① $\vec{F} = \langle y, xz, x^2 \rangle$

C: boundary of triangle cut from $4x+y+z=4$ by the first octant
(counterclockwise when viewed from above)

$\oint_C \vec{F} \cdot d\vec{r} = ?$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & x^2 \end{vmatrix} = \langle -x, -2x, z-1 \rangle$$

$y = 4-4x$

$z = 4-y$

$z = 4-4x$

Stokes: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, d\sigma$

S = triangle bounded by C

S = implicit surface $f(x,y,z) = 4$,

$\quad f = 4x+y+z$.

So $\nabla f = \langle 4,1,1 \rangle$ and $|\nabla f| = \sqrt{18}$.

Outer normal to S: $\vec{n} = \dfrac{\pm \nabla f}{|\nabla f|} = \pm \dfrac{1}{\sqrt{18}} \langle 4,1,1 \rangle$

$$\boxed{\vec{n} = \frac{1}{\sqrt{18}} \langle 4,1,1 \rangle.}$$

$y = 4-4x$

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \dfrac{1}{\sqrt{18}} (-4x - 2x + z - 1) \, d\sigma$

$\quad = \iint_R \dfrac{1}{\sqrt{18}} (-6x + z - 1) \dfrac{|\nabla f|}{|\nabla f \cdot \vec{k}|} \, dA = \iint_R \dfrac{1}{\sqrt{18}} (-6x + z - 1) \dfrac{\sqrt{18}}{1} \, dA$
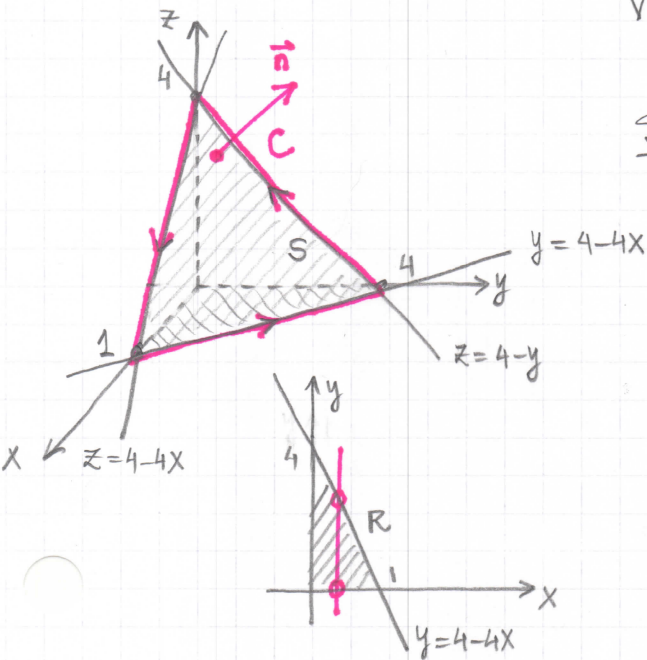
$\quad = \iint_{\underset{\text{xy-plane}}{R}} (-6x + z - 1) \, dA = \iint_R \left(-6x + \underbrace{(4 - 4x - y)}_{\text{from } f=4} - 1 \right) dA$

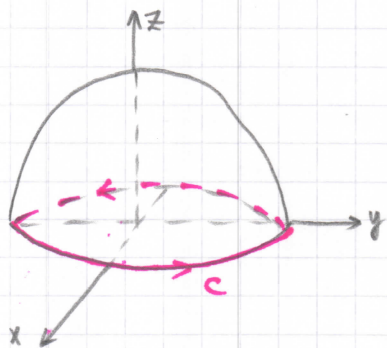$\quad = \iint_R (-10x - y + 3) \, dA = \int_0^1 \int_0^{4-4x} (-10x - y + 3) \, dy \, dx$

$\quad = \int_0^1 \left(-10xy - \dfrac{y^2}{2} + 3y \right) \Big|_{y=0}^{y=4-4x} dx = \int_0^1 \left(-10x(4-4x) - \dfrac{(4-4x)^2}{2} + 3(4-4x) \right) dx$

$\quad = 4\int_0^1 (-10x + 10x^2 - 2 + 4x - 2x^2 + 3 - 3x) \, dx = 4\int_0^1 (8x^2 - 9x + 1) \, dx$

$\quad = 4\left(\dfrac{8x^3}{3} - \dfrac{9x^2}{2} + x \right) \Big|_0^1 = 4\left(\dfrac{8}{3} - \dfrac{9}{2} + 1 \right) = 4 \cdot \dfrac{-5}{6} = \boxed{-\dfrac{10}{3}}$

② $\iint_S \nabla \times (3y\vec{u}) \cdot \vec{n} \, d\sigma$, $S: x^2 + y^2 + z^2 = 1$, $z \geq 0$.

$$\vec{F} = \langle 3y, 0, 0 \rangle$$

$$\iint_S \nabla \times (3y\vec{u}) \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

$$C: \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle, \quad 0 \leq t \leq 2\pi$$
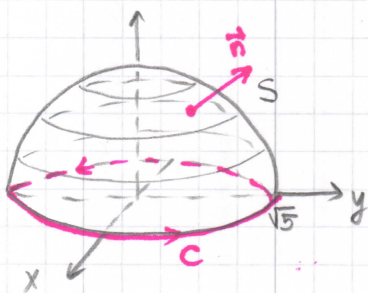
$$d\vec{r}(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 3\sin t, 0, 0 \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -3\sin^2 t \, dt = -\frac{3}{2}\left(t - \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} = \boxed{-3\pi}$$

③ $\vec{F} = \langle 4y, \ 5-5x, \ z^2 - 2 \rangle$

$S: \vec{r}(\phi, \theta) = \langle \sqrt{5}\sin\phi\cos\theta, \ \sqrt{5}\sin\phi\sin\theta, \ \sqrt{5}\cos\phi \rangle, \quad 0 \leq \phi \leq \pi/2, \ 0 \leq \theta \leq 2\pi$

S: upper hemisphere of sphere centered at 0 w/radius $\sqrt{5}$.

$$C: \vec{r}(\theta) = \langle \sqrt{5}\cos\theta, \sqrt{5}\sin\theta, 0 \rangle, \quad 0 \leq \theta \leq 2\pi$$

$$d\vec{r}(\theta) = \langle -\sqrt{5}\sin\theta, \sqrt{5}\cos\theta, 0 \rangle$$

$$\vec{F}(\vec{r}(\theta)) = \langle 4\sqrt{5}\sin\theta, \ 5 - 5\sqrt{5}\cos\theta, \ -2 \rangle$$

$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(-20\sin^2\theta + 5\sqrt{5}\cos\theta - 25\cos^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(-20 + 5\sqrt{5}\cos\theta - 5\cos^2\theta\right) d\theta$$

$$= \left(-20\theta + 5\sqrt{5}\sin\theta - \frac{5}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right)\Big|_0^{2\pi}$$

$$= -40\pi - \frac{5}{2} \cdot 2\pi$$

$$= \boxed{-45\pi}$$