Notes on Limits of Sequences (I)

Some terminology:

- $\boxed{p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_2 n^2 + a_1 n + a_0}$ (polynomial in n)
  - Degree of p: $\boxed{\deg(p) = k}$ (highest power of n)
  - Leading Coefficient of p: $\boxed{l.c.(p) = a_k}$ (coefficient of n^k)

Example: $p(n) = 8n^7 + 10n^2 + 2n + 1$; $\deg(p) = 7$; $l.c.(p) = 8$

- $\boxed{f(n) = c_1 \, [p(n)]^a}$ where $a > 0$, c is a real number, $p(n) =$ polynomial
  - Dominating power of f:

$$\boxed{D_{power}(f) = a \cdot \deg(p)}$$

  - Dominating coefficient of f:

$$\boxed{D_{coeff}(f) = c_1 \cdot [l.c.(p)]^a}$$

Example: $f(n) = \sqrt{7n^5 + 20n^3 + 2n^2 + 5n + 1}$ $\quad (a = \frac{1}{2})$

$D_{coeff}(f) = \sqrt{7}$; $D_{power}(f) = \frac{5}{2}$

Example: $f(n) = -3\sqrt{5n^{10} + 2n^2 + 1}$

$D_{coeff}(f) = -3 \cdot \sqrt{5}$; $D_{power}(f) = \frac{10}{2} = 5$

Example: $f(n) = -2\left(-3n^2 + 6n - 1\right)^3$

$D_{coeff}(f) = -2(-3)^3 = -2(-27) = \underline{\underline{54}}$; $D_{power}(f) = 2 \cdot 3 = 6$

Example: $f(n) = 5\left(n^7 + 20n^2 - 1\right)^3$

$D_{coeff}(f) = 5$; $D_{power}(f) = 7 \cdot 3 = \underline{\underline{21}}$

Limits of the type $\boxed{\lim_{n\to\infty} \frac{p(n)}{q(n)}}$ where p, q are polynomials:

- Case 1: $\deg(p) = \deg(q) \Rightarrow \boxed{\lim_{n\to\infty} \frac{p(n)}{q(n)} = \frac{l.c.(p)}{l.c.(q)}}$

Examples: $\lim_{n\to\infty} \frac{-10n^3+20n^2+3n}{7n^3+2} = \boxed{-\frac{10}{7}}$

$\lim_{n\to\infty} \frac{20n^{10}+30n+2}{3n^{10}-2n^5+6n} = \boxed{\frac{20}{3}}$

- Case 2: $\deg(p) < \deg(q) \Rightarrow \boxed{\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0}$

Examples: $\lim_{n\to\infty} \frac{3n^6+2}{n^7+2n-6} = 0$; $\lim_{n\to\infty} \frac{n^{10}+1}{n^{12}-6n^2+1} = 0$

- Case 3: $\deg(p) > \deg(q) \Rightarrow \boxed{\lim_{n\to\infty} \frac{p(n)}{q(n)} = \pm\infty}$ where the sign is determined by the sign of $\frac{l.c.(p)}{l.c.(q)}$

Examples: $\lim_{n\to\infty} \frac{20n^3-3n^2+1}{-n^2+n+1} = \boxed{-\infty}$ b/c $\frac{20}{-1} < 0$

$\lim_{n\to\infty} \frac{3n^6+20n^2-30}{2n^3+n^2-6n} = \boxed{\infty}$ b/c $\frac{3}{2} > 0$

Limits of the type $\boxed{\lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{c_1[p(n)]^a}{c_2[q(n)]^b}}$

where c_1, c_2 are real numbers; $a, b > 0$; p, q are polynomials:

- **Case 1:** $D_{power}(f) = D_{power}(g) \Rightarrow \boxed{\lim_{n\to\infty}\frac{f(n)}{g(n)} = \frac{D_{coeff}(f)}{D_{coeff}(g)}}$

Examples:

$$\lim_{n\to\infty}\frac{-2\sqrt{3n^5+20n+1}}{3\sqrt{7n^5-6n^2+3n}} = \frac{-2\sqrt{3}}{3\sqrt{7}} \quad ; \quad D_{power}(f) = D_{power}(g) = 5/2$$

$$\lim_{n\to\infty}\frac{(2n^3-6n^2+3n+1)^3}{2(-n^3+2n^2-6)^3} = \frac{2^3}{2(-1)^3} = -4 \quad ; \quad D_{power}(f) = D_{power}(g) = 9$$

- **Case 2:** $D_{power}(f) < D_{power}(g) \Rightarrow \boxed{\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0}$

Examples:

$$\lim_{n\to\infty}\frac{\sqrt{3n^5+20n^2-1}}{\sqrt{n^7+5n^2-6n}} = 0 \quad ; \quad D_{power}(f) = \frac{5}{2} < D_{power}(g) = \frac{7}{2}$$

$$\lim_{n\to\infty}\frac{-2(6n^2+2n-1)^2}{3n^6} = 0 \quad ; \quad D_{power}(f) = 4 < D_{power}(g) = 6$$

- **Case 3:** $D_{power}(f) > D_{power}(g) \Rightarrow \boxed{\lim_{n\to\infty}\frac{f(n)}{g(n)} = \pm\infty}$ where the sign is determined by the sign of $\frac{D_{coeff}(f)}{D_{coeff}(g)}$

Examples:

$$\lim_{n\to\infty}\frac{-\sqrt{3n^7+20n^2-1}}{n^2+1} = -\infty \quad ; \quad D_{power}(f) = \frac{7}{2} > D_{power}(g) = 2$$

$$\frac{D_{coeff}(f)}{D_{coeff}(g)} = \frac{-\sqrt{3}}{1} < 0$$

$$\lim_{n\to\infty}\frac{(3n^2+6n-4)^5}{10n^9+5} = \infty \quad ; \quad D_{power}(f) = 10 > D_{power}(g) = 9$$

$$\frac{D_{coeff}(f)}{D_{coeff}(g)} = \frac{3^5}{10} > 0$$

$$\lim_{n\to\infty}\frac{(-2n^3+6n^2-5)^2}{-7n+2} = -\infty \quad ; \quad D_{power}(f) = 6 > D_{power}(g) = 1$$

$$\frac{D_{coeff}(f)}{D_{coeff}(g)} = \frac{(-2)^2}{-7} = \frac{4}{-7} < 0$$