Name: Solutions

September 10^{th}, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
Exam 1

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	20	
2	20	
3	10	
4	20	
5	20	
6	10	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

[6pts.] 1. [20 pts.] Given the vectors $\vec{v_1} = \langle 1, 0, 2\rangle$ and $\vec{v_2} = \langle -1, 2, 3\rangle$:
a). Find $\vec{v_1} \cdot \vec{v_2}$.

$$\vec{v_1} \cdot \vec{v_2} = 1(-1) + 0\cdot 2 + 2\cdot 3 = -1+6 = \boxed{5}$$ 6pts.

[6pts.] b). Find $\vec{v_1} \times \vec{v_2}$.

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ -1 & 2 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= \vec{i}(0-4) - \vec{j}(3+2) + \vec{k}(2-0)$$

$$= \boxed{-4\vec{i} - 5\vec{j} + 2\vec{k}} \text{ or } \boxed{\langle -4, -5, 2\rangle}$$

3pts. - Setting up determinant

1pt. - each component

[5pts.] c). Find the angle between $\vec{v_1}$ and $\vec{v_2}$. Give an exact answer.

$$\theta = \cos^{-1}\left(\frac{\vec{v_1}\cdot\vec{v_2}}{|\vec{v_1}||\vec{v_2}|}\right) = \boxed{\cos^{-1}\left(\frac{5}{\sqrt{5}\sqrt{14}}\right)} \text{ or } \boxed{\cos^{-1}\left(\frac{\sqrt{5}}{\sqrt{14}}\right)} \text{ or } \boxed{\cos^{-1}\left(\frac{5}{\sqrt{70}}\right)}$$

$|\vec{v_1}| = \sqrt{1+4} = \sqrt{5}$ 1pt.

$|\vec{v_2}| = \sqrt{1+4+9} = \sqrt{14}$ 1pt.

2pts. - formula

1pt. - final answer

[3pts.] d). Find a (simplified) component equation for the plane determined by the points $(0, 0, 0)$, $(1, 0, 2)$ and $(-1, 2, 3)$.

$\vec{v_1} = \langle 1,0,2\rangle$ and $\vec{v_2} = \langle -1,2,3\rangle$ are vectors on the plane

Normal vector: $\vec{v_1} \times \vec{v_2} = \langle -4, -5, 2\rangle$ 1pt.

Point : $(0,0,0)$ 1pt.

Component Equation: (simplified) $\boxed{-4x - 5y + 2z = 0}$ or $\boxed{4x + 5y - 2z = 0}$ 1pt.

2. [20 pts.] Find parametric equations for the line that is tangent to the curve:

$$\vec{r}(t) = \left(\ln\frac{t}{3}\right)\vec{i} + \left(\frac{t-3}{t+6}\right)\vec{j} + \left(t\ln\frac{t}{3}\right)\vec{k},$$

at the point on the curve where $t = 3$.

$$\vec{v}(t) = \left\langle \frac{1}{t/3}\cdot\frac{1}{3},\ \frac{t+6-(t-3)}{(t+6)^2},\ \ln\left(\frac{t}{3}\right) + t\cdot\frac{1}{t/3}\,\frac{1}{3} \right\rangle$$

$$= \boxed{\left\langle \frac{1}{t},\ \frac{9}{(t+6)^2},\ \ln\left(\frac{t}{3}\right)+1 \right\rangle}$$ 9pts. - 3pts./component

$$\vec{v}(3) = \left\langle \frac{1}{3},\ \frac{9}{9^2},\ \ln(1)+1 \right\rangle = \boxed{\left\langle \frac{1}{3},\ \frac{1}{9},\ 1 \right\rangle}$$ 4pts. - 4/3 pts./comp.

$$\vec{r}(3) = \left\langle \ln(1),\ \frac{3-3}{9},\ 3\ln(1) \right\rangle = \boxed{\langle 0,0,0 \rangle}$$ 4pts. - 4/3 pts./comp.

Tangent line at $t=3$:

Parallel vector: $\vec{v}(3) = \langle 1/3, 1/9, 1 \rangle$

Point: $(0,0,0)$

Equations:
$$\boxed{\begin{aligned} x &= \tfrac{1}{3}t \\ y &= \tfrac{1}{9}t \\ z &= t \end{aligned}}$$ 3 pts. - 1pt./eqn.

3. [10 pts.] Express the vector $\overrightarrow{P_1P_2}$ in the form $a\vec{i} + b\vec{j} + c\vec{k}$, where P_1 is the point $(4, -3, 8)$ and P_2 is the point $(-9, -9, 6)$.

$$\overrightarrow{P_1P_2} = \langle -9-4,\ -9+3,\ 6-8 \rangle = \boxed{\langle -13, -6, -2 \rangle}$$

Correct order of subtraction: 1pt.

Each component - 3 pts.

4. [20 pts.] Evaluate the integral:

$$\int_0^1 \left[\left(6te^{3t^2}\right)\vec{i} + \left(6e^{-6t}\right)\vec{j} + 5\pi\vec{k}\right]\, dt.$$

Give exact answers.

$$\int_0^1 6te^{3t^2}dt = \int_0^1 3e^{3u}du = e^{3u}\Big|_0^1 = \boxed{e^3-1} \quad \text{6pts.}$$

$u = t^2$ $\quad$ $t=0 \Rightarrow u=0$

$du = 2t\,dt$ $\quad$ $t=1 \Rightarrow u=1$

$$\int_0^1 6e^{-6t}dt = -e^{-6t}\Big|_0^1 = \boxed{-e^{-6}+1} \quad \text{6 pts.}$$

$$\int_0^1 5\pi\,dt = 5\pi t\Big|_0^1 = \boxed{5\pi} \quad \text{6pts.}$$

Final answer: $\boxed{\langle e^3-1, -e^{-6}+1, 5\pi\rangle}$ or $\boxed{(e^3-1)\vec{i} - (e^{-6}-1)\vec{j} + (5\pi)\vec{k}}$

2pts.

5. [20 pts.] Given the curve:

$$\vec{r}(t) = \left\langle -\sqrt{2}e^t\cos(t),\ -\sqrt{2}e^t\sin(t),\ 2\right\rangle,$$

find:

[8pts.] a). The unit tangent vector $\vec{T}(t)$.

$$\vec{v}(t) = \boxed{\left\langle -\sqrt{2}e^t\cos(t) + \sqrt{2}e^t\sin(t),\ -\sqrt{2}e^t\sin(t) - \sqrt{2}e^t\cos(t),\ 0\right\rangle}$$ 3pts. 1pt./comp.

$$|\vec{v}(t)|^2 = 2e^{2t}\cos^2(t) + 2e^{2t}\sin^2(t) - \cancel{4e^{2t}\sin(t)\cos(t)} +$$
$$+ 2e^{2t}\sin^2(t) + 2e^{2t}\cos^2(t) \cancel{+ 4e^{2t}\sin(t)\cos(t)}$$
$$= 2e^{2t} + 2e^{2t} = 4e^{2t}$$

$$|\vec{v}(t)| = \boxed{2e^t}$$ 3pts.

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \boxed{\left\langle -\frac{1}{\sqrt{2}}\cos(t) + \frac{1}{\sqrt{2}}\sin(t),\ -\frac{1}{\sqrt{2}}\sin(t) - \frac{1}{\sqrt{2}}\cos(t),\ 0\right\rangle}$$

2pts.

1/2pt. formula

1/2pt./comp.

[8pts.] b). The unit normal vector $\vec{N}(t)$.

$$\frac{d\vec{T}}{dt} = \boxed{\left\langle \frac{1}{\sqrt{2}}\sin(t) + \frac{1}{\sqrt{2}}\cos(t),\ -\frac{1}{\sqrt{2}}\cos(t) + \frac{1}{\sqrt{2}}\sin(t),\ 0\right\rangle}$$ 3pts. 1pt./comp.

$$\left|\frac{d\vec{T}}{dt}\right|^2 = \frac{1}{2}\sin^2(t) + \frac{1}{2}\cos^2(t) + \cancel{\sin(t)\cos(t)} +$$
$$+ \frac{1}{2}\cos^2(t) + \frac{1}{2}\sin^2(t) - \cancel{\sin(t)\cos(t)}$$
$$= 1/2 + 1/2 = \boxed{1}$$ 3pts.

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \boxed{\left\langle \frac{1}{\sqrt{2}}(\sin(t) + \cos(t)),\ -\frac{1}{\sqrt{2}}(\cos(t) - \sin(t)),\ 0\right\rangle}$$

2pts.

1pt. formula

1/3pt./comp.

[4pts.] c). The curvature κ.

$$\kappa = \frac{|d\vec{T}/dt|}{|\vec{v}|} = \boxed{\frac{1}{2e^t}}$$

2pts. - formula

2pts. - answer

6. [10 pts.] Consider the curve:

$$\vec{r}(t) = \langle 0,\ \cos^3(t),\ \sin^3(t) \rangle, \quad -\frac{\pi}{2} \le t \le 0.$$

Find the length of the curve on the given parameter domain.

$\vec{v}(t) = \langle 0,\ -3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t) \rangle$ (3pts.) - 1pt. /component

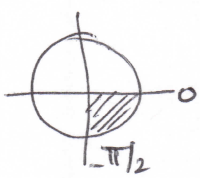
$|\vec{v}(t)|^2 = 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)$ (1pt.)

$= 9\cos^2(t)\sin^2(t)\left(\cos^2(t) + \sin^2(t)\right)$ (1pt.)

$= 9\cos^2(t)\sin^2(t).$ (1pt.)

$|\vec{v}(t)| = \sqrt{9\cos^2(t)\sin^2(t)} = \boxed{-3\cos(t)\sin(t)}$ (2pts.) — (1pt.) "$-$" sign; (1pt.) $-\cos(t)\sin(t)$

because the angle t is in the fourth quadrant where $\sin(t) \le 0$ and $\cos(t) \ge 0$, so



$|3\cos(t)\sin(t)| = -3\cos(t)\sin(t).$

$$L = \int_{-\pi/2}^{0} |\vec{v}(t)|\,dt = \int_{-\pi/2}^{0} -3\cos(t)\sin(t)\,dt =$$

$$= -\frac{3}{2}\sin^2(t)\Big|_{-\pi/2}^{0} = -\frac{3}{2}\sin^2(0) + \frac{3}{2}\sin^2(-\pi/2)$$

$$= \boxed{3/2}$$ (2 pts.) — (1pt.) set up integral; (1pt.) answer

or $$= +\frac{3}{2}\cos^2(t)\Big|_{-\pi/2}^{0} = \frac{3}{2}\cos^2(0) - \frac{3}{2}\cos^2(-\pi/2)$$

$$= \boxed{3/2}$$