Notes on Limits of Sequences (II)

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$ ← must know

Proof: Recall that if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$. This limit comes from:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$$ ← also must know...

Let $f(x) = \left(1+\frac{1}{x}\right)^x$. Then

$$\lim_{x\to\infty} \ln(f(x)) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

[Indeterminate case $\frac{0}{0}$, l'Hospital]

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$

So $\boxed{\lim_{x\to\infty} \ln[f(x)] = 1}$.

Recall now that for any real number $y>0$: $y = e^{\ln(y)}$

So:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln[f(x)]} = e^{\lim_{x\to\infty}\ln[f(x)]} = e^1 = \boxed{e}.$$

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Limits using the fact that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$: $[1^\infty$ indeterminate$]$

Exam Tip: You must show your work with these limits; you should not just memorize that $\lim_{n\to\infty}\left(1+\frac{☺}{n}\right)^n = e^{☺}$

① $$\lim_{n\to\infty}\left(1+\frac{6}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\left(\frac{n}{6}\right)}\right)^n = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{\left(\frac{n}{6}\right)}\right)^{\frac{n}{6}}}_{\downarrow n\to\infty \atop e}\right]^6 = \boxed{e^6}$$

② $$\lim_{n\to\infty}\left(1+\frac{2}{n^2+n}\right)^{n^2+n+3} = \lim_{n\to\infty}\left(1+\frac{2}{n^2+n}\right)^{n^2+n}\cdot\left(1+\frac{2}{n^2+n}\right)^3$$

$$= \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{\frac{n^2+n}{2}}\right)^{\frac{n^2+n}{2}}}_{\downarrow n\to\infty \atop e}\right]^2\cdot\underbrace{\left(1+\frac{2}{n^2+n}\right)}_{\downarrow n\to\infty \atop 1}^3 = e^2\cdot 1^3 = \boxed{e^2}$$

③ $$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3n} = \lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^{3n} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{3n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{3n}} = \lim_{n\to\infty}\frac{1}{\left[\underbrace{\left(1+\frac{1}{n}\right)^n}_{\downarrow n\to\infty \atop e}\right]^3} = \boxed{\frac{1}{e^3}}$$

④ $$\lim_{n\to\infty}\left(1+\frac{3}{n}+\frac{5}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{3n+5}{n^2}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{3n+5}}\right)^n$$ Note that $\lim_{n\to\infty}\frac{n^2}{3n+5} = \infty$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n^2}{3n+5}}\right)^{\frac{n^2}{3n+5}}\right]^{\frac{3n+5}{n^2}\cdot n}$$

$$= \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{1}{\frac{n^2}{3n+5}}\right)^{\frac{n^2}{3n+5}}}_{\xrightarrow[n\to\infty]{} e}\Bigg]^{\frac{3n+5}{n}\ \xrightarrow[n\to\infty]{}\ 3} = \boxed{e^3}$$

⑤ $$\lim_{n\to\infty}\left(\frac{2n+9}{2n+5}\right)^{2n+1} = \lim_{n\to\infty}\left(\frac{2n+5+4}{2n+5}\right)^{2n+1} =$$

$$= \lim_{n\to\infty}\left(1+\frac{4}{2n+5}\right)^{2n+1} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{2n+5}{4}}\right)^{2n+1}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{2n+5}{4}}\right)^{\frac{2n+5}{4}}\right]^{\frac{4}{2n+5}\cdot(2n+1)}$$

$$= \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{1}{\frac{2n+5}{4}}\right)^{\frac{2n+5}{4}}}_{\xrightarrow[n\to\infty]{} e}\Bigg]^{\frac{8n+4}{2n+5}\ \xrightarrow[n\to\infty]{}\ \frac{8}{2}=4} = \boxed{e^4}$$