10/9/2014

Class Example #3

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

I Substitution:

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{y/x} \end{cases}$$

II Find the inverse transformation (x, y in terms of u, v)

$$\begin{cases} u = \sqrt{x}\sqrt{y} \\ \sqrt{y} = v\sqrt{x} \end{cases}$$

$u = \sqrt{x} \cdot v\sqrt{x} \Rightarrow \boxed{x = \frac{u}{v}}$

$y = v^2 x = v^2 \cdot \frac{u}{v} \Rightarrow \boxed{y = uv}$

III Find the Jacobian:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + v \cdot \frac{u}{v^2} = \frac{2u}{v}$$

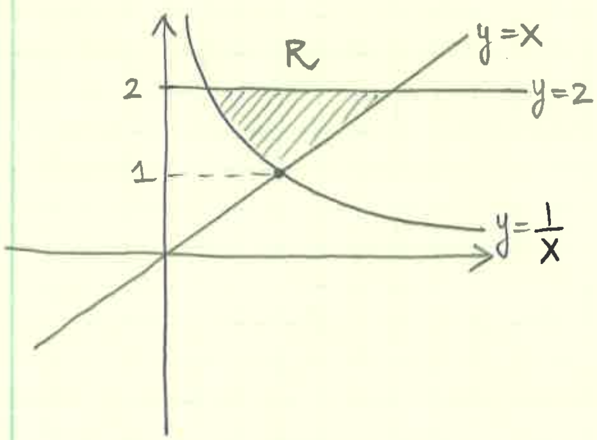
What do we have so far?

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} \cdot e^{\sqrt{xy}}\, dx\, dy = \int_?^? \int_?^? \cancel{v} \cdot e^u \cdot \frac{2u}{\cancel{v}}\, du\, dv$$

IV Transform the region R in the (x, y)-plane to the region G in the (u, v)-plane

IVa Sketch R:

$\frac{1}{y} \le x \le y \;;\; 1 \le y \le 2$



IVb Boundaries of R → Boundaries of G

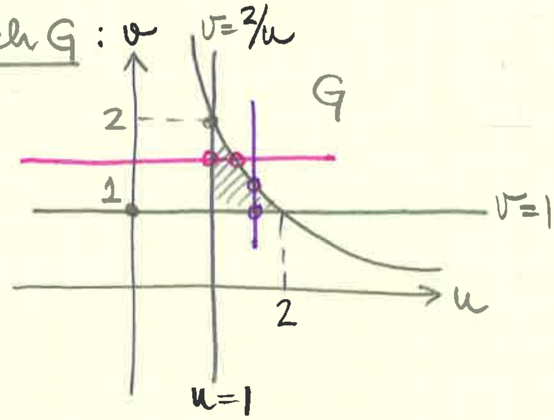
$\boxed{y = x} \longrightarrow \frac{y}{x} = 1 \Rightarrow v^2 = 1 \Rightarrow \boxed{v = 1}$

$\boxed{y = \frac{1}{x}} \longrightarrow xy = 1 \Rightarrow u^2 = 1 \Rightarrow \boxed{u = 1}$

(because u, v are both equal to square roots ⇒ positive)

$\boxed{y = 2} \longrightarrow uv = 2 \Rightarrow \boxed{v = \frac{2}{u}}$

IVc. Sketch G :



V Evaluate Integral: $\iint_G 2ue^u \, du\, dv$

Horizontal Cross-Sections: $\int_1^2 \int_1^{2/v} 2ue^u \, du\, dv$

$$= \int_1^2 \int_1^{2/v} 2u(e^u)' \, du\, dv = \int_1^2 \left[2ue^u \Big|_{u=1}^{u=2/v} - \int_1^{2/v} 2e^u \, du \right] dv$$

$$= \int_1^2 \left(2 \cdot \frac{2}{v} e^{2/v} - \cancel{2e} - 2e^{2/v} + \cancel{2e} \right) dv$$

$$= \int_1^2 \left(\frac{4}{v} - 2 \right) e^{2/v} \, dv \rightsquigarrow \text{bad integral}$$

Try Vertical Cross-Sections

Vertical Cross-Sections: $\int_1^2 \int_1^{2/u} 2ue^u \, dv\, du$

$$= \int_1^2 \left(2ue^u \cdot v \Big|_{v=1}^{v=2/u} \right) du = \int_1^2 \left(\cancel{2}\cancel{u}e^u \cdot \frac{2}{\cancel{u}} - 2ue^u \cdot 1 \right) du$$

$$= \int_1^2 \left(4e^u - 2ue^u \right) du = 4e^u \Big|_1^2 - 2 \int_1^2 u(e^u)' \, du$$

$$= 4e^2 - 4e - 2\left(ue^u \Big|_1^2 - \int_1^2 e^u \, du \right)$$

$$= 4e^2 - 4e - 2\left(2e^2 - e - e^u \Big|_1^2 \right)$$

$$= \cancel{4e^2} - 4e - \cancel{4e^2} + \cancel{2e} + 2e^2 - \cancel{2e}$$

$$= \boxed{2e^2 - 4e}$$