## Comparison Test:

Suppose that $\{a_n\}$, $\{b_n\}$ are sequences with positive terms, and:

$$a_n \le b_n \quad \text{for all } n$$

Then:

$$\boxed{\sum_{n=1}^{\infty} b_n \text{ converges}} \Rightarrow \boxed{\sum_{n=1}^{\infty} a_n \text{ converges}}$$

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges}} \Rightarrow \boxed{\sum_{n=1}^{\infty} b_n \text{ diverges}}$$

**! CAUTION: !** $\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow$ no conclusion

$\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ no conclusion

## Exercises:

1). $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges by the Comparison Test:

$$\frac{1}{2^n+1} < \frac{1}{2^n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is a convergent geometric series.}$$

2). $\sum_{n=1}^{\infty} \frac{n}{4n^3+1}$ converges by the Comparison Test:

$$\frac{n}{4n^3+1} < \frac{n}{4n^3} = \frac{1}{4n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

(p-series, $p = 2 > 1$)

3). $\sum_{n=1}^{\infty} \frac{n^3}{6n^4-2}$ diverges by the Comparison Test:

$$\frac{n^3}{6n^4-2} > \frac{n^3}{6n^4} = \frac{1}{6n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{6n} = \frac{1}{6} \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

(harmonic series)

4). $\sum_{n=1}^{\infty} \frac{n+4}{n\sqrt{n}}$ diverges by the Comparison Test:

$$\frac{n+4}{n\sqrt{n}} > \frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (p-series with } p = \tfrac{1}{2} < 1)$$

$-\frac{\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}$ for all x

5). $\sum_{n=1}^{\infty} \frac{\arctan(3n)}{n^{2.4}}$ converges by the Comparison Test

$$\frac{\arctan(3n)}{n^{2.4}} \leq \frac{\pi/2}{n^{2.4}} \text{ and } \sum_{n=1}^{\infty} \frac{\pi/2}{n^{2.4}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{2.4}} \text{ converges}$$

(p-series w/ $p=2.4>1$)

6). $\sum_{n=1}^{\infty} \frac{9^{n+1}}{8^n - 7}$ diverges by the Comparison Test:

$$\frac{9^{n+1}}{8^n-7} > \frac{9^{n+1}}{8^n} = 9\cdot\left(\frac{9}{8}\right)^n \text{ and } \sum_{n=1}^{\infty} 9\cdot\left(\frac{9}{8}\right)^n \text{ diverges}$$

(geometric series w/ ratio $\frac{9}{8}>1$)

7). $\sum_{n=1}^{\infty} \frac{e^{2/n}}{n}$ diverges by the Comparison Test

$$\frac{e^{2/n}}{n} > \frac{1}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic series)}$$

8). $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^4+2}$ converges by the Comparison Test

$$\frac{\cos^2 n}{n^4+2} \leq \frac{1}{n^4+2} < \frac{1}{n^4} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ converges (p-series w/ } p=4)$$

9). $\sum_{n=1}^{\infty} \frac{4+\sin(n)}{3^n}$ converges by the Comparison Test.

$$\frac{4+\sin(n)}{3^n} \leq \frac{4+1}{3^n} = \frac{5}{3^n} \text{ and } \sum_{n=1}^{\infty} \frac{5}{3^n} \text{ converges}$$

(geometric series w/ ratio $\frac{1}{3}<1$)

## Limit Comparison Test: (LCT)

Suppose $\{a_n\}$, $\{b_n\}$ are sequences with positive terms. If:

$$\infty > \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ either both converge OR both diverge.

### Exercises:

1). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+2}$

Cannot do the trick with $\frac{1}{3n+2} \overset{?}{>} \frac{1}{3n}$, because $3n+2 > 3n$
so in fact $\frac{1}{3n+2} < \frac{1}{3n}$, and the Comparison Test is inconclusive.
($\sum_{n=1}^{\infty} b_n$ diverges => no conclusion)

But we do see that the terms of the series remind us of the harmonic series... So let

$$a_n = \frac{1}{3n+2} \quad ; \quad b_n = \frac{1}{n}$$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} > 0$

So by LCT, $\sum_{n=1}^{\infty} a_n$ diverges, because $\sum_{n=1}^{\infty} b_n$ diverges.

2). $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-6n}{n^3+4n+3}$

Let $a_n = \dfrac{n^2-6n}{n^3+4n+3}$ and $b_n = \dfrac{1}{n}$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-6n}{n^3+4n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n^2-6n)}{n^3+4n+3} = 1 > 0$

So $\sum_{n=1}^{\infty} a_n$ diverges by LCT, because $\sum_{n=1}^{\infty} b_n$ diverges.

3). $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+2}{n^6+2n^2+1}$

Let $a_n = \dfrac{5n^2+2}{n^6+2n^2+1}$ and $b_n = \dfrac{1}{n^4}$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4(5n^2+2)}{n^6+2n^2+1} = 5 > 0$

So $\sum_{n=1}^{\infty} a_n$ converges by LCT since $\sum_{n=1}^{\infty} b_n$ converges (p-series w/ $p = 4 > 1$).