7.3 Trigonometric Substitution

Expression	Substitution	What happens?
$\sqrt{a^2-x^2}$	$x=a\sin\theta$, $\theta\in[-\pi/2,\pi/2]$ ↙ Range of arcsin	$dx=a\cos\theta\,d\theta$; $\theta=\arcsin(x/a)$ $\sqrt{a^2-x^2}=\sqrt{a^2(1-\sin^2\theta)}=a\sqrt{\cos^2\theta}=a\lvert\cos\theta\rvert$ (b/c $\theta\in[-\pi/2,\pi/2]$:) $=a\cos\theta$. $\boxed{\sqrt{a^2-x^2}=a\cos\theta}$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$, $\theta\in(-\pi/2,\pi/2)$ ↙ Range of arctan	$dx=a\sec^2\theta\,d\theta$; $\theta=\arctan(x/a)$ $\sqrt{a^2+x^2}=\sqrt{a^2(1+\tan^2\theta)}=a\sqrt{\sec^2\theta}=a\lvert\sec\theta\rvert$ (b/c $\theta\in(-\pi/2,\pi/2)$:) $=a\sec\theta$ $\boxed{\sqrt{a^2+x^2}=a\sec\theta}$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$, $\theta\in[0,\pi/2)\cup[\pi,3\pi/2)$ ↙ Range of arcsec	$dx=a\sec\theta\tan\theta\,d\theta$; $\theta=\operatorname{arcsec}(x/a)$ $\sqrt{x^2-a^2}=\sqrt{a^2(\sec^2\theta-1)}=a\sqrt{\tan^2\theta}=a\lvert\tan\theta\rvert$ (b/c $\theta\in$ Quad I or III) $=a\tan\theta$ $\boxed{\sqrt{x^2-a^2}=a\tan\theta}$

1. $\int\sqrt{4-x^2}\,dx$

$x=2\sin\theta$, $\theta\in[-\pi/2,\pi/2]$

$dx=2\cos\theta\,d\theta$

$\sqrt{4-x^2}=2\cos\theta$

$$=\int 2\cos\theta\cdot 2\cos\theta\,d\theta=4\int\cos^2\theta\,d\theta$$

$$=4\int\frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$=2\left(\theta+\frac{1}{2}\sin(2\theta)\right)=2\theta+\sin(2\theta)$$

$$=2\theta+2\sin\theta\cos\theta$$

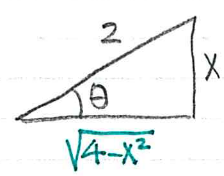
Needs to be put back in terms of x:

$\frac{x}{2}=\sin\theta$

$\boxed{\theta=\arcsin(x/2)}$

$$=2\arcsin(x/2)+2\underbrace{\sin(\arcsin(x/2))}_{x/2}\underbrace{\cos(\arcsin(x/2))}_{\frac{1}{2}\sqrt{4-x^2}}$$

$$=\boxed{2\arcsin(x/2)+\frac{x}{2}\sqrt{4-x^2}}+C$$



2. $\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sin^3\theta\cdot\cos\theta\cdot\cos\theta\,d\theta$

$x=\sin\theta$, $\underline{\underline{\theta\in[0,\pi/2]}}$ **!**

because $x\in[0,1]$!

$dx=\cos\theta\,d\theta$; $\sqrt{1-x^2}=\cos\theta$

$$= \int_0^{\pi/2} \sin^3\theta\cdot\cos^2\theta\,d\theta = \int_0^{\pi/2}\sin\theta\cdot\sin^2\theta\cdot\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2}(1-\cos^2\theta)\cos^2\theta\,(\sin\theta\,d\theta)$$

$$= -\left(\frac{1}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta\right)\Big|_0^{\pi/2}$$

$$= -\left(\frac{1}{3}\cdot 0-\frac{1}{5}\cdot 0-\frac{1}{3}\cdot 1+\frac{1}{5}\cdot 1\right)=\frac{1}{3}-\frac{1}{5}=\boxed{\frac{2}{15}}$$

3. $\int\sqrt{1-4x^2}\,dx = \int\sqrt{4\left(\frac{1}{4}-x^2\right)}\,dx = 2\int\sqrt{\frac{1}{4}-x^2}\,dx$

$x=\frac{1}{2}\sin\theta$, $\theta\in[-\pi/2,\pi/2]$

$dx=\frac{1}{2}\cos\theta\,d\theta$

$\sqrt{\frac{1}{4}-x^2}=\frac{1}{2}\cos\theta$

$$=2\int\frac{1}{2}\cos\theta\cdot\frac{1}{2}\cos\theta\,d\theta = \frac{1}{2}\int\cos^2\theta\,d\theta$$

$$=\frac{1}{2}\int\frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$=\frac{1}{4}\left(\theta+\frac{1}{2}\sin(2\theta)\right)$$

$\theta=\arcsin(2x)$

$$=\frac{1}{4}\theta+\frac{1}{8}\cdot 2\sin\theta\cdot\cos\theta$$

$$=\frac{1}{4}\arcsin(2x)+\frac{1}{4}(2x)\left(\sqrt{1-4x^2}\right)+C$$

$$=\boxed{\frac{1}{4}\arcsin(2x)+\frac{x}{2}\sqrt{1-4x^2}+C}$$

4. $\int\sqrt{5+4x-x^2}\,dx = \int\sqrt{9-4+4x-x^2}\,dx = \int\sqrt{9-(x-2)^2}\,dx$

$x-2=3\sin\theta$, $\theta\in[-\pi/2,\pi/2]$

$dx=3\cos\theta\,d\theta$

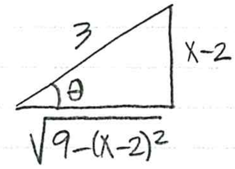
$\sqrt{9-(x-2)^2}=3\cos\theta$

$$=\int 3\cos\theta\cdot 3\cos\theta\,d\theta = \frac{9}{2}\int(1+\cos(2\theta))\,d\theta$$

$$=\frac{9}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)$$

$\theta=\arcsin\left(\frac{x-2}{3}\right)$

$$=\frac{9}{2}\left(\theta+\sin\theta\cos\theta\right)$$



$$=\frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right)+\frac{x-2}{3}\cdot\frac{\sqrt{9-(x-2)^2}}{3}\right)+C$$

$$=\boxed{\frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right)+\frac{x-2}{9}\sqrt{5+4x-x^2}\right)+C}$$

5. $\displaystyle\int \frac{\sqrt{x^2-25}}{x^3}\,dx$

$x = 5\sec\theta \;,\; \theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$

$dx = 5\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-25} = 5\tan\theta$

$$= \int \frac{5\tan\theta}{5^3\sec^3\theta} \cdot 5\sec\theta\tan\theta\, d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{5}\int \cos^2\theta\tan^2\theta\, d\theta$$

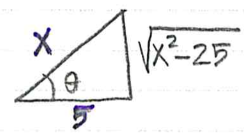
$$= \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{5}\int \frac{1}{2}(1-\cos(2\theta))\,d\theta = \frac{1}{10}\left(\theta - \frac{1}{2}\sin(2\theta)\right)$$

$$= \frac{1}{10}\left(\theta - \sin\theta\cos\theta\right)$$

$$= \frac{1}{10}\left(\sec^{-1}\left(\frac{x}{5}\right) - \frac{\sqrt{x^2-25}}{x}\,\frac{5}{x}\right)$$

$$= \boxed{\frac{1}{10}\left(\sec^{-1}\left(\frac{x}{5}\right) - \frac{5\sqrt{x^2-25}}{x^2}\right) + C}$$

$\frac{x}{5} = \sec\theta \Rightarrow \cos\theta = \frac{5}{x}$



6. $\displaystyle\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{1}{\sqrt{(x-1)^2+4}}\,dx$

$x-1 = 2\tan\theta \,,\; \theta \in (-\pi/2, \pi/2)$

$dx = 2\sec^2\theta\, d\theta$

$\sqrt{(x-1)^2+4} = 2\sec\theta$

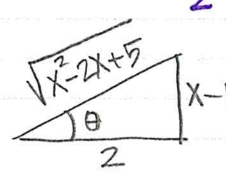
$$= \int \frac{1}{2\sec\theta}\, 2\sec^2\theta\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$$

$$= \boxed{\ln\left|\frac{1}{2}\sqrt{x^2-2x+5} + \frac{x-1}{2}\right| + C}$$

$$= \boxed{\ln\left|\sqrt{x^2-2x+5} + x - 1\right| + C}$$

$\tan\theta = \frac{x-1}{2}$



$\cos\theta = \frac{2}{\sqrt{x^2-2x+5}}$

7. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

$x = 2\tan\theta,\ \theta \in (-\pi/2, \pi/2)$
$dx = 2\sec^2\theta\, d\theta;\ \sqrt{x^2+4} = 2\sec\theta$

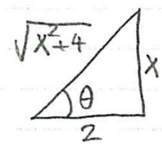
$$= \int \frac{2\sec^2\theta}{4\tan^2\theta \cdot 2\sec\theta}\, d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\, \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, \underbrace{\cos\theta\, d\theta}_{du} \qquad u = \sin\theta$$

$$= -\frac{1}{4}\,\frac{1}{\sin\theta} + C$$

$$= \boxed{-\frac{\sqrt{x^2+4}}{4x} + C}$$

$\tan\theta = \dfrac{x}{2}$



$\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$

8. $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\, dx$

Regular Substitution: $u = x^2+4$
$du = 2x\,dx$

$$= \int \frac{1}{2\sqrt{u}}\, du = \sqrt{u} + C = \boxed{\sqrt{x^2+4} + C}$$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2-2x-15}} = \int \frac{1}{\sqrt{(x^2-2x+1)-1-15}}\, dx = \int \frac{1}{\sqrt{(x-1)^2-16}}\, dx$

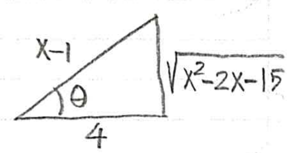
$x - 1 = 4\sec\theta,\ \theta \in [0, \pi/2) \cup [$
$dx = 4\sec\theta\tan\theta\, d\theta$
$\sqrt{(x-1)^2-16} = 4\tan\theta$

$$= \int \frac{\cancel{4}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{4\tan\theta}} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x-1}{4} + \frac{\sqrt{x^2-2x-15}}{4}\right| + C} = \boxed{\ln\left|x-1+\sqrt{x^2-2x-15}\right| + C}$$

$\sec\theta = \dfrac{x-1}{4}$



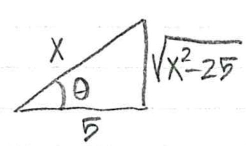
10. $\displaystyle\int \frac{1}{(x^2-25)\sqrt{x^2-25}}\, dx$

$x = 5\sec\theta,\ \theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$
$dx = 5\sec\theta\tan\theta;\ \sqrt{x^2-25} = 5\tan\theta$

$$= \int \frac{\cancel{5}\sec\theta\,\cancel{\tan\theta}}{25\tan^2\theta \cdot \cancel{5\tan\theta}}\, d\theta = \frac{1}{25}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= -\frac{1}{25}\,\frac{1}{\sin\theta} + C = \boxed{-\frac{x}{25\sqrt{x^2-25}} + C}$$

$\sec\theta = \dfrac{x}{5}$



$\sin\theta = \dfrac{\sqrt{x^2-25}}{x}$