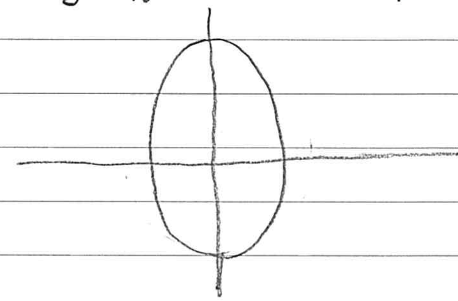
§2.6 Implicit Differentiation

* y defined explicitly in terms of x: $y=\sqrt{x^5+1}$; $y=\tan(x^2+1/x)$; $y = x\cos x$
* y defined implicitly in terms of x: $x^2+y^2=25$; $\underbrace{x^3+y^3=6xy}_{\text{Very difficult to solve explicitly.}}$

① Curve: $\frac{x^2}{4}+\frac{y^2}{9}=1$.



a). $\frac{dy}{dx}=?$ Differentiate both sides wrt x:

$$\frac{2x}{4}+\frac{2y\cdot y'}{9}=0$$

Chain Rule: $\frac{d}{dx}(y^2)=2y(x)\cdot y'(x)$

Solve for y':

$$\frac{1}{2}x+\frac{2}{9}y\cdot y'=0 \Rightarrow \frac{2}{9}y\cdot y' = -\frac{1}{2}x \Rightarrow y'=\frac{9}{2}\frac{1}{y}\left(-\frac{1}{2}x\right)=\frac{-9}{4}\frac{x}{y}$$

b). Eqn. of tangent line to curve @ $\left(\frac{1}{2}, \frac{3}{4}\sqrt{15}\right)$?

Point: $(x_0, y(x_0)) = \left(\frac{1}{2}, \frac{3}{4}\sqrt{15}\right)$

Slope: $-\frac{9}{4}\frac{x}{y(x)}\Big|_{x=x_0=1/2} = -\frac{\cancel{9}^{3}}{\cancel{4}}\cdot\frac{1/2}{\frac{\cancel{3}}{\cancel{4}}\sqrt{15}} = -\frac{3}{2\sqrt{15}}$

$$y-\frac{3}{4}\sqrt{15} = -\frac{3}{2\sqrt{15}}\left(x-\frac{1}{2}\right)$$

② Curve: $x^2+2xy=5y^3+3$

a). $\frac{dy}{dx}=?$

$$2x+2y+2xy'=15y^2y'$$

$$2x+2y=(15y^2-2x)y' ; \quad y'=\frac{2x+2y}{15y^2-2x}$$

b). Tangent line @ $(2,1)$?

Point: $(x_0,y_0)=(2,1)$

Slope: $y'(x_0)=\frac{2x_0+2y_0}{15y_0^2-2x_0}=\frac{2\cdot 2+2\cdot 1}{15\cdot 1^2-2\cdot 2}=\frac{6}{11}$

$$y-1=\frac{6}{11}(x-2)$$

③ Curve: $2xy^3 = xy + 5$

Tangent line at $(5,1)$?

$$2y^3 + 6xy^2y' = y + xy'$$
$$(6xy^2 - x)y' = y - 2y^3$$
$$\boxed{y' = \frac{y-2y^3}{6xy^2-x}}$$

$$y'(5) = \frac{1-2\cdot 1^3}{6\cdot 5\cdot 1^2 - 5} = \frac{-1}{25}$$

Tgt. line: $\boxed{y-1 = \frac{-1}{25}(x-5)}$

④ Curve: $2xy - \pi\cos(y) = 13\pi$

Tangent line at $(6,\pi)$?

$$2y + 2xy' + \pi\sin(y)\cdot y' = 0$$
$$(2x + \pi\sin y)y' = -2y$$
$$\boxed{y' = \frac{-2y}{2x+\pi\sin y}}$$

$$y'(6) = \frac{-2\pi}{12+\pi\sin\pi} = \frac{-\pi}{6}$$

Tgt. line: $\boxed{y-\pi = \frac{-\pi}{6}(x-6)}$

⑤ Curve: $y^2 + 5x = x^2y - 40$, and $y(4) = 6$. Tangent line at $(4,6)$?

$$2yy' + 5 = 2xy + x^2y'$$
$$(2y - x^2)y' = 2xy - 5$$
$$\boxed{y' = \frac{2xy-5}{(2y-x^2)}}$$

$$y'(4) = \frac{2\cdot 24 - 5}{(12-16)} = \frac{43}{-4}$$

Tgt. line: $y - 6 = -\frac{43}{4}(x-4)$

⑥ Curve: $x^2 + 2x + xy = 8$

a). $\frac{dy}{dx} = ?$ $\quad 2x + 2 + y + xy' = 0 \Rightarrow \boxed{y' = \frac{-2x-2-y}{x}}$

b). Express y' as a function of x only.

$$x^2 + 2x + xy = 8 \Rightarrow y = \frac{8-x^2-2x}{x} = \frac{8}{x} - x - 2 \Rightarrow y' = \frac{-2x-\cancel{2}-\frac{8}{x}+x+\cancel{2}}{x}$$
$$= \frac{-x-\frac{8}{x}}{x} = \boxed{-1-\frac{8}{x^2}}$$

c). Express y'' as a function of x only.

$$y' = -1 - \frac{8}{x^2} \Rightarrow \boxed{y'' = \frac{16}{x^3}}$$

⑦ Curve: $\sqrt{x} + \sqrt{y} = 4$.

$y' = ?$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \Rightarrow \boxed{y' = \frac{-\sqrt{y}}{\sqrt{x}}}$$

⑧ Curve: $-16(x^2+y^2)^2 = 200(x^2-y^2)$. Find slope of tgt. line at $(1,-3)$?

$$-16\cdot 2(x^2+y^2)(2x+2yy') = 200(2x-2yy') \quad \Big|\cdot\frac{1}{8}$$
$$-8x(x^2+y^2) - 8yy'(x^2+y^2) = 50x - 50yy'$$
$$(50y - 8y(x^2+y^2))y' = 50x + 8x(x^2+y^2)$$
$$\boxed{y' = \frac{50x+8x(x^2+y^2)}{50y-8y(x^2+y^2)}} \Rightarrow y'(1) = \frac{50+8\cdot 10}{-150+24\cdot 10} = \frac{130}{90} = \boxed{\frac{13}{9}}$$