Notes on Limits of Sequences (III)
(Other Topics)

- **FACTORIALS:** $n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n$ [Convention: $0! = 1$]
  - $3! = 1\cdot 2\cdot 3 = 6$; $5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$
  - $$\lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{(2n-1)!}{\underbrace{1\cdot 2\cdot 3\cdot \ldots \cdot (2n-1)}_{(2n-1)!}\cdot 2n\cdot(2n+1)}$$
    $$= \lim_{n\to\infty} \frac{1}{2n(2n+1)} = \boxed{0}$$
  - $$\lim_{n\to\infty} \frac{3n^2(3n-1)!}{(3n+1)!} = \lim_{n\to\infty} \frac{3n^2(3n-1)!}{(3n-1)!\cdot 3n\cdot(3n+1)}$$
    $$= \lim_{n\to\infty} \frac{3n}{3(3n+1)} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

- **SQUEEZE THEOREM:** If $a_n \le b_n \le c_n$ for all $n \geq n_0$, and:
  $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$
  then $\lim_{n\to\infty} b_n = L$.

  $a_n \le b_n \le c_n$

  $a_n \xrightarrow{n\to\infty} L$, $b_n \to L$, $c_n \xrightarrow{n\to\infty} L$

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

<u>Proof</u>: For any real number c: $\boxed{-|c| \le c \le |c|}$

So for all n:
$$-|a_n| \le a_n \le |a_n|$$
$$-|a_n| \xrightarrow{n\to\infty} 0 \xleftarrow{n\to\infty} |a_n|$$

So $\lim_{n\to\infty} a_n = 0$ by the Squeeze Thm. ∎

- $\lim_{n\to\infty} \frac{(-1)^n}{3\sqrt{n}} = 0$ because $\lim_{n\to\infty} \left|\frac{(-1)^n}{3\sqrt{n}}\right| = \lim_{n\to\infty} \frac{1}{3\sqrt{n}} = 0$

- $\lim_{n\to\infty} \frac{\sin(2n)}{\sqrt{n}} = 0$ because $\lim_{n\to\infty} \frac{-1}{\sqrt{n}} \le \frac{\sin(2n)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ (Squeeze Thm.)

  $\frac{-1}{\sqrt{n}} \xrightarrow{n\to\infty} 0 \xleftarrow{n\to\infty} \frac{1}{\sqrt{n}}$, $\frac{\sin(2n)}{\sqrt{n}} \to 0$

Theorem: If $|a_n| \leq b_n$ for all $n \geq n_0$ and $b_n \to 0$, then $|a_n| \to 0$ (and also $a_n \to 0$).

Proof: $0 \leq |a_n| \leq b_n$ for all $n \geq n_0$ (Squeeze Thm).

$$0 \xrightarrow{n\to\infty} 0, \quad |a_n| \to 0, \quad b_n \xrightarrow{n\to\infty} 0$$

- Find $\boxed{\lim_{n\to\infty} \frac{(-2)^n}{6n!}}$:

$$a_n = \frac{(-2)^n}{6n!} \Rightarrow |a_n| = \frac{2^n}{6n!}$$

$$|a_n| = \frac{2^n}{6n!} = \frac{1}{6}\cdot\frac{2^n}{n!} = \frac{1}{6}\cdot\frac{2\cdot2\cdot2\cdot\ldots\cdot2}{1\cdot2\cdot3\cdot\ldots\cdot n}$$

$$= \frac{1}{6}\cdot\frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{5}\cdot\ldots\cdot\frac{2}{n-1}\cdot\frac{2}{n}$$

$$\leq \frac{1}{6}\cdot\frac{2}{1}\cdot\frac{2}{2}\cdot\cancel{\frac{2}{3}}\cdot\cancel{\frac{3}{4}}\cdot\cancel{\frac{4}{5}}\cdot\ldots\cdot\cancel{\frac{n-2}{n-1}}\cdot\frac{\cancel{n-1}}{n} = \frac{2}{3n} \xrightarrow{n\to\infty} 0$$

so $|a_n| \leq \frac{2}{3n} \xrightarrow[n\to\infty]{} 0$, so $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} a_n = \boxed{0}$

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } -1<r<1 \\ 1, & \text{if } r=1 \\ \infty, & \text{if } r>1 \end{cases}}$$

(DNE, if $r \leq -1$)

- $a_n = 2-(0.4)^n$; $\lim_{n\to\infty} a_n = 2-0=2$

- $\lim_{n\to\infty} \frac{2^{n+2}}{5^n} = \lim_{n\to\infty} \frac{2^n \cdot 2^2}{5^n} = \lim_{n\to\infty} 4\cdot\left(\frac{2}{5}\right)^n = 0$

$$\left(\frac{2}{5}\right)^n \xrightarrow{n\to\infty} 0$$