Name: Solutions

December 9th, 2014.
Math 2401; Sections D1, D2, D3.
Georgia Institute of Technology
FINAL EXAM

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

1. [10 points] Consider the lines:

$$L_1 : x = 1 + t; \;\; y = 2 - t; \;\; z = 3t$$

$$L_2 : x = 2 - s; \;\; y = 1 + 2s; \;\; z = 3 + s.$$

(a). Find the point of intersection of these lines.
(b). Find an equation for the plane determined by these lines.

(a). $\begin{cases} 1+t = 2-s \\ 2-t = 1+2s \\ 3t = 3+s \end{cases}$ $\begin{cases} t = 1-s \\ 2-(1-s) = 1+2s \\ 3t = 3+s \end{cases}$ $\Rightarrow 1+s = 1+2s \Rightarrow \boxed{s=0}\ \boxed{t=1}$

(2 pts.) $\quad 3\cdot 1 = 3+0 \quad$ (2 pts.)

Point of intersection: $\boxed{(2,1,3)}$ (1pt.)

(b). Vector parallel to L_1: $\vec{v_1} = \langle 1, -1, 3 \rangle$ (½pt.)

Vector parallel to L_2: $\vec{v_2} = \langle -1, 2, 1 \rangle$. (½ pt.)

Vector normal to the plane: $\vec{n} = \vec{v_1} \times \vec{v_2}$ (½ pt.)

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ -1 & 2 & 1 \end{vmatrix} = \langle -7, -4, 1 \rangle$$ (2pts.)

The plane passes through $(2,1,3)$. (½ pt.)

Plane equation: $-7(x-2) - 4(y-1) + 1(z-3) = 0$ (1 pt.)

$$7(x-2) + 4(y-1) - z + 3 = 0$$

$$7x - 14 + 4y - 4 - z + 3 = 0$$

$$\boxed{7x + 4y - z = 15}$$

2. [10 points] Consider the curve:

$$\vec{r}(t) = (2\sin t)\vec{i} + (t^4 - 4\cos t)\vec{j} + (e^{2t})\vec{k}.$$

(a). Find the velocity vector $\vec{v}(t)$ for this curve.
(b). Find the acceleration vector $\vec{a}(t)$ for this curve.
(c). Find the angle θ between $\vec{v}(0)$ and $\vec{a}(0)$.

(a). $\vec{v}(t) = \langle \underset{(1\text{pt.})}{2\cos t},\ \underset{(1\text{pt.})}{4t^3 + 4\sin t},\ \underset{(1\text{pt.})}{2e^{2t}} \rangle$

(b). $\vec{a}(t) = \langle \underset{(1\text{pt.})}{-2\sin t},\ \underset{(1\text{pt.})}{12t^2 + 4\cos t},\ \underset{(1\text{pt.})}{4e^{2t}} \rangle$

(c). $\vec{v}(0) = \langle 2, 0, 2 \rangle$ (1pt.)

$\vec{a}(0) = \langle 0, 4, 4 \rangle$ (1 pt.)

$\vec{v}(0) \cdot \vec{a}(0) = 8$ (1/3 pt.)

$|\vec{v}(0)| = \sqrt{4+4} = 2\sqrt{2}$ (1/3 pt.)

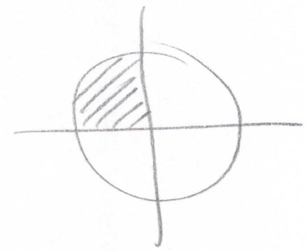
$|\vec{a}(0)| = \sqrt{16+16} = 4\sqrt{2}$ (1/3 pt.)

$$\cos\theta = \underset{(1/3\text{pt.})}{\frac{\vec{v}(0)\cdot\vec{a}(0)}{|\vec{v}(0)|\,|\vec{a}(0)|}} = \frac{8}{2\sqrt{2}\cdot 4\sqrt{2}} = \frac{1}{2} \quad (1/3\text{ pt.})$$

$\boxed{\theta = \pi/3}$ (1/3 pt.)

3. [10 points] Find the length of the curve:

$$\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad \frac{\pi}{2} \le t \le \pi.$$



$\vec{v}(t) = \langle -3\cos^2 t \sin t, \; 3\sin^2 t \cos t \rangle$ (2 pts.)

$|\vec{v}(t)| = \sqrt{9\cos^4 t \sin^2 t + 9\cos^2 t \sin^4 t}$ (1 pt.)

$= \sqrt{9\cos^2 t \sin^2 t \,(\sin^2 t + \cos^2 t)}$ (1 pt.)

$= 3\,|\sin t \cos t|$

$= \underbrace{-}_{2\text{pts.}}\underbrace{3\sin t \cos t}_{1\text{pt.}}$ because $\frac{\pi}{2} \le t \le \pi$, so t is in Quadrant II where $\sin t \ge 0$ and $\cos t \le 0$.

$L = \int_{\pi/2}^{\pi} |\vec{v}(t)|\, dt$ (1 pt.)

$= \int_{\pi/2}^{\pi} -3\sin t \cos t \, dt$

$= \left. \frac{3\cos^2 t}{2} \right|_{\pi/2}^{\pi}$ (1 pt.)

$= \boxed{\frac{3}{2}}$ (1 pt.)

4. [10 points] Find the direction $\vec{u}$ such that the directional derivative:

$$D_{\vec{u}}f(1,1) = -1,$$

where $f(x,y) = x^2y - 2y + xy^2$.

$\nabla f(x,y) = \langle 2xy + y^2,\ x^2 - 2 + 2xy \rangle$ (2 pts.)

$\nabla f(1,1) = \langle 2+1,\ 1-2+2 \rangle = \langle 3, 1 \rangle$ (1 pt.)

$D_{\vec{u}}f(1,1) = \nabla f(1,1) \cdot \vec{u}$ (1 pt.)

$\vec{u} = \langle u_1, u_2 \rangle$

$D_{\vec{u}}f(1,1) = \langle 3, 1 \rangle \cdot \langle u_1, u_2 \rangle = 3u_1 + u_2$

(1pt.) $\begin{cases} 3u_1 + u_2 = -1 \\ u_1^2 + u_2^2 = 1 \end{cases}$ (1pt.) $\qquad \begin{cases} u_2 = -1 - 3u_1 \\ u_1^2 + (1+3u_1)^2 = 1 \end{cases}$

$u_1^2 + 1 + 6u_1 + 9u_1^2 = 1$

$10u_1^2 + 6u_1 = 0$

$2u_1(5u_1+3) = 0 \Rightarrow u_1 = \boxed{0, -3/5}$ (2 pts.)

$u_1 = 0 \Rightarrow u_2 = -1 \quad \boxed{\vec{u} = \langle 0, -1 \rangle}$ (1pt.)

$u_1 = -3/5 \Rightarrow u_2 = -1 + \frac{9}{5} = \frac{4}{5} \quad \boxed{\vec{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle}$ (1pt.)

5. [10 points] Find the points where extrema occur for the function:

$$f(x, y) = x^3 y^5,$$

subject to the constraint:

$$x + y = 8.$$

$g(x,y) = x + y$

$\nabla g = \langle 1, 1 \rangle$ (1 pt.)

$\nabla f = \langle 3x^2y^5, 5x^3y^4 \rangle$ (2pts.)

(1 pt.) $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 8 \end{cases}$ $\begin{cases} 3x^2y^5 = \lambda \\ 5x^3y^4 = \lambda \\ x + y = 8 \end{cases}$ $\Rightarrow 3x^2y^5 = 5x^3y^4$

$3x^2y^5 - 5x^3y^4 = 0$

$x^2y^4(3y - 5x) = 0$

(2pts.) $\boxed{x = 0} \Rightarrow \boxed{y = 8}$

(2pts.) $\boxed{y = 0} \Rightarrow \boxed{x = 8}$

$3y - 5x = 0 \Rightarrow y = \frac{5}{3}x \Rightarrow \frac{5}{3}x + x = 8$

$\Rightarrow \frac{8x}{3} = 8$

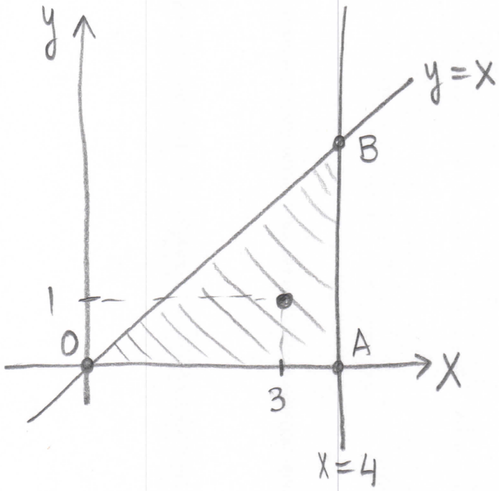
(2pts.) $\Rightarrow \boxed{x = 3} \Rightarrow \boxed{y = 5}$

Points of Extrema: $\boxed{(0,8),\ (8,0),\ (3,5)}$

6. [10 points] Find the minimum and maximum of the function:

$$f(x, y) = (x - 3)^2 + (y - 1)^2$$

on the triangular plate in the first Quadrant determined by the lines $y = 0$, $x = 4$, and $y = x$.



$f_x = 2(x-3)$ (½ pt.)

$f_y = 2(y-1)$ (½ pt.)

Critical Points: $\begin{cases} 2(x-3)=0 \\ 2(y-1)=0 \end{cases}$ $\begin{cases} x=3 \\ y=1 \end{cases}$

$\boxed{(3,1)}$ inside region (1pt.)

Boundaries:

$\boxed{OA}$: $y=0$, $0 \le x \le 4$

$f(x,0) = (x-3)^2+1$, $0 \le x \le 4$

$f'(x,0) = 2(x-3)$

$2(x-3)=0 \Rightarrow x=3 \Rightarrow (3,0)$

$f(0,0)$ and $f(4,0)$

$\boxed{AB}$: $x=4$, $0 \le y \le 4$

$f(4,y) = 1+(y-1)^2$

$f'(4,y) = 2(y-1)$

$2(y-1)=0 \Rightarrow y=1 \Rightarrow f(4,1)$

$f(4,4)$

$\boxed{OB}$: $x=y$, $0 \le x \le 4$

$f(x,x) = (x-3)^2+(x-1)^2$

$f'(x,x) = 2(x-3)+2(x-1)$

$2(x-3)+2(x-1)=0$

$x-3+x-1=0$

$2x-4=0$

$x=2 \Rightarrow f(2,2)$

(1 pt.) $f(3,1) = \boxed{0}$ ← min (½ pt.)

(1 pt.) $f(3,0) = 1$

(1 pt.) $f(0,0) = \boxed{10}$ ← max (½ pt.)

(1 pt.) $f(4,0) = 2$

(1 pt.) $f(4,1) = 1$

(1 pt.) $f(4,4) = 10$

(1 pt.) $f(2,2) = 2$

7. [10 points] Reverse the order of integration:

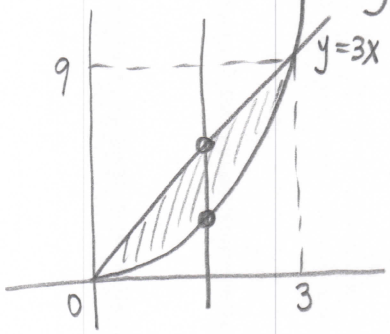
$$\int_0^9 \int_{y/3}^{\sqrt{y}} dx\, dy.$$

Do not evaluate the integrals.

$$\frac{y}{3} \le x \le \sqrt{y}$$

$x = \sqrt{y}\ ;\ x^2 = y$

$y = 3x\ ;$



(2pts.)

$$\begin{cases} y = 3x \\ y = x^2 \end{cases}$$

$3x = x^2$

$x = 0, 3$

$y = 0, 9$

$$\int_0^9 \int_{y/3}^{\sqrt{y}} dx\,dy = \boxed{\int_0^3 \int_{x^2}^{3x} dy\,dx}$$

(4 pts.) (4pts.)

8. [10 points] Evaluate:

$$\int_1^{e^{10}} \int_1^{e^4} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz.$$

$$\int_1^{e^{10}} \int_1^{e^4} \frac{1}{yz} \ln(x) \Big|_{x=1}^{e^3} dy\, dz \qquad \text{(2 pts.)}$$

$$= \int_1^{e^{10}} \int_1^{e^4} \frac{3}{yz}\, dy\, dz \qquad \text{(2 pts.)}$$

$$= \int_1^{e^{10}} \frac{3}{z} \ln(y) \Big|_{y=1}^{e^4} dz \qquad \text{(2 pts.)}$$

$$= \int_1^{e^{10}} \frac{12}{z}\, dz \qquad \text{(2 pts.)}$$

$$= 12 \ln(z) \Big|_{z=1}^{e^{10}} \qquad \text{(1 pt.)}$$

$$= \boxed{120} \qquad \text{(1 pt.)}$$

9. [10 points] The following field is conservative:

$$\vec{F}(x,y,z) = (2xy + \cos(x))\,\vec{i} + (x^2)\,\vec{j} + \left(-\sin(z)e^{\cos(z)}\right)\vec{k}.$$

(a). Find a potential function f for this field.
(b). Evaluate:

$$\int_C \vec{F}\cdot d\vec{r},$$

where C is a smooth curve from $(\pi, 0, 0)$ to $(\pi, 0, \pi)$.

(a). $\frac{\partial f}{\partial y} = x^2 \Rightarrow f = x^2 y + g(x,z)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial x} &= 2xy + \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial x} &= 2xy + \cos(x) \end{aligned}\right\} \Rightarrow \frac{\partial g}{\partial x} = \cos(x) \Rightarrow \boxed{g = \sin(x) + h(z)}$$

(3pts.)

$$\Rightarrow f = x^2 y + \sin(x) + h(z)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial f}{\partial z} &= h'(z) \\ \frac{\partial f}{\partial z} &= -\sin(z)e^{\cos(z)} \end{aligned}\right\} \Rightarrow h'(z) = -\sin(z)e^{\cos(z)}$$

$$\Rightarrow \boxed{h(z) = e^{\cos(z)} + C}$$

(3 pts.)

$$\boxed{f(x,y,z) = x^2 y + \sin(x) + e^{\cos(z)} + C}$$ (1pt.)

(b). $\int_C \vec{F}\cdot d\vec{r} = f(\pi,0,\pi) - f(\pi,0,0)$ (2pts.)

$= \boxed{e^{-1} - e}$ (1pt.)

10. [10 points] Find the outward flux of the field:

$$\vec{F}(x,y,z) = \sqrt{x^2+y^2+z^2}(x\vec{i}+y\vec{j}+z\vec{k})$$

over the boundary of the region D in space, given by:

$$D: 4 \leq x^2+y^2+z^2 \leq 5.$$

$$\nabla\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 3\sqrt{x^2+y^2+z^2} + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= 4\sqrt{x^2+y^2+z^2} \quad (5\text{ pts.})$$

$$\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\cdot x\right) = \sqrt{x^2+y^2+z^2} + x\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}$$

$$= \sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}}$$

Divergence Theorem:

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV \quad (1\text{ pt.})$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_2^{\sqrt{5}} 4\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \quad (1\text{ pt}) \ (1\text{ pt.})$$

(1pt.)

$$= 2\pi\left(-\cos\phi\right)\Big|_0^{\pi}\left(\rho^4\right)\Big|_2^{\sqrt{5}}$$

$$= 2\pi\left(-(-1)+1\right)\left(25-16\right)$$

$$= 4\pi\cdot 9 = \boxed{36\pi} \quad (1\text{ pt.})$$

11. [10 points] Find the circulation of the field:

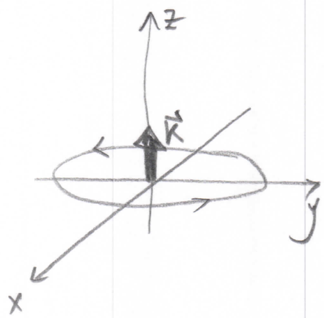
$$\vec{F}(x,y,z) = x^2\vec{i} + 3x\vec{j} + z^2\vec{k}$$

around the ellipse:

$$C : 16x^2 + y^2 = 3$$

in the xy-plane, oriented counterclockwise when viewed from above (from the positive side of the z-axis). You may use the fact that the area of an ellipse with axes a and b is πab.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3x & z^2 \end{vmatrix} = \langle 0, 0, 3 \rangle = 3\vec{k} \quad \text{(3pts.)}$$



Stokes' Theorem: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S 3\vec{k}\cdot\vec{n}\,d\sigma$ (2pts.)

S = surface in xy-plane bounded by the ellipse.

$\vec{n} = \vec{k}$ (1pt.)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S 3\vec{k}\cdot\vec{k}\,d\sigma = \iint_S 3\,d\sigma \quad \text{(1pt.)}$$

$$= 3\cdot \text{Area}(S) \quad \text{(1pt.)}$$

$$= 3\pi ab$$

$$= 3\pi \frac{\sqrt{3}}{4}\sqrt{3} = \boxed{\frac{9\pi}{4}} \quad \text{(1pt.)}$$

$$C: 16x^2 + y^2 = 3$$

$$\frac{16}{3}x^2 + \frac{1}{3}y^2 = 1$$

$$\left(\frac{x}{\sqrt{3}/4}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1 \quad \text{(1pt.)}$$

$$a = \frac{\sqrt{3}}{4}$$

$$b = \sqrt{3}$$

12. [10 points] Find:

$$\oint_C x^2y^3\,dx + (x^3y^2 + x)\,dy,$$

where C is the counterclockwise oriented boundary of a square of side length 3 in the xy-plane.

$M = x^2y^3$ (1 pt.)

$N = x^3y^2 + x$ (1 pt.)

Green's Theorem: $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ (3 pts.)

$= \iint_R \left((3x^2y^2+1) - (3x^2y^2)\right) dA$ (3 pts.)

$= \iint_R dA$ (1 pt.)

$= \text{Area}(R) = \boxed{9}$ (1 pt.)

13. [10 points] Find the limit, if it exists:

$$\lim_{(x,y)\to(1,1)} \frac{x^2y - xy^2}{\sqrt{x} - \sqrt{y}}.$$

$$\lim_{(x,y)\to(1,1)} \frac{x^2y - xy^2}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(1,1)} \frac{xy(x-y)}{\sqrt{x}-\sqrt{y}}$$ (3 pts.)

$$= \lim_{(x,y)\to(1,1)} \frac{xy(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}}$$ (3 pts.)

$$= \lim_{(x,y)\to(1,1)} xy(\sqrt{x}+\sqrt{y})$$ (3 pts.)

$$= \boxed{2}$$ (1 pt.)

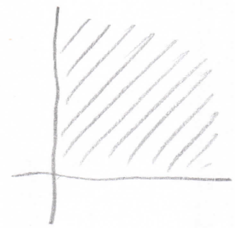
14. [10 points] Find:

$$\int_0^\infty e^{-3x^2}\, dx.$$

$$I = \int_0^\infty e^{-3x^2} dx$$

$$I^2 = \left(\int_0^\infty e^{-3x^2} dx\right)\left(\int_0^\infty e^{-3y^2} dy\right) \qquad \text{(2pts.)}$$

$$= \int_0^\infty \int_0^\infty e^{-3(x^2+y^2)} dx\, dy \qquad \text{(2 pts.)}$$



$$= \int_0^{\pi/2} \int_0^\infty e^{-3r^2} r\, dr\, d\theta \qquad \text{(3 pts.)}$$

$$= \int_0^{\pi/2} \left(-\frac{1}{6} e^{-3r^2}\right)\Bigg|_{r=0}^{\infty} d\theta \qquad \text{(1 pt.)}$$

$$= \frac{\pi}{2}\left(0 + \frac{1}{6}\right) = \frac{\pi}{12} \qquad \text{(1 pt.)}$$

$$I^2 = \frac{\pi}{12} \Rightarrow \boxed{I = \sqrt{\frac{\pi}{12}}} \qquad \text{(1pt.)}$$