Class Example #3 (3/10/2016) – Substitution in double integrals

$$\boxed{\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy}$$

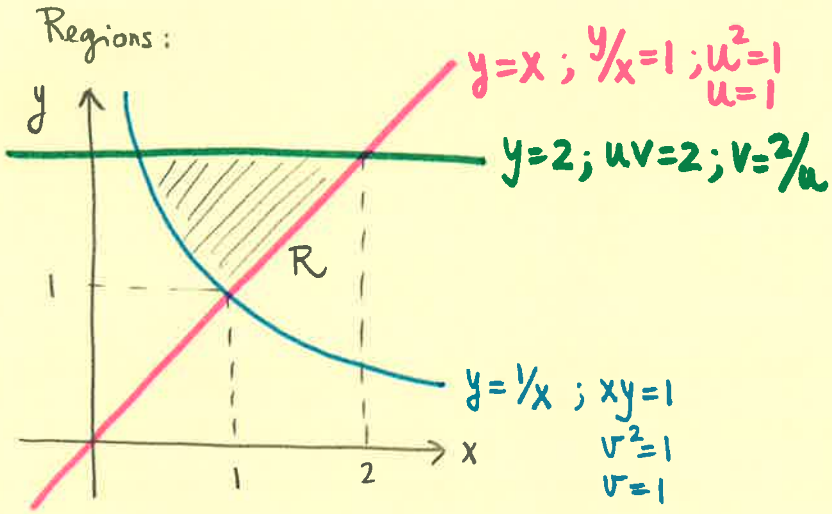
Substitutions: $\begin{cases} u = \sqrt{\frac{y}{x}} & \Rightarrow y = u^2 x \\ v = \sqrt{xy} & \Rightarrow v^2 = u^2 x^2 \end{cases}$

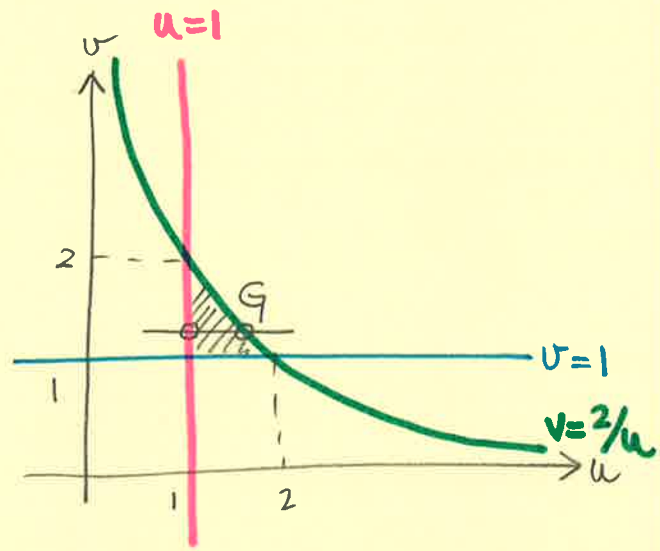
$\Rightarrow x = \frac{v}{u} \Rightarrow y = uv$

(because all quantities are positive)

$$\Rightarrow \begin{cases} x = \frac{v}{u} \\ y = uv \end{cases} \Rightarrow J = \begin{vmatrix} -v/u^2 & 1/u \\ v & u \end{vmatrix} = -2\frac{v}{u} < 0 \Rightarrow \text{Take absolute value later!}$$

Regions:





$$\Rightarrow \int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G u e^v \left|-2\frac{v}{u}\right| d(u,v) = 2\iint_G v e^v\, d(u,v)$$

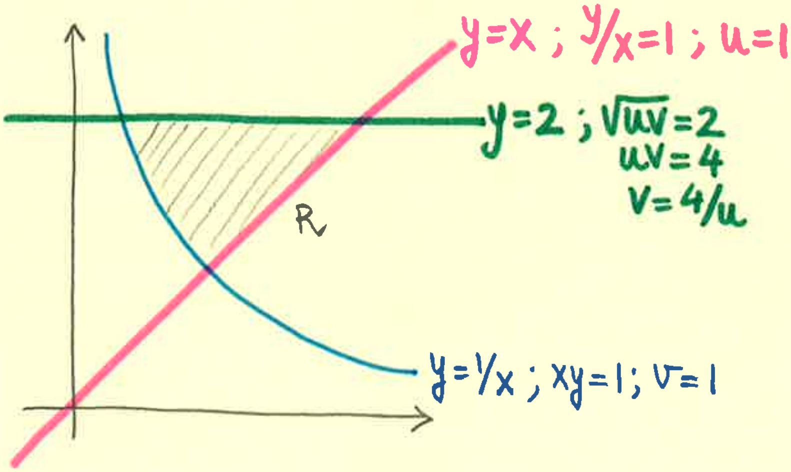
$$= 2\int_1^2 \int_1^{2/v} v e^v\, du\, dv = 2\int_1^2 v e^v u \Big|_{u=1}^{u=2/v} dv = 2\int_1^2 (2e^v - ve^v)\, dv$$

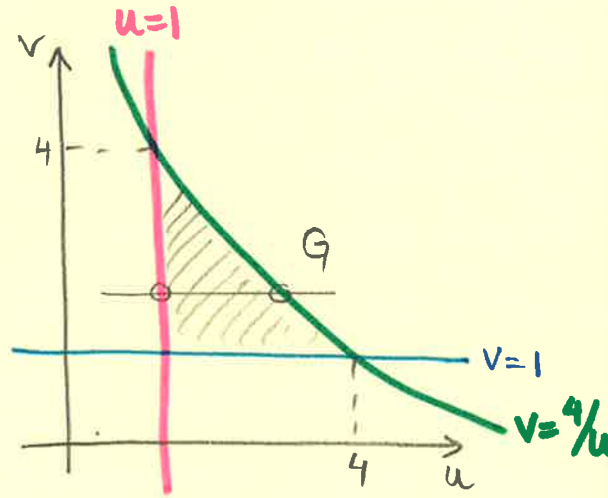
$$= 2\left(2e^v\Big|_1^2 - \int_1^2 v(e^v)'\, dv\right) = 2\left(2e^2 - 2e - ve^v\Big|_1^2 + \int_1^2 e^v\, dv\right)$$

$$= 2\left(\cancel{2e^2} - 2e - \cancel{2e^2} + e + e^v\Big|_1^2\right) = 2\left(-e + e^2 - e\right) = \boxed{2e^2 - 4e}$$

Try another substitution: $\begin{cases} u = y/x \implies y = xu \\ v = xy \implies v = x^2 u \implies x = \sqrt{\frac{v}{u}} \implies y = \sqrt{uv} \end{cases}$

$$\implies \begin{cases} x = \sqrt{\frac{v}{u}} \\ y = \sqrt{uv} \end{cases} \implies J = \begin{vmatrix} -\frac{\sqrt{v}}{2u\sqrt{u}} & \frac{1}{2\sqrt{uv}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u}$$





$$\implies \iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, d(x,y) = \iint_G \sqrt{u}\, e^{\sqrt{v}} \cdot \frac{1}{2u}\, d(u,v) = \iint_G \frac{1}{2\sqrt{u}}\, e^{\sqrt{v}}\, d(u,v)$$

take horizontal cross-sections

$$= \int_1^4 \int_1^{4/v} \frac{1}{2\sqrt{u}} e^{\sqrt{v}}\, du\, dv$$

$$= \int_1^4 e^{\sqrt{v}} \left(\sqrt{u}\right)\Big|_{u=1}^{u=4/v} dv = \int_1^4 e^{\sqrt{v}} \left(\frac{2}{\sqrt{v}} - 1\right) dv$$

$$= \int_1^4 \left(e^{\sqrt{v}} \frac{2}{\sqrt{v}} - e^{\sqrt{v}}\right) dv$$

$$= 4e^{\sqrt{v}}\Big|_1^4 - \int_1^4 e^{\sqrt{v}}\, dv$$

$$= 4(e^2 - e) - 2e^2$$

$$= \boxed{2e^2 - 4e}$$

$$\int_1^4 e^{\sqrt{v}}\, dv$$

$w = \sqrt{v}$

$dw = \frac{1}{2\sqrt{v}} dv = \frac{1}{2w} dv$

$\implies dv = 2w\, dw$

$$= \int_1^2 2w e^w\, dw$$

$$= 2\left(we^w\Big|_1^2 - \int_1^2 e^w\, dw\right)$$

$$= 2\left(2e^2 - \cancel{e} - e^2 + \cancel{e}\right)$$

$$= 2e^2$$

(Still works, but integration is a bit harder)