Name: Solutions

November 19^{th}, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
Exam 4

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	25	
2	25	
3	20	
4	15	
5	15	
Total	100	

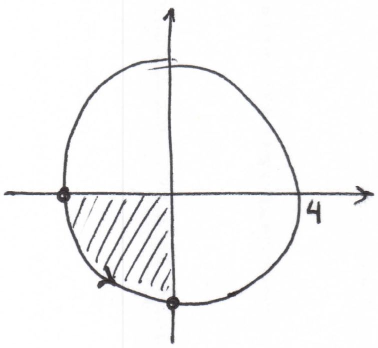
Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [25 points] Find:

$$\int_C f\,ds,$$

where $f(x, y) = x + y$, and C, positively oriented, is the quarter of the circle $x^2 + y^2 = 16$ lying in the third quadrant.



(4pts.) (3pts.)

$C: \vec{r}(t) = \langle 4\cos t, 4\sin t \rangle, \quad \pi \leq t \leq \frac{3\pi}{2}$

$x = 4\cos t$

$y = 4\sin t$

$f(\vec{r}(t)) = 4\cos t + 4\sin t$ (3pts.)

$\vec{v}(t) = \langle -4\sin t, 4\cos t \rangle$ (3pts.)

$|\vec{v}(t)| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$ (3pts.)

$\int_C f\,ds = \int_\pi^{\frac{3\pi}{2}} f(\vec{r}(t))\,|\vec{v}(t)|\,dt$ (2pts.)

$= \int_\pi^{\frac{3\pi}{2}} (4\cos t + 4\sin t) \cdot 4\,dt$ (2pts.)

$= 16(\sin t - \cos t)\Big|_\pi^{3\pi/2}$ (3pts.)

$= 16[(-1-0) - (0-(-1))]$

$= \boxed{-32}$ (2pts.)

2. [25 points] Recall Green's formulas:

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \int\!\!\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA},$$

$$\boxed{\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \int\!\!\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA}.$$

Let the field:

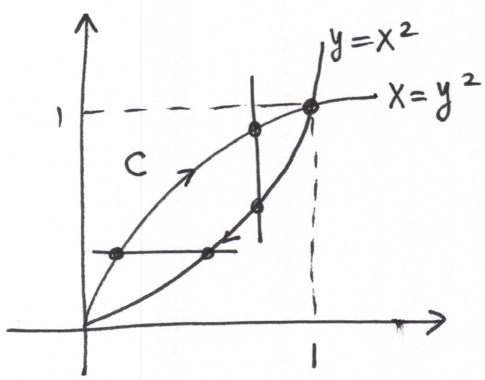
$$\vec{F} = (5xy + y^2)\vec{i} + (5x - y)\vec{j},$$

and let C be the boundary of the region in the first quadrant enclosed by the curves $x = y^2$ and $y = x^2$. Use Green's Theorem to fill in the blanks below, expressing the flux and circulation of $\vec{F}$ over C as double integrals over the region enclosed by C in the plane.

Flux of $\vec{F}$ over $C = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{x^2}}^{\boxed{\sqrt{x}}} \boxed{5y-1}\, dy\, dx.$ (12 pts.) (8pts.) bounds (2 pts. each) (4pts.) $5y-1$

Circulation of $\vec{F}$ over $C = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{y^2}}^{\boxed{\sqrt{y}}} \boxed{5-5x-2y}\, dx\, dy.$ (12pts.) (8pts.) bounds (2 pts. each) (4pts.) $5-5x-2y$

Be careful at the order of integration!
Do not compute the values of the integrals.



$M = 5xy + y^2$
$N = 5x - y$ (1pt.)

$$\text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$
$$= \iint_R (5y-1)\,dA$$
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (5y-1)\,dy\,dx$$

$$\text{Circulation} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \iint_R \big(5-(5x+2y)\big)\,dA$$
$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (5-5x-2y)\,dx\,dy$$

3. [20 points] The following field:

$$\vec{F}(x,y,z) = \left\langle 2xye^{x^2y},\ \ x^2e^{x^2y} + \ln(z),\ \ \frac{y}{z} \right\rangle$$

is conservative.

(10 pts.) a). Find a potential function f for this field.

$$\frac{\partial f}{\partial x} = 2xye^{x^2y} \Rightarrow f = e^{x^2y} + g(\cancel{y,z}) \quad \text{(3pts.)}$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial y} &= x^2e^{x^2y} + \frac{\partial g}{\partial y} \\ &= x^2e^{x^2y} + \ln(z) \end{aligned}\right\} \Rightarrow \frac{\partial g}{\partial y} = \ln(z) \Rightarrow g = y\ln(z) + h(z)$$

(3pts.)

$$f = e^{x^2y} + y\ln(z) + h(z)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial z} &= \frac{y}{z} + h'(z) \\ &= \frac{y}{z} \end{aligned}\right\} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \quad \text{(2pts.)}$$

$$\boxed{f(x,y,z) = e^{x^2y} + y\ln(z) + C} \quad \text{(2pts.)}$$

(5pts.) b). Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is a smooth path from $(1,0,3)$ to $(0,1,e^3)$.

$$\int_C \vec{F} \cdot d\vec{r} = f(0,1,e^3) - f(1,0,3) \quad \text{(3pts.)}$$

$$= (1+3) - (1+0)$$

$$= \boxed{3} \quad \text{(2pts.)}$$

(5pts.) c). Compute $\oint_C \vec{F} \cdot d\vec{r}$, where C is a simple closed curve (loop).

$\oint_C \vec{F} \cdot d\vec{r} = \boxed{0}$ because $\vec{F}$ is conservative. (5pts.)

4. [15 points] Find the area of the region enclosed by the sinusoidal curve:

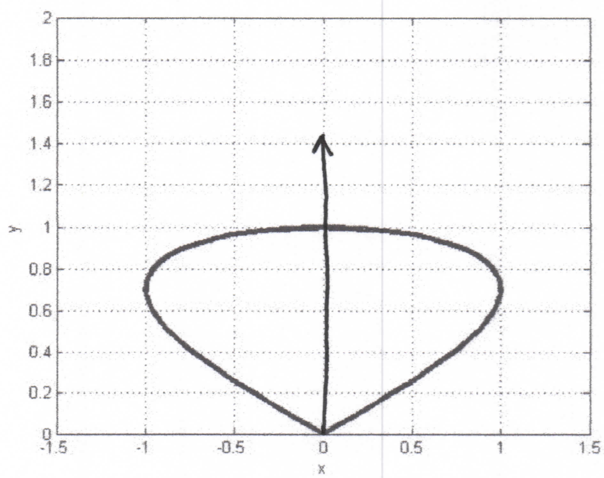
$$\vec{r}(t) = \sin(2t)\vec{i} + \sin(t)\vec{j},\ 0 \le t \le \pi$$

in the xy-plane, pictured below. You may use Green's area formula:

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx,$$

or any other method you like. You may need to recall that:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \text{ and } \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).$$



Method 1: Green's Theorem

$x = \sin(2t)$; (1pt.)

$y = \sin(t)$; (1pt.)

$dx = 2\cos(2t)$; (2pts.)

$dy = \cos(t)$ (2pts.)

$$A = \frac{1}{2}\int_0^{\pi}\left[\sin(2t)\cos(t) - \sin(t)\cdot 2\cos(2t)\right]dt$$ (2pts.)

$$= \frac{1}{2}\int_0^{\pi}\left[2\sin(t)\cos^2(t) - 2\sin(t)\left(2\cos^2 t - 1\right)\right]dt$$

$$= \int_0^{\pi}\left[\sin(t)\cos^2(t) - 2\sin(t)\cos^2(t) + \sin(t)\right]dt$$

$$= \int_0^{\pi}\left[-\sin(t)\cos^2(t) + \sin(t)\right]dt$$

$$= \left(\frac{\cos^3(t)}{3} - \cos(t)\right)\Bigg|_0^{\pi}$$

$$= \left(\frac{-1}{3} - (-1)\right) - \left(\frac{1}{3} - 1\right) = \frac{2}{3} + \frac{2}{3}$$

$$= \boxed{4/3}$$ (1pt.)

(6pts.) computation

Method 2: $\text{Area} = \iint_R dA$

$x = \sin(2t)$

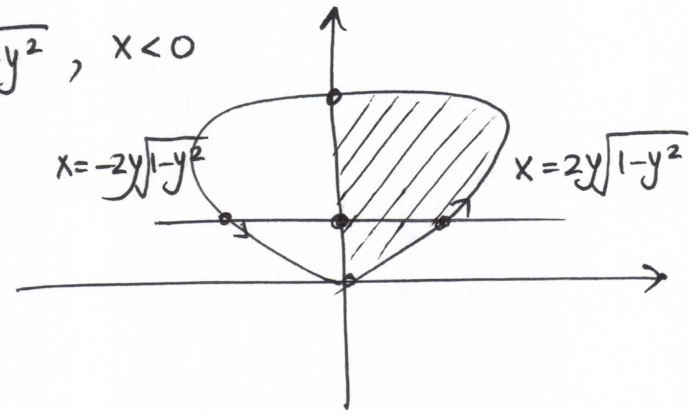
$y = \sin(t), \quad 0 \le t \le \pi \quad \Rightarrow \boxed{y \geq 0}$

$$x = 2\sin(t)\cos(t) \quad \Rightarrow \quad x = \begin{cases} 2y\sqrt{1-y^2}, & x \geq 0 \\ -2y\sqrt{1-y^2}, & x < 0 \end{cases}$$

$= 2y\cos(t)$

$= 2y\left(\pm\sqrt{1-\sin^2(t)}\right)$

(9 pts.) $= 2y\left(\pm\sqrt{1-y^2}\right)$



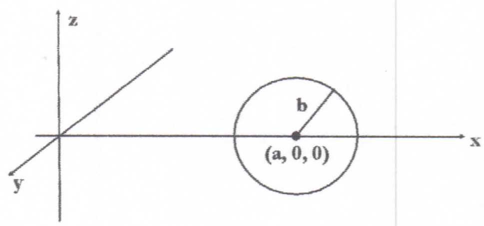
(2 pts.) $A = 2\int_0^1 \int_0^{2y\sqrt{1-y^2}} dx\, dy$

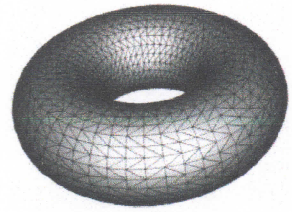
$= 2\int_0^1 2y\sqrt{1-y^2}\, dy$

$= 2 \cdot \frac{-2}{3}(1-y^2)^{3/2}\Big|_0^1 = -\frac{4}{3}(0-1)$

Computation (3 pts.)

$= \boxed{4/3}$ (1 pt.)





5. [15 points] Let $0 < b < a$. A *torus* is the "doughnut" surface obtained by revolving the circle in the xz-plane centered at $(a, 0, 0)$, with radius b, around the z axis (pictured above). Taking $a = 2$ and $b = 1$, a parametrization of this surface is:

$$\vec{r}(\phi, \theta) = \langle (2 + \cos\phi)\cos\theta, \quad (2 + \cos\phi)\sin\theta, \quad \sin\phi \rangle,$$

$$0 \le \phi \le 2\pi;\ 0 \le \theta \le 2\pi.$$

Use this parametrization, along with the surface area differential:

$$d\sigma = |\vec{r}_\phi \times \vec{r}_\theta|\, d\phi\, d\theta,$$

to find the surface area of the torus.

$$\vec{r}_\phi = \langle -\sin\phi\cos\theta,\ -\sin\phi\sin\theta,\ \cos\phi \rangle \qquad (3\text{pts.})$$

$$\vec{r}_\theta = \langle -(2+\cos\phi)\sin\theta,\ (2+\cos\phi)\cos\theta,\ 0 \rangle \qquad (3\text{pts.})$$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle -(2+\cos\phi)\cos\theta\cos\phi,\ -(2+\cos\phi)\sin\theta\cos\phi,\ -(2+\cos\phi)\sin\phi\cos^2\theta - (2+\cos\phi)\sin\phi\sin^2\theta \rangle$$

$$= -(2+\cos\phi)\langle \cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi \rangle \qquad (3\text{pts.})$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = (2+\cos\phi)\sqrt{\cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi}$$

$$= (2+\cos\phi)\sqrt{\cos^2\phi + \sin^2\phi}$$

$$= (2+\cos\phi) \qquad (3\text{pts.})$$

$$\text{Area} = \int_0^{2\pi}\int_0^{2\pi} (2+\cos\phi)\, d\phi\, d\theta$$

$$= 2\pi\left(2\phi + \sin\phi\right)\Big|_0^{2\pi} \qquad (3\text{pts.})$$

$$= 2\pi\,(2 \cdot 2\pi)$$

$$= \boxed{8\pi^2}$$