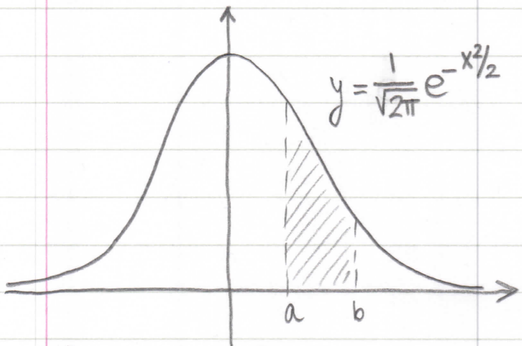
The Gaussian Function / Bell Curve



"Bell Curve"

Gaussian function : $\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

A random variable X is said to be "normally / Gaussian distributed" if:

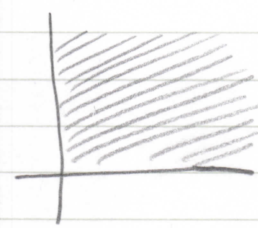
$$\mathbb{P}[a \le X \le b] = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

The Gaussian Integral : $\boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1}$

Proof : Let $\boxed{I := \int_0^{\infty} e^{-x^2/2} dx}$

and consider:

$$\begin{aligned}
I^2 &= \left(\int_0^{\infty} e^{-x^2/2} dx\right)\left(\int_0^{\infty} e^{-y^2/2} dy\right) \\
&= \int_0^{\infty}\int_0^{\infty} e^{-\frac{x^2}{2} - \frac{y^2}{2}} dx\, dy \\
&= \int_0^{\infty}\int_0^{\infty} e^{-\frac{x^2+y^2}{2}} dx\, dy \quad \text{(Change to Polar Coordinates)} \\
&= \int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2} \cdot r\, dr\, d\theta \\
&= \int_0^{\pi/2} -e^{-r^2/2}\Big|_0^{\infty} d\theta \\
&= \int_0^{\pi/2} (0 - (-1))\, d\theta = \int_0^{\pi/2} 1\, d\theta = \theta\Big|_0^{\pi/2} = \boxed{\frac{\pi}{2}}
\end{aligned}$$



$$I^2 = \frac{\pi}{2} \Rightarrow I = \frac{\sqrt{\pi}}{\sqrt{2}} \Rightarrow \int_0^{\infty} e^{-x^2/2} dx = \frac{\sqrt{\pi}}{\sqrt{2}} \Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2} dx = 2 \cdot \frac{\sqrt{\pi}}{\sqrt{2}} = \sqrt{2\pi}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1} \quad //$$