

Notes on Limits of Sequences (III)
(Other Topics)

• FACTORIALS: $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ [Convention: $0! = 1$]

• $3! = 1 \cdot 2 \cdot 3 = 6$; $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

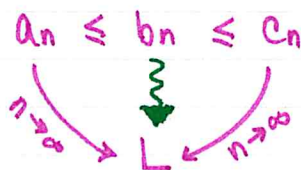
•
$$\lim_{n \rightarrow \infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n \rightarrow \infty} \frac{(2n-1)!}{\underbrace{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n-1)}_{(2n-1)!} \cdot 2n \cdot (2n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2n(2n+1)} = \boxed{0}$$

•
$$\lim_{n \rightarrow \infty} \frac{3n^2 (3n-1)!}{(3n+1)!} = \lim_{n \rightarrow \infty} \frac{3n^2 (3n-1)!}{(3n-1)! \cdot 3n \cdot (3n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{3n}{3n(3n+1)} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

• SQUEEZE THEOREM : If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$, and :
 $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$
 then $\lim_{n \rightarrow \infty} b_n = L$.



Theorem : If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof : For any real number ϵ : $\boxed{-\epsilon \leq a_n \leq \epsilon}$

So for all n : $-|a_n| \leq a_n \leq |a_n|$

So $\lim_{n \rightarrow \infty} a_n = 0$ by the Squeeze Thm. ■

• $\lim_{n \rightarrow \infty} \frac{(-1)^n}{3\sqrt{n}} = 0$ because $\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{3\sqrt{n}} \right| = \lim_{n \rightarrow \infty} \frac{1}{3\sqrt{n}} = 0$

• $\lim_{n \rightarrow \infty} \frac{\sin(2n)}{\sqrt{n}} = 0$ because $\lim_{n \rightarrow \infty} \frac{-1}{\sqrt{n}} \leq \frac{\sin(2n)}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$
 (Squeeze Thm.)

Theorem: || If $|a_n| \leq b_n$ for all $n \geq n_0$ and $b_n \rightarrow 0$, then $|a_n| \rightarrow 0$ (and also $a_n \rightarrow 0$).

Proof: $0 \leq |a_n| \leq b_n$ for all $n \geq n_0$ (Squeeze Thm).
 $\begin{matrix} \nearrow_{n \rightarrow \infty} & & \searrow_{n \rightarrow \infty} \\ & 0 & \end{matrix}$

• Find $\lim_{n \rightarrow \infty} \frac{(-2)^n}{6n!}$:

$$a_n = \frac{(-2)^n}{6n!} \Rightarrow |a_n| = \frac{2^n}{6n!}$$

$$\begin{aligned} |a_n| &= \frac{2^n}{6n!} = \frac{1}{6} \cdot \frac{2^n}{n!} = \frac{1}{6} \cdot \frac{2 \cdot 2 \cdot 2 \cdot \dots \cdot 2}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \\ &= \frac{1}{6} \cdot \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} \cdot \dots \cdot \frac{2}{n-1} \cdot \frac{2}{n} \\ &\leq \frac{1}{6} \cdot \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} \cdot \dots \cdot \frac{n-2}{n-1} \cdot \frac{n-1}{n} = \frac{2}{3n} \end{aligned}$$

$\downarrow_{n \rightarrow \infty}$
0

So $|a_n| \leq \frac{2}{3n} \xrightarrow{n \rightarrow \infty} 0$, so $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} a_n = \underline{\underline{0}}$

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \\ \infty, & \text{if } r > 1 \end{cases}$$

(DNE, if $r \leq -1$)

• $a_n = 2 - (0.4)^n$; $\lim_{n \rightarrow \infty} a_n = 2 - 0 = 2$

• $\lim_{n \rightarrow \infty} \frac{2^{n+2}}{5^n} = \lim_{n \rightarrow \infty} \frac{2^n \cdot 2^2}{5^n} = \lim_{n \rightarrow \infty} 4 \cdot \left(\frac{2}{5}\right)^n = 0$
 $\swarrow_{n \rightarrow \infty}$
0