mathrm{pr}_B \wedge \mathrm{pr}_B} & & & & \\ B \wedge B & & & & \\ \big\downarrow{\scriptstyle \mu_B} & & & & \\ B & & & & \end{array}$$

For $X = S^0$ we obtain

$$\mathcal{L}_{S^0}(A \times B) = (A \times B) \wedge (A \times B)$$

and this is equivalent to $A \wedge A \vee A \wedge B \vee B \wedge A \vee B \wedge B$, whereas $\mathcal{L}_{S^0}(A) \times \mathcal{L}_{S^0}(B)$ is equivalent to $A \wedge A \vee B \wedge B$, so in this case $\mathcal{L}_{S^0}(A \times B)$ is not equivalent to $\mathcal{L}_{S^0}(A) \vee \mathcal{L}_{S^0}(B)$. In general, if a simplicial set has finitely many connected components, say $X = X_1 \sqcup \cdots \sqcup X_n$, then

$$\mathcal{L}_X(A \times B) \simeq \mathcal{L}_{X_1}(A \times B) \wedge \cdots \wedge \mathcal{L}_{X_n}(A \times B),$$

so it suffices to study $\mathcal{L}_X(A \times B)$ for connected simplicial sets X. We will first consider the case $X = S^1$, where $\mathcal{L}_{S^1}$ with respect to the minimal simplicial model of the circle is $\mathsf{THH} = \mathsf{THH}^{[1]}$, and then use that special case to prove the result for general connected finite simplicial sets X.

We thank Mike Mandell, who suggested to use Brooke Shipley's version of THH in the setting of symmetric spectra. Shipley shows in [36] that a variant of Bökstedt's model for THH in symmetric spectra of simplicial sets is equivalent to the version that mimics the Hochschild complex and she proves several important features of this construction. See also [31] for a correction of the proof of the comparison.

Proposition 8.1 *For symmetric ring spectra A and B, the product of the projections induces a stable equivalence* $\mathsf{THH}(A \times B) \xrightarrow{\simeq} \mathsf{THH}(A) \times \mathsf{THH}(B)$.

Schwede establishes a Quillen equivalence between symmetric ring spectra and S–algebras in [35, Theorem 5.1]. This allows a transfer of the above result to the setting of [17]:

Corollary 8.2 *Let A and B be two cofibrant S–algebras and let $(A \times B)^c$ denote a cofibrant replacement of $A \times B$. Then the product of the projection maps induces a weak equivalence*

$$\mathsf{THH}((A \times B)^c) \xrightarrow{\sim} \mathsf{THH}(A) \times \mathsf{THH}(B).$$

In the following proof we denote by THH the model of THH defined in Definition 4.2.6 of [36]. This is no abuse of notation: Let A and B be S–cofibrant symmetric ring spectra [37, Theorem 2.6]. Then, by [36, Theorem 4.2.8] and [31, Theorem 3.6], $\mathsf{THH}(A)$ and $\mathsf{THH}(B)$ are stably equivalent to $\mathsf{THH}(A)$ and $\mathsf{THH}(B)$ in our sense. As $\mathsf{THH}(-)$ sends stable equivalences to stable equivalences [36, Corollary 4.2.9] we can choose an S–cofibrant replacement of $A \times B$, $(A \times B)^c$, and get that $\mathsf{THH}(A \times B)$ is stably equivalent to $\mathsf{THH}((A \times B)^c)$ and this in turn is stably equivalent to our notion of THH.

Proof of Proposition 8.1 For any symmetric ring spectrum R, $\mathsf{THH}(R)$ is defined as the diagonal of a bisimplicial symmetric spectrum $\mathsf{THH}_\bullet(R)$ [36, Definition 4.2.6], where one of the simplicial directions comes from the THH–construction and the other one comes from the fact that we are working with symmetric spectra in simplicial sets. In [31, page 4101] the authors use the geometric realization instead of the diagonal, but this does not cause any difference in the arguments.

We will start by showing that there is a chain of stable equivalences between $\mathsf{THH}(A \times B)$ and $\mathsf{THH}(A) \times \mathsf{THH}(B)$. We use the chain of identifications of Figure 2.

(1) With [36, Proposition 4.2.7] we get a level equivalence between $\mathsf{THH}(A \times B)$ and $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathsf{THH}_\bullet(A \times B)$. Similarly, in the bottom row we can identify the homotopy colimit with THH. This does not need any cofibrancy assumptions.

(2) Let Δ_f denote the subcategory of Δ containing all objects but only injective maps. The induced map on homotopy colimits

$$\mathrm{hocolim}_{\Delta_f^{\mathrm{op}}} \mathsf{THH}_\bullet(A \times B) \to \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathsf{THH}_\bullet(A \times B)$$

is an equivalence because the homotopy colimit of symmetric spectra is defined levelwise [36, Definition 2.2.1] and in every simplicial degree p and every level ℓ, $\mathsf{THH}_p(A \times B)(\ell)$ is cofibrant (because it is just a simplicial set) and hence the claim

$$
\begin{array}{ccccc}
\mathsf{THH}(A\times B) & \xleftarrow[(1)]{\simeq} & \operatorname{hocolim}_{\Delta^{op}}\mathsf{THH}_\bullet(A\times B) & \xleftarrow[(2)]{\simeq} & \operatorname{hocolim}_{\Delta_f^{op}}\mathsf{THH}_\bullet(A\times B) \\
\vdots & & & & \uparrow{\scriptstyle (3)\ \simeq} \\
\vdots & & & & \operatorname{hocolim}_{\Delta_f^{op}}\mathsf{THH}_\bullet(A\vee B) \\
\vdots & & & & {\scriptstyle J}\uparrow\ \downarrow{\scriptstyle R} \\
\vdots{\scriptstyle (\mathrm{pr}_A,\mathrm{pr}_B)} & & & & \operatorname{hocolim}_{\Delta_f^{op}}(\mathsf{THH}_\bullet(A)\vee\mathsf{THH}_\bullet(B)) \\
\vdots & & & & \downarrow{\scriptstyle (4)\ \simeq} \\
\vdots & & & & \operatorname{hocolim}_{\Delta_f^{op}}\mathsf{THH}_\bullet(A)\vee\operatorname{hocolim}_{\Delta_f^{op}}\mathsf{THH}_\bullet(B) \\
\downarrow & & & & \downarrow{\scriptstyle (2)\ \simeq} \\
\mathsf{THH}(A)\times\mathsf{THH}(B) & \xleftarrow[(5)]{\simeq} & \mathsf{THH}(A)\vee\mathsf{THH}(B) & \xleftarrow[(1)]{\simeq} & \operatorname{hocolim}_{\Delta^{op}}\mathsf{THH}_\bullet(A)\vee\operatorname{hocolim}_{\Delta^{op}}\mathsf{THH}_\bullet(B)
\end{array}
$$

Figure 2

follows as in [11, Proposition 20.5]. An analogous argument applies in the second occurrence of (2).

(3) We consider $A\vee B$ as a nonunital ring spectrum via the multiplication map

$$(A\vee B)\wedge(A\vee B)\simeq(A\wedge A)\vee(A\wedge B)\vee(B\wedge A)\vee(B\wedge B)\to(A\wedge A)\vee(B\wedge B)\to A\vee B,$$

where the first map sends the mixed terms to the terminal spectrum and the second one uses the multiplication in A and B. Correspondingly, $\mathsf{THH}_\bullet(A\vee B)$ is a presimplicial spectrum that only uses the face maps of the structure maps of THH.

The canonical map $A\vee B\to A\times B$ is a stable equivalence of nonunital symmetric ring spectra and hence by adapting the argument in the proof of [36, Corollary 4.2.9] we get a π_*–equivalence of the corresponding presimplicial objects,

$$\mathsf{THH}_\bullet(A\vee B)\simeq\mathsf{THH}_\bullet(A\times B).$$

Pointwise level equivalences give level equivalences on homotopy colimits [36, Proposition 2.2.2], so the map in (3) is a level equivalence.

(4) Homotopy colimits commute with sums.

(5) The product is stably equivalent to the sum.

To have a chain of stable equivalences between $\mathsf{THH}(A\times B)$ and $\mathsf{THH}(A)\times\mathsf{THH}(B)$, it remains to understand the effect of the maps J and R and we will control them in Lemma 8.3 below.

We claim that the product of the projections

$$\mathrm{pr}_A\colon A\times B\to A,\quad \mathrm{pr}_B\colon A\times B\to B$$

produces an equivalence. Observe that we can apply the projection pr_A to every stage in our diagram. On $A\vee B$, this will induce the collapse map $A\vee B\to A$. By applying pr_A to the entire chain of equivalences, we get a diagram of equivalences between various versions of $\mathsf{THH}(A)$. We can do the same for pr_B. This gives a commutative diagram

$$\begin{array}{ccc} \mathsf{THH}(A\times B) & \text{chain of maps} & \mathsf{THH}(A)\times\mathsf{THH}(B) \\ \big\downarrow{\scriptstyle(\mathrm{pr}_A,\mathrm{pr}_B)} & & \big\downarrow{\scriptstyle(\mathrm{pr}_A,\mathrm{pr}_B)} \\ \mathsf{THH}(A)\times\mathsf{THH}(B) & (\mathrm{pr}_A(\text{the chain}),\mathrm{pr}_B(\text{the chain})) & \mathsf{THH}(A)\times\mathsf{THH}(B) \end{array}$$

where the chain is a zigzag of arrows going both ways. The chain on top has all its stages equivalences. By the above discussion, so is the product of the chains on the bottom. And the map on the right is the identity. So, working step by step in the zigzag from the right, we show that the pair of projections $(\mathrm{pr}_A,\mathrm{pr}_B)$ induces equivalences at all the intermediate steps, until we get to the leftmost $(\mathrm{pr}_A,\mathrm{pr}_B)$, which is therefore also an equivalence. □

We consider the map

$$j\colon \mathsf{THH}_\bullet(A)\vee\mathsf{THH}_\bullet(B)\to\mathsf{THH}_\bullet(A\vee B)$$

that is induced by the inclusions $A\hookrightarrow A\vee B$ and $B\hookrightarrow A\vee B$. We let J be the induced map on the homotopy colimit. It has a retraction $R=\mathrm{hocolim}_{\Delta_f^{\mathrm{op}}} r$ with

$$r\colon \mathsf{THH}_\bullet(A\vee B)\to\mathsf{THH}_\bullet(A)\vee\mathsf{THH}_\bullet(B)$$

that sends all mixed smash products to the terminal object. Note that $R\circ J=\mathrm{id}$.

Lemma 8.3 *There is a presimplicial homotopy $j\circ r\simeq\mathrm{id}$.*

Proof We consider the n^{th} presimplicial degree of $\mathsf{THH}_n(A\vee B)$,

$$\mathsf{THH}_n(A\vee B)=(A\vee B)^{n+1}.$$

This is a sum of terms of the form $A^{\wedge i_1}\wedge B^{\wedge i_2}\wedge\cdots\wedge B^{\wedge i_k}\wedge A^{\wedge i_{k+1}}$ for suitable k with $0\leqslant i_1,i_{k+1}$ and $0<i_j$ for $1<j<k+1$ such that $\sum_{j=1}^{k+1} i_j=n+1$.

Restricted to such a summand we define $h_j\colon (A\vee B)^{n+1}\to (A\vee B)^{n+2}$ for $0\leqslant j\leqslant n$ as

$$h_j|_{A^{\wedge i_1}\wedge B^{\wedge i_2}\wedge\cdots\wedge B^{\wedge i_k}\wedge A^{\wedge i_{k+1}}} = \begin{cases} \mathrm{id}_{A^{\wedge j+1}}\wedge \eta_A\wedge \mathrm{id} & \text{if } j+1\leqslant i_1,\\ \mathrm{id}_{B^{\wedge j+1}}\wedge \eta_B\wedge \mathrm{id} & \text{if } i_1=0 \text{ and } j+1\leqslant i_2,\\ * & \text{otherwise.}\end{cases}$$

Then $d_0h_0=\mathrm{id}$, $d_{n+1}h_n=j\circ r$ and

$$d_ih_j=\begin{cases} h_{j-1}d_i & \text{if } i<j,\\ d_ih_{j-1} & \text{if } i=j\neq 0,\\ h_jd_{i-1} & \text{if } i>j+1.\end{cases}$$ □

This implies that $J\circ R\simeq \mathrm{id}$, so we get that

$$\mathrm{hocolim}_{\Delta_f^{\mathrm{op}}}\mathsf{THH}_\bullet(A\vee B)\simeq \mathrm{hocolim}_{\Delta_f^{\mathrm{op}}}(\mathsf{THH}_\bullet(A)\vee \mathsf{THH}_\bullet(B)).$$

For the general setting we work with *commutative* ring spectra and we return to the setting of [17]. Note that the naturality of cofibrant replacements ensures that we get morphisms of commutative ring spectra

$$A^c\leftarrow (A\times B)^c\to B^c$$

and hence a weak equivalence (because the product of acyclic fibrations is an acyclic fibration)

$$(A\times B)^c\to A^c\times B^c.$$

This proves the case of $X=*$ of the following proposition and is needed in the proof.

Proposition 8.4 *For any connected finite simplicial set X and commutative ring spectra A and B, the projection maps $\mathrm{pr}_A\colon A\times B\to A$ and $\mathrm{pr}_B\colon A\times B\to B$ induce an equivalence*

$$\mathcal{L}_X((A\times B)^c)\simeq \mathcal{L}_X(A^c)\times \mathcal{L}_X(B^c)$$

and in particular, for all $n\geqslant 1$,

$$\mathsf{THH}^{[n]}((A\times B)^c)\simeq \mathsf{THH}^{[n]}(A^c)\times \mathsf{THH}^{[n]}(B^c).$$

Proof We prove the result for all finite connected simplicial sets X by induction on the dimension n of the top nondegenerate simplex in X. Since the only finite connected simplicial set with its only nondegenerate simplices in dimension zero is a point, the result is obvious for $n=0$.

For higher n, the crucial observation is that if we have simplicial sets X, Y and Z such that Z is a nonempty subset of both X and Y, then if the projection maps $(A \times B)^c \to A^c$ and $(A \times B)^c \to B^c$ induce equivalences as given in the statement of this proposition for X, Y and Z, then we also obtain an equivalence

$$\mathcal{L}_{X \cup_Z Y}((A \times B)^c) \xrightarrow{\simeq} \mathcal{L}_{X \cup_Z Y}(A^c) \times \mathcal{L}_{X \cup_Z Y}(B^c). \tag{8.5}$$

This is because then

$$\begin{aligned} \mathcal{L}_{X \cup_Z Y}((A \times B)^c) &\simeq \mathcal{L}_X((A \times B)^c) \wedge_{\mathcal{L}_Z((A \times B)^c)} \mathcal{L}_Y((A \times B)^c) \\ &\simeq (\mathcal{L}_X(A^c) \times \mathcal{L}_X(B^c)) \wedge^L_{\mathcal{L}_Z(A^c) \times \mathcal{L}_Z(B^c)} (\mathcal{L}_Y(A^c) \times \mathcal{L}_Y(B^c)) \\ &\simeq (\mathcal{L}_X(A^c) \wedge_{\mathcal{L}_Z(A^c)} \mathcal{L}_Y(A^c)) \times (\mathcal{L}_X(B^c) \wedge_{\mathcal{L}_Z(B^c)} \mathcal{L}_Y(B^c)) \\ &\simeq \mathcal{L}_{X \cup_Z Y}(A^c) \times \mathcal{L}_{X \cup_Z Y}(B^c). \end{aligned}$$

For the first and last equivalence we use that the Loday construction sends pushouts to homotopy pushouts and that for a cofibrant commutative ring spectrum the map $\mathcal{L}_Z(R) \to \mathcal{L}_Y(R)$ is a cofibration. For the second equivalence we use our assumption that the proposition holds for X, Y and Z.

The third equivalence holds because $\mathcal{L}_Z(A^c)$ acts trivially on $\mathcal{L}_X(B^c)$ and on $\mathcal{L}_Y(B^c)$, thus it sends the corresponding factors to the terminal ring spectrum; the same holds for the action of $\mathcal{L}_Z(B^c)$ on $\mathcal{L}_X(A^c)$ and on $\mathcal{L}_Y(A^c)$. Therefore a Künneth spectral sequence argument shows that we obtain a weak equivalence.

For $n = 1$, we use homotopy invariance of the Loday construction and the fact that any finite connected simplicial set with nondegenerate simplices only in dimensions 0 and 1 is homotopy equivalent to $\bigvee_{i=1}^m S^1$ for some $m \geqslant 0$. If $m = 0$, we deduce the proposition from the $n = 0$ case above; if $m = 1$, we use Proposition 8.1, and for $m > 1$, we use induction and (8.5).

For the inductive step, assume that $n > 1$ and that the proposition holds for any finite connected simplicial set with nondegenerate cells in dimensions $< n$, and in particular for $\partial\Delta^n$, the boundary of the standard n–simplex. Assume that we have a finite simplicial set X for which the proposition holds. We then prove that the proposition also holds for $X \cup_{\partial\Delta^n} \Delta^n$, that is, X with an additional n–simplex glued to it along the boundary. Without loss of generality, we may assume that the boundary of the new n–simplex is embedded in X: if it is not, apply four-fold edgewise subdivision to everything, and then $X \cup_{\partial\Delta^n} \Delta^n$ will consist of the central small n–simplex inside the original n–simplex that was added that does not touch the boundary of the originally added n–simplex and all the rest of the subdivided complex. But the rest of the subdivided complex is homotopy equivalent to the original X, so the proposition holds for it, and the central small n–simplex does indeed have its boundary embedded in the four-fold edgewise subdivision of $X \cup_{\partial\Delta^n} \Delta^n$.

Then, by assumption the proposition holds for X, by the inductive hypothesis it holds for $\partial\Delta^n$, by homotopy invariance it holds for $\Delta^n \simeq *$, and so by (8.5) it holds for $X \cup_{\partial\Delta^n} \Delta^n$. □

For later use we need a version of Proposition 8.4 with coefficients. Again, we choose cofibrant models $(A\times B)^c$, A^c and B^c and we assume M^c is a cofibrant A^c–module spectrum, N^c is a cofibrant B^c–module spectrum and $(M\times N)^c$ is a cofibrant $(A\times B)^c$–module spectrum such that these cofibrant replacements are compatible with the projection maps on $(A\times B)^c$ and $(M\times N)^c$.

Corollary 8.6 *For all connected pointed finite simplicial sets there is an equivalence*

$$\mathcal{L}_X((A\times B)^c;(M\times N)^c) \to \mathcal{L}_X(A^c;M^c)\times\mathcal{L}_X(B^c;N^c).$$

Proof The argument in the proof of Proposition 8.4 can be adapted to pointed finite simplicial sets. We know that

$$\mathcal{L}_X((A\times B)^c;(M\times N)^c) \simeq \mathcal{L}_X((A\times B)^c)\wedge_{(A\times B)^c}((M\times N)^c)$$

and by the result above this is equivalent to

$$(\mathcal{L}_X(A^c)\times\mathcal{L}_X(B^c))\wedge^L_{A^c\times B^c}(M^c\times N^c).$$

Again, we can identify the coequalizers because the action of A^c on N^c and the one of B^c on M^c is trivial and obtain

$$(\mathcal{L}_X(A^c)\wedge_{A^c} M^c)\times(\mathcal{L}_X(B^c)\wedge_{B^c} N^c) \simeq \mathcal{L}_X(A^c;M^c)\times\mathcal{L}_X(B^c;N^c).$$ □

9 Applications

9.1 $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m;\mathbb{Z}/p)$

This example was our original motivation for obtaining the splitting result of Theorem 7.6. We apply it to the case where $R=\mathbb{Z}$, p is a prime and $m \geqslant 2$. As a special case of Theorem 7.6 we obtain the following splitting:

Theorem 9.1 *We have*

$$\begin{aligned}\mathsf{THH}^{[n]}(\mathbb{Z}/p^m;\mathbb{Z}/p) &\simeq \mathsf{THH}^{[n]}(\mathbb{Z};\mathbb{Z}/p)\wedge^L_{H\mathbb{Z}/p}\mathsf{THH}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p)\\ &\cong \mathsf{THH}^{[n]}(\mathbb{Z};\mathbb{Z}/p)\wedge^L_{H\mathbb{Z}/p}\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p).\end{aligned}$$

This gives a direct calculation of

$$\mathsf{THH}^{[n]}_*(\mathbb{Z}/p^m;\mathbb{Z}/p)\cong \mathsf{THH}^{[n]}_*(\mathbb{Z};\mathbb{Z}/p)\otimes \mathsf{Sh}^{[n],\mathbb{Z}}_*(\mathbb{Z}/p^m;\mathbb{Z}/p)$$

for all n because in [13, Theorem 3.1] we determine $\mathsf{THH}^{[n]}_*(\mathbb{Z};\mathbb{Z}/p)$ to be an iterated Tor algebra

$$B^n_{\mathbb{F}_p}(x)\otimes_{\mathbb{F}_p} B^{n+1}_{\mathbb{F}_p}(y)$$

where $|x|=2p$, $|y|=2p-2$, $B^1_{\mathbb{F}_p}(z)=\mathbb{F}_p[z]$ and $B^n_{\mathbb{F}_p}(z)=\mathrm{Tor}_*^{B^{n-1}_{\mathbb{F}_p}(z)}(\mathbb{F}_p,\mathbb{F}_p)$. We determined $\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p)$ in Proposition 5.3.

The case $n=1$ was calculated in [32; 9] and later as well in [1].

Remark 9.2 We cannot use the sequence of canonical projection maps

$$\cdots\to\mathbb{Z}/p^{m+1}\mathbb{Z}\to\mathbb{Z}/p^m\mathbb{Z}\to\cdots\to\mathbb{Z}/p^2\mathbb{Z}\to\mathbb{Z}/p\mathbb{Z}$$

in order to compare the groups $\mathsf{THH}(\mathbb{Z}/p^m;\mathbb{Z}/p)$ for varying m because, while the tensor factors coming from $\mathsf{THH}_*(\mathbb{Z};\mathbb{Z}/p)$ are mapped isomorphically from $\mathsf{THH}(\mathbb{Z}/p^{m+1};\mathbb{Z}/p)$ to $\mathsf{THH}(\mathbb{Z}/p^m;\mathbb{Z}/p)$, the tensor factor $\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^{m+1};\mathbb{Z}/p)$ is mapped via the augmentation map to $\mathsf{Sh}^{[n],\mathbb{Z}}(\mathbb{Z}/p^m;\mathbb{Z}/p)$ in each step of the sequence. This is straightforward to see with the help of the explicit resolutions used in the proof of Lemma 7.3.

9.2 Number rings

As a warm-up we consider $R=\mathbb{Z}[i]$, $p=1-i$ and $2\in(p)^2$. Then we get that

$$\begin{aligned}&\mathsf{THH}^{[n]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i))\\ &\qquad\simeq \mathsf{THH}^{[n]}(\mathbb{Z}[i];\mathbb{Z}[i]/(1-i))\wedge_{H\mathbb{Z}[i]/(1-i)}\mathsf{Sh}^{[n],\mathbb{Z}[i]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i)).\end{aligned}$$

Note that $\mathbb{Z}[i]/(1-i)\cong\mathbb{Z}/2$ and $\mathbb{Z}[i]/2\cong\mathbb{F}_2[x]/x^2$. Thus we can calculate

$$\mathsf{THH}^{[n]}(\mathbb{Z}[i]/2;\mathbb{Z}[i](1-i))\cong\mathsf{THH}^{[n]}(\mathbb{F}_2[x]/x^2;\mathbb{F}_2)$$

using the flow chart in [6] and we know from [13, Theorem 4.3] that

$$\mathsf{THH}^{[n]}_*(\mathbb{Z}[i];\mathbb{Z}[i]/(1-i))$$

can also be computed using iterated Tor's. The term $\mathsf{Sh}^{[n],\mathbb{Z}[i]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i))$ can be computed as an iterated Tor by Proposition 5.3. Thus all of the components of the above expression are known. What was not previously known is that $\mathsf{THH}^{[n]}(\mathbb{Z}[i]/2;\mathbb{Z}[i]/(1-i))$ splits in the above manner.

The general case is as follows: Consider $p \in \mathbb{Z}$ a prime, and let K be a number field such that p is ramified in $\mathcal{O}_K$, so $p = \mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_r^{e_r}$ where $e_i > 1$ for at least one i. The Chinese remainder theorem lets us split $\mathcal{O}_K/p$ as a ring as

$$\mathcal{O}_K/p \cong \prod_{j=1}^{r} \mathcal{O}_K/\mathfrak{p}_j^{e_j}$$

and $\mathcal{O}_K/\mathfrak{p}_i$ as an $\mathcal{O}_K/p$–module is then isomorphic to $0\times\cdots\times\mathcal{O}_k/\mathfrak{p}_i\times\cdots\times 0$ with the nontrivial component sitting in spot number i. With Corollary 8.6 we obtain the following result:

Theorem 9.3 *We have*

(9.4)
$$\mathsf{THH}^{[n]}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i) \simeq \mathsf{THH}^{[n]}(\mathcal{O}_K;\mathcal{O}_K/\mathfrak{p}_i)\wedge_{H\mathcal{O}_K/\mathfrak{p}_i}\mathsf{THH}^{[n],\mathcal{O}_K}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i).$$

Again, $\mathcal{O}_K/p \cong (\mathcal{O}_K)_{\mathfrak{p}_i}/p$ is isomorphic to $\mathcal{O}_K/\mathfrak{p}_i[\pi]/\pi^{e_i}$, where π is the uniformizer, hence $\mathcal{O}_K/\mathfrak{p}_i[\pi]/\pi^{e_i} \cong \mathcal{O}_K/\mathfrak{p}_i[x]/x^{e_i}$, so we can calculate $\mathsf{THH}^{[n]}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$ using Proposition 2.5. We can determine $\mathsf{THH}^{[n]}(\mathcal{O}_K;\mathcal{O}_K/\mathfrak{p}_i)$ using [13, Theorem 4.3] and we calculated $\mathsf{THH}^{[n],\mathcal{O}_K}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$ in Proposition 5.3. Using these calculations one can deduce right away that there is a splitting on the level of homotopy groups of $\mathsf{THH}^{[n]}_*(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$. But (9.4) yields a splitting of $\mathsf{THH}^{[n]}(\mathcal{O}_K/p;\mathcal{O}_K/\mathfrak{p}_i)$ on the level of augmented commutative $H\mathcal{O}_K/\mathfrak{p}_i$–algebra spectra.

9.3 Galois descent

In [34, Definition 9.2.1] John Rognes defines a map of commutative S–algebras $f\colon A \to B$ to be formally THH–étale if the unit map $B \to \mathsf{THH}^A(B)$ is a weak equivalence. Note that this implies that the augmentation map $\mathsf{THH}^A(B) \to B$ that is induced by multiplying all B–entries in $\mathsf{THH}^A(B)$ together is also a weak equivalence because the composition $B \to \mathsf{THH}^A(B) \to B$ is the identity on B.

Therefore, applying the Brun juggling formula of Theorem 4.1 in this case to $X = S^1$, we obtain

$$\mathsf{THH}^{[2]}(A) \wedge^L_A B \simeq \mathsf{THH}^{[2]}(A;B) \simeq B \wedge_{\mathsf{THH}(B)} \mathsf{THH}^A(B) \simeq B \wedge^L_{\mathsf{THH}(B)} B \simeq \mathsf{THH}^{[2]}(B).$$

We can slightly generalize this:

Definition 9.5 Let X be a pointed simplicial set. A morphism $f\colon A \to B$ is formally X–étale if the unit map $B \to \mathcal{L}^A_X(B)$ is a weak equivalence.

For formally X–étale morphisms $f\colon A \to B$, the Brun juggling formula of Theorem 4.1 for X implies

$$\mathcal{L}_{\Sigma X}(A) \wedge^L_A B \simeq \mathcal{L}_{\Sigma X}(A;B) \simeq B \wedge_{\mathcal{L}_X(B)} \mathcal{L}^A_X(B) \simeq B \wedge_{\mathcal{L}_X(B)} B \simeq \mathcal{L}_{\Sigma X}(B).$$

This statement is related to Akhil Mathew's result [28, Proposition 5.2], where he shows that $\mathcal{L}_Y(A;B) \simeq \mathcal{L}_Y(B)$ if $f\colon A \to B$ is a faithful finite G–Galois extension and if Y is a simply connected pointed simplicial set. Such Galois extensions are formally THH–étale by [34, Lemma 9.2.6].

9.4 Algebraic function fields over $\mathbb{F}_p$

In several of our splitting formulas, higher THH of the ground field is a tensor factor. So far we have only considered prime fields or rather simple-minded algebraic extensions of those. Topological Hochschild homology groups of algebraic function fields are an important class of examples.

Let L be an algebraic function field over $\mathbb{F}_p$. Then there is a transcendence basis $(x_1, \dots, x_d)$ such that L is a finite separable extension of $\mathbb{F}_p(x_1, \dots, x_d)$ [30, Theorem 9.27]. As separable extensions do not contribute anything substantial to topological Hochschild homology we obtain the following result:

Theorem 9.6 *Let L be an algebraic function field over $\mathbb{F}_p$; then*

$$\mathsf{THH}_*(L) \cong L \otimes_{\mathbb{F}_p} \mathsf{THH}_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon x_1, \dots, \varepsilon x_d).$$

Proof McCarthy and Minasian show in [29, Lemma 5.5 and Remark 5.6] that THH has étale descent in our case. Therefore

$$\begin{aligned}\mathsf{THH}(L) &\simeq HL \wedge^L_{H\mathbb{F}_p(x_1,\dots,x_d)} \mathsf{THH}(\mathbb{F}_p(x_1,\dots,x_d)) \\ &\simeq HL \wedge^L_{H\mathbb{F}_p(x_1,\dots,x_d)} H\mathbb{F}_p(x_1,\dots,x_d) \wedge^L_{H\mathbb{F}_p[x_1,\dots,x_d]} \mathsf{THH}(\mathbb{F}_p[x_1,\dots,x_d]) \\ &\simeq HL \wedge^L_{H\mathbb{F}_p[x_1,\dots,x_d]} \mathsf{THH}(\mathbb{F}_p[x_1,\dots,x_d]).\end{aligned}$$

But the topological Hochschild homology of monoid rings is known by Theorem 7.1 of [20] and hence $\pi_*\mathsf{THH}(\mathbb{F}_p[x_1,\dots,x_d]) \cong \mathsf{THH}_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathsf{HH}_*(\mathbb{F}_p[x_1,\dots,x_d])$. As

$$\mathsf{HH}_*(\mathbb{F}_p[x_1,\dots,x_d]) \cong \mathbb{F}_p[x_1,\dots,x_d] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon x_1,\dots,\varepsilon x_d),$$

we get the result. □

McCarthy and Minasian actually show more in [29, Lemma 5.5 and Remark 5.6], and we can adapt the above proof to a more general situation.

Theorem 9.7 *Let X be a connected pointed simplicial set X. Then*

$$\mathcal{L}_X(HL) \simeq HL \wedge^L_{H\mathbb{F}_p[x_1,\dots,x_d]} \mathcal{L}_X(\mathbb{F}_p[x_1,\dots,x_d]).$$

The Loday construction on pointed monoid algebras satisfies a splitting of the form

$$\mathcal{L}_X(H\mathbb{F}_p[\Pi_+]) \simeq \mathcal{L}_X(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathcal{L}_X^{H\mathbb{F}_p}(H\mathbb{F}_p[\Pi_+]);$$

see [20, Theorem 7.1]. Therefore $\mathcal{L}_X(\mathbb{F}_p[x_1,\dots,x_d])$ splits as

$$\mathcal{L}_X(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathcal{L}_X^{H\mathbb{F}_p}(H\mathbb{F}_p[x_1,\dots,x_d]).$$

In particular, for $X = S^n$ we get an explicit formula for $\mathsf{THH}^{[n]}(L)$:

Corollary 9.8 *For all $n \geqslant 1$,*

$$\mathsf{THH}^{[n]}(L) \simeq HL \wedge^L_{H\mathbb{F}_p[x_1,\dots,x_d]} (\mathsf{THH}^{[n]}(\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathsf{THH}^{[n],\mathbb{F}_p}(\mathbb{F}_p[x_1,\dots,x_d])).$$

Recall that we know

$$\begin{aligned}\pi_*\mathsf{THH}^{[n],\mathbb{F}_p}(\mathbb{F}_p[x_1,\dots,x_d]) &= \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x_1,\dots,x_d]) \\ &\cong \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x]^{\otimes_{\mathbb{F}_p} d}) \\ &\cong \mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x])^{\otimes_{\mathbb{F}_p} d}\end{aligned}$$

and we determined $\mathsf{HH}_*^{[n],\mathbb{F}_p}(\mathbb{F}_p[x])$ in [6, Theorem 8.6].

Remark 9.9 Topological Hochschild homology of L considers HL as an S–algebra and this allows us to consider L over the prime field. The Hochschild homology of an algebraic function field L over a general field K was for instance determined in [8, Corollary 5.3] and is more complicated.

Remark 9.10 Theorem 9.6 contradicts the statement of [18, Remark 7.2]. In that remark, it is crucial to assume that one works in an *augmented* setting; in the above situation, this is not the case.

9.5 $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m)$

We close with the open problem of computing $\mathsf{THH}^{[n]}(\mathbb{Z}/p^m)$ for higher n.

The juggling formula of Theorem 3.3 applied to the sequence $S \to H\mathbb{Z} \to H\mathbb{Z}/p^m = H\mathbb{Z}/p^m$ for $m \geqslant 2$ yields the equivalence

$$\mathsf{THH}^{[n]}(\mathbb{Z}/p^m) \simeq \mathsf{THH}^{[n]}(\mathbb{Z};\mathbb{Z}/p^m) \wedge^L_{\mathsf{THH}^{[n-1],\mathbb{Z}}(H\mathbb{Z}/p^m)} H\mathbb{Z}/p^m.$$

Up to $n = 2$ we know $\mathsf{THH}^{[n]}_*(H\mathbb{Z})$: the case $n = 1$ is Bökstedt's calculation [7] and $n = 2$ is [14, Theorem 2.1]. Therefore we can determine $\mathsf{THH}^{[n]}_*(\mathbb{Z};\mathbb{Z}/p^m)$ up to $n = 2$. As p^m is regular in $\mathbb{Z}$,

$$\mathsf{THH}^{\mathbb{Z}}_*(\mathbb{Z}/p^m) = \mathsf{Sh}^{\mathbb{Z}}_*(\mathbb{Z}/p^m) \cong \Gamma_{\mathbb{Z}/p^m}(x_2)$$

with $|x_2| = 2$. If we could determine the right $\mathsf{Sh}^{\mathbb{Z}}_*(\mathbb{Z}/p^m)$–module structure on $\mathsf{THH}^{[2]}_*(\mathbb{Z};\mathbb{Z}/p^m)$, then this would allow us to calculate the E^2–term of the Künneth spectral sequence for $\mathsf{THH}^{[2]}_*(\mathbb{Z}/p^m)$,

$$E^2_{p,q} = \mathrm{Tor}^{\Gamma_{\mathbb{Z}/p^m}(x_2)}_{p,q}(\mathsf{THH}^{[2]}_*(\mathbb{Z};\mathbb{Z}/p^m), \mathbb{Z}/p^m) \Rightarrow \mathsf{THH}^{[2]}_*(\mathbb{Z}/p^m).$$

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