Infinite Series:

Definition: An infinite series is obtained by summing the terms of an infinite sequence $\{a_n\}$:

$$a_1 + a_2 + a_3 + \dots$$

This is denoted

$$\boxed{\sum_{n=1}^{\infty} a_n} \quad \text{or} \quad \boxed{\sum a_n}$$

- There are two sequences associated with any series $\sum_{n=1}^{\infty} a_n$:
  - The sequence $\{a_n\}$ of its individual terms
  - The sequence $\{S_N\}$ of partial sums:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$S_N = a_1 + a_2 + \dots + a_N = \sum_{n=1}^{N} a_n$$
$$\vdots$$

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We say that a series $\sum_{n=1}^{\infty} a_n$ is convergent to some number S, and write:

$$\sum_{n=1}^{\infty} a_n = S$$

if and only if the sequence $\{S_N\}$ of partial sums converges to S as $N \to \infty$.

$$\boxed{\sum_{n=1}^{\infty} a_n = S} \iff \boxed{\lim_{N \to \infty} S_N = S}$$

Example: Suppose the sequence of partial sums of a series $\sum a_n$ is given by:

$$S_N = \frac{N-2}{N+2}$$

a). Find a_n:

$$a_1 = S_1 = \frac{1-2}{1+2} = \frac{-1}{3}$$

For $n > 1$: $a_n = S_n - S_{n-1} = \frac{n-2}{n+2} - \frac{(n-1)-2}{(n-1)+2} = \frac{n-2}{n+2} - \frac{n-3}{n+1}$

$$= \frac{(n-2)(n+1) - (n-3)(n+2)}{(n+2)(n+1)}$$

$$= \frac{\cancel{n^2}+n-2n-2-\cancel{n^2}-2n+3n+6}{(n+2)(n+1)}$$

$$= \frac{4}{(n+2)(n+1)}$$

So: $a_1 = \frac{-1}{3}$ and $a_n = \frac{4}{(n+2)(n+1)}$ for $n > 1$.

b). Find $\sum_{n=1}^{\infty} a_n$, if this converges.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N-2}{N+2} = \boxed{1}.$$

Types of Series you must know:

① Geometric Series : $\boxed{a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1}}$ ($r =$ "ratio")

- converges if and only if $|r| < 1$, and then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.
- diverges if $|r| \geq 1$.

Examples:

1). $10 + 9 + \frac{81}{10} + \frac{729}{100} + \ldots = 10 + 10 \cdot \frac{9}{10} + 10 \cdot \frac{81}{100} + 10 \cdot \frac{729}{1000} + \ldots$

$$= 10 + 10 \cdot \left(\frac{9}{10}\right) + 10 \cdot \left(\frac{9}{10}\right)^2 + 10 \cdot \left(\frac{9}{10}\right)^3 + \ldots$$

$$= \sum_{n=1}^{\infty} 10 \cdot \left(\frac{9}{10}\right)^{n-1}$$ geometric series w/ ratio $r = \frac{9}{10}$

$|r| = \frac{9}{10} < 1 \Rightarrow$ convergent

$$= \frac{10}{1 - \frac{9}{10}} = \frac{10}{\frac{1}{10}} = \boxed{100}$$

2). $\sum_{n=1}^{\infty} \frac{10^n}{(-7)^{n-1}} = \sum_{n=1}^{\infty} 10 \cdot \frac{10^{n-1}}{(-7)^{n-1}} = \sum_{n=1}^{\infty} 10 \cdot \left(\frac{10}{-7}\right)^{n-1}$

geometric series w/ ratio $r = \frac{10}{-7}$

$|r| = \frac{10}{7} > 1 \Rightarrow$ divergent

3). $\sum_{n=1}^{\infty} \frac{e^n}{8^{n-1}} = \sum_{n=1}^{\infty} e \cdot \frac{e^{n-1}}{8^{n-1}} = \sum_{n=1}^{\infty} e \cdot \left(\frac{e}{8}\right)^{n-1}$

geometric series w/ ratio $r = \frac{e}{8}$ and $a = e$

$|r| = \frac{e}{8} < 1 \Rightarrow$ convergent

$$= \frac{e}{1 - \frac{e}{8}} = \boxed{\frac{8e}{8-e}}$$

4). $\sum_{K=1}^{\infty} (\cos(9))^K = \sum_{K=1}^{\infty} \cos(9) \cdot (\cos(9))^{K-1}$

geometric series w/ ratio $r = \cos(9)$ and $a = \cos(9)$

$|r| = |\cos(9)| < 1 \Rightarrow$ convergent

$$= \boxed{\frac{\cos(9)}{1 - \cos(9)}}$$

5). For what values of x does the series $\sum_{n=1}^{\infty}(-2)^n x^n$ converge? What does it converge to for those values of x?

$$\sum_{n=1}^{\infty}(-2)^n x^n = \sum_{n=1}^{\infty}(-2x)(-2x)^{n-1}$$

geometric series w/ ratio $r=-2x$ and $a=-2x$

This series converges if and only if $|r| = 2|x| < 1$, so if $|x| < \frac{1}{2}$
So:

$$\boxed{x \in \left(-\frac{1}{2}, \frac{1}{2}\right)}$$

For x in $\left(-\frac{1}{2}, \frac{1}{2}\right)$, this series converges to $\frac{-2x}{1-(-2x)} = \boxed{\frac{-2x}{1+2x}}$

6). Express the number $0.\overline{2} = 0.2222\ldots$ as a ratio of integers:

$$0.\overline{2} = 0.2 + 0.02 + 0.002 + 0.0002 + \ldots$$

$$= \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \ldots$$

$$= \sum_{n=1}^{\infty}\frac{2}{10^n} = \sum_{n=1}^{\infty}\frac{2}{10}\cdot\frac{1}{10^{n-1}} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{5}\cdot\left(\frac{1}{10}\right)^{n-1}$$ geometric series with ratio $r=\frac{1}{10}$ and $a=\frac{1}{5}$

$|r| = \frac{1}{10} < 1 \Rightarrow$ convergent

$$= \frac{\frac{1}{5}}{1-\frac{1}{10}} = \frac{\frac{1}{5}}{\frac{9}{10}} = \frac{1}{5}\cdot\frac{10}{9} = \boxed{\frac{2}{9}}$$

So $\boxed{0.\overline{2} = \frac{2}{9}}$

② Harmonic Series: $\boxed{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n}}$

DIVERGENT

Alternating Harmonic Series: $\boxed{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}}$

CONVERGENT

(by the Alternating Series Test)

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③ The p-series: $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots}$

converges for $p>1$

diverges for $p\leq 1$

Examples:

1). $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\ldots=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots,$

$=\sum_{n=1}^{\infty}\frac{1}{n^4}$ convergent p-series with $p=4>1$

2). $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ divergent p-series with $p=\frac{1}{2}<1$