Name: Solutions

October 1st, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	20	
2	15	
3	15	
4	15	
5	20	
6	15	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [20 points] Consider the function:

$$h(x,y) = \frac{x^2+y}{y}$$

[5 pts.] a. Find the limit of $h(x,y)$ as $(x,y) \to (0,0)$ along linear paths $y = kx$.

$$h(x,y)\big|_{y=kx} = \frac{x^2+kx}{kx} = \frac{x+k}{k} \quad \text{if } x \neq 0$$

3pts. – correct expression of $h(x,y)\big|_{y=kx}$

2pts. – limit / answer

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} h(x,y) = \boxed{1}$$

[5pts.] b. Can you conclude from part a. that:

$$\lim_{(x,y)\to(0,0)} h(x,y) = 1?$$

Justify your answer briefly.

3pts. – "no"

2pts. – justification

No, because all part a. shows is that the limit is 1 along linear paths. The limit must be the same along all paths along which (x,y) approaches $(0,0)$.

[5pts.] c. Find the limit of $h(x,y)$ as $(x,y) \to (0,0)$ along parabolic paths $y = kx^2$.

$$h(x,y)\big|_{y=kx^2} = \frac{x^2+kx^2}{kx^2} = \frac{1+k}{k}$$

3pts. – correct expression of $h(x,y)\big|_{y=kx^2}$

2pts. – limit / answer

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2}} h(x,y) = \boxed{\frac{1+k}{k}}$$

[5pts.] d. What conclusions can you draw from the results you obtained in part c. about $\lim_{(x,y)\to(0,0)} h(x,y)$?

$\lim_{(x,y)\to(0,0)} h(x,y)$ does not exist, by the Two Path Test. 5pts.

2. [15 points] Given that y is defined implicitly in terms of x by:

$$y = \sin(3x + 4y),$$

find $\frac{dy}{dx}$.

Method I : Partial Derivatives

$F(x,y) = y - \sin(3x+4y) = 0$ 3.5 pts

$$\underbrace{\frac{dy}{dx} = -\frac{F_x}{F_y}}_{3.5\text{ pts.}} = -\frac{-\cos(3x+4y)\cdot 3}{1-\cos(3x+4y)\cdot 4}$$

$$= \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}$$

Correct $F_x \to 4.5$ pts.

Correct $F_y \to 4.5$ pts.

Method II : Implicit Differentiation

3 pts. $\frac{dy}{dx} = \frac{d}{dx}\sin(3x+4y)$

3 pts. $\frac{dy}{dx} = \cos(3x+4y)\left(3+4\frac{dy}{dx}\right)$

3 pts. $\frac{dy}{dx} = 3\cos(3x+4y) + 4\cos(3x+4y)\cdot\frac{dy}{dx}$

3 pts. $\left(1-4\cos(3x+4y)\right)\frac{dy}{dx} = 3\cos(3x+4y)$

3 pts. $\frac{dy}{dx} = \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}$

3. [15 points] Determine whether or not the function $f(x,y) = e^{-2y}\cos(2x)$ satisfies the two-dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

3.5 pts. $\frac{\partial f}{\partial x} = -2e^{-2y}\sin(2x)$

$\frac{\partial f}{\partial y} = -2e^{-2y}\cos(2x)$ 3.5 pts.

3.5 pts. $\frac{\partial^2 f}{\partial x^2} = -4e^{-2y}\cos(2x)$

$\frac{\partial^2 f}{\partial y^2} = 4e^{-2y}\cos(2x)$ 3.5 pts.

$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is true 1 pt.

4. [15 points] Consider the function:

$$f(x,y) = \ln(x^2 + y^4).$$

[6pts.] a. Find the gradient of f.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{2x}{x^2+y^4}, \frac{4y^3}{x^2+y^4} \right\rangle$$

3pts. 3pts.

[9pts.] b. Find the directions $\vec{u}$ where $D_{\vec{u}} f(P_0) = 0$, where $P_0(1,1)$.

$$(\nabla f)_{P_0} = \left\langle \frac{2\cdot 1}{1+1}, \frac{4\cdot 1}{1+1} \right\rangle = \langle 1, 2 \rangle$$

2pts. - evaluating ∇f at P_0

$$D_{\vec{u}} f(P_0) = (\nabla f)_{P_0} \cdot \vec{u}$$

2pts. - using correct formula for $D_{\vec{u}} f$

$$\vec{u} = \langle u_1, u_2 \rangle$$

2pts. - setting up correct system of equations

$$(\nabla f)_{P_0} \cdot \vec{u} = u_1 + 2u_2$$

2pts. - Solving the system of equations

$$\begin{cases} u_1 + 2u_2 = 0 \\ u_1^2 + u_2^2 = 1 \end{cases} \qquad \begin{cases} u_1 = -2u_2 \\ u_1^2 + u_2^2 = 1 \end{cases} \qquad (-2u_2)^2 + u_2^2 = 1$$

$$4u_2^2 + u_2^2 = 1$$

$$5u_2^2 = 1$$

$$u_2^2 = \frac{1}{5}$$

$$u_2 = \pm \frac{1}{\sqrt{5}}$$

$$u_2 = \frac{1}{\sqrt{5}} \Rightarrow u_1 = -\frac{2}{\sqrt{5}}$$

$$u_2 = \frac{-1}{\sqrt{5}} \Rightarrow u_1 = \frac{2}{\sqrt{5}}$$

Directions: $\boxed{\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)}$

1pt. - final answer

5. [20 points] Consider the function:

$$f(x, y) = x^3 + y^3 + 6x^2 - 3y^2 - 5.$$

[10 pts.] a. Find the critical points of f.

$f_x = 3x^2 + 12x$ 2 pts.

$f_y = 3y^2 - 6y$ 2 pts.

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ 2 pts. setting up system of equations $\begin{cases} 3x^2 + 12x = 0 \\ 3y^2 - 6y = 0 \end{cases}$ $\begin{cases} 3x(x+4) = 0 \\ 3y(y-2) = 0 \end{cases}$ $\begin{cases} x = 0, -4 \\ y = 0, 2 \end{cases}$ 2 pts. Solving system

Critical Points: $(0,0)$, $(0,2)$, $(-4,0)$, $(-4,2)$

½ pt. ½ pt. ½ pt. ½ pt.

[10 pts.] b. Use the Second Derivative Test to classify each critical point as a saddle point, a local minimum, or a local maximum.

$f_{xx} = 6x + 12$ 1 pt.

$f_{yy} = 6y - 6$ 1 pt.

$f_{xy} = 0$ 1 pt.

$f_{xx} f_{yy} - f_{xy}^2 = (6x+12)(6y-6)$ 1 pt.

$\boxed{(0,0)} \to \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(0,0)} = 12(-6) < 0$ saddle point 1 pt.

$\boxed{(0,2)} \to \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(0,2)} = 12(12-6) > 0$ 1 pt.

$f_{xx}\Big|_{(0,2)} = 0 + 12 > 0$ local min 1 pt.

$\boxed{(-4,0)} \to \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(-4,0)} = (-24+12)(-6) > 0$ 1 pt.

$f_{xx}\Big|_{(-4,0)} = -24 + 12 < 0$ local max 1 pt.

$\boxed{(-4,2)} \to \left(f_{xx} f_{yy} - f_{xy}^2\right)\Big|_{(-4,2)} = (-24+12)(12-6) < 0$ saddle point 1 pt.

6. [15 points] Find the point on the sphere $x^2 + y^2 + z^2 = 4$ that is farthest from the point $(-1, -1, -1)$.

Maximize : $f(x,y,z) = (x+1)^2 + (y+1)^2 + (z+1)^2$ 1pt.

Subject to : $g(x,y,z) = x^2+y^2+z^2-4=0$ 1pt.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z)=0 \end{cases}$$ 1pt.

$\nabla f = \langle 2(x+1), 2(y+1), 2(z+1) \rangle$ 2pts.

$\nabla g = \langle 2x, 2y, 2z \rangle$ 2pts.

1pt. Setting up system

$$\begin{cases} 2(x+1) = 2\lambda x \\ 2(y+1) = 2\lambda y \\ 2(z+1) = 2\lambda z \\ x^2+y^2+z^2 = 4 \end{cases} \quad \begin{cases} x+1 = \lambda x \\ y+1 = \lambda y \\ z+1 = \lambda z \\ x^2+y^2+z^2=4 \end{cases} \quad \begin{cases} (1-\lambda)x = -1 \\ (1-\lambda)y = -1 \\ (1-\lambda)z = -1 \\ x^2+y^2+z^2=4 \end{cases}$$

$x = y = z = -\frac{1}{1-\lambda}$ 3pts. – solving system

$x^2+y^2+z^2=4$ becomes $x^2+x^2+x^2=4$

$$3x^2 = 4$$
$$x^2 = \frac{4}{3}$$
$$x = \pm \frac{2}{\sqrt{3}}$$

Possible Solutions: $\underbrace{\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}_{\text{maximum}}$, $\underbrace{\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)}_{\text{minimum}}$

maximum 1pt. minimum 1pt.

Answer : $\boxed{\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}$ 2pts. – choosing correct solution