7.8 Improper Integrals

Type 1:

- If $\int_a^t f(x)dx$ exists for every number $t \geq a$: $\boxed{\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx}$
- If $\int_t^b f(x)dx$ exists for every number $t \leq b$: $\boxed{\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx}$

The improper integrals above are called <u>convergent</u> if the limit exists & is finite; <u>divergent</u> otherwise.

- If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent:

$\boxed{\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx}$ (Any number may be used for a).

1. $$\int_{-\infty}^0 \frac{1}{5-6x}dx = \lim_{t\to-\infty}\left(\int_t^0 \frac{1}{5-6x}dx\right) = \lim_{t\to-\infty}\left(-\frac{1}{6}\ln|5-6x|\Big|_t^0\right)$$
$$= \lim_{t\to-\infty}\left(-\frac{1}{6}\ln(5) + \frac{1}{6}\ln|5-6t|\right) = \boxed{\infty}\ \underline{\text{Divergent}}$$

2. $$\int_2^\infty e^{-2x}dx = \lim_{t\to\infty}\left(\int_2^t e^{-2x}dx\right) = \lim_{t\to\infty}\left(\left(-\frac{1}{2}e^{-2x}\right)\Big|_2^t\right)$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4}\right) = \boxed{\frac{1}{2}e^{-4}}\ \underline{\text{Convergent}}$$

3. $$\int_{-\infty}^\infty \frac{5}{1+36x^2}dx = \frac{5}{6}\frac{\pi}{2}\cdot 2 = \boxed{\frac{5\pi}{6}}\ \underline{\text{Convergent}}$$

$$\int_0^\infty \frac{5}{1+36x^2}dx = \lim_{t\to\infty}\left(\int_0^t \frac{5}{1+(6x)^2}dx\right) = \lim_{t\to\infty}\left(\frac{5}{6}\arctan(6x)\Big|_0^t\right)$$
$$= \lim_{t\to\infty}\left(\frac{5}{6}\arctan(6t) - \frac{5}{6}\underbrace{\arctan(0)}_{0}\right) = \frac{5}{6}\frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{5}{1+(6x)^2}dx = \lim_{t\to-\infty}\left(\frac{5}{6}\arctan(6x)\Big|_t^0\right) = \lim_{t\to-\infty}\left(\frac{5}{6}\arctan(0) - \frac{5}{6}\arctan(6t)\right) = \frac{5}{6}\frac{\pi}{2}$$

4. $$\int_0^\infty \frac{dx}{(x+6)(x^2+1)}$$

$$\frac{1}{(x+6)(x^2+1)} = \frac{A}{x+6} + \frac{Bx+C}{x^2+1}$$
$$1 = A(x^2+1) + (x+6)(Bx+C)$$
$$x=-6:\ 1 = 37A \quad \boxed{A = 1/37}$$
$$x=0:\ 1 = \frac{1}{37} + 6C;\ \frac{36}{37} = 6C \quad \boxed{C = \frac{6}{37}}$$
$$x=6:\ 1 = 1 + 12\left(6B + \frac{6}{37}\right) \quad \boxed{B = -\frac{1}{37}}$$

$$\int_0^t \frac{dx}{(x+6)(x^2+1)} = \int_0^t \left(\frac{1}{37}\frac{1}{x+6} - \frac{1}{37}\frac{x-6}{x^2+1}\right)dx$$
$$= \frac{1}{37}\int_0^t \left(\frac{1}{x+6} - \frac{x}{x^2+1} + \frac{6}{x^2+1}\right)dx$$
$$= \frac{1}{37}\left(\ln|x+6| - \frac{1}{2}\ln(x^2+1) + 6\arctan(x)\right)\Big|_0^t$$
$$= \frac{1}{37}\left(\ln\left(\frac{x+6}{\sqrt{x^2+1}}\right) + 6\arctan x\right)\Big|_0^t$$
$$= \frac{1}{37}\left(\ln\underbrace{\left(\frac{t+6}{\sqrt{t^2+1}}\right)}_{\downarrow t\to\infty\ 1} + 6\underbrace{\arctan(t)}_{\downarrow t\to\infty\ \pi/2} - \ln(6) - 6\underbrace{\arctan(0)}_{0}\right) \xrightarrow[t\to\infty]{} \boxed{\frac{1}{37}(3\pi - \ln 6)}$$

<u>Convergent.</u>

Type 2:

- If f is continuous on $[a,b)$ but <u>discontinuous at $x=b$</u>: $\boxed{\int_a^b f(x)dx = \lim_{t\to b_-} \int_a^t f(x)dx}$
- If f is continuous on $(a,b]$ but <u>discontinuous at $x=a$</u>: $\boxed{\int_a^b f(x)dx = \lim_{t\to a_+} \int_t^b f(x)dx}$

Again these are <u>convergent</u> (if the limit exists & is finite) or <u>divergent</u> (otherwise)

- If f has a discontinuity at $x=c\in(a,b)$ and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge, then $\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx}$

1. $\int_0^{\ln 8} \frac{1}{x^2} e^{-\frac{1}{x}} dx$ Special attention to the case <u>$x=0$</u> (function undefined)

$$= \lim_{t\to 0_+}\left(\int_t^{\ln 8} \frac{1}{x^2} e^{-\frac{1}{x}}dx\right) = \lim_{t\to 0_+}\left(e^{-\frac{1}{x}}\Big|_t^{\ln 8}\right) = \lim_{t\to 0+}\left(e^{-\frac{1}{\ln 8}} - e^{-\frac{1}{t}}\right) = \boxed{e^{-\frac{1}{\ln 8}}}$$

<u>Convergent</u>

$$e^{-\frac{1}{t}} \xrightarrow[t\to 0_+]{} e^{-\infty} = 0$$

2. $\int_{-1}^1 \frac{1}{|x|^{4/5}} dx$ Special attention to <u>$x=0$</u> (function undefined + branch function)

$$\int_{-1}^0 |x|^{-4/5}dx = \int_{-1}^0 (-x)^{-4/5}dx = \lim_{t\to 0_-}\left(\int_{-1}^t (-x)^{-4/5}dx\right) = \lim_{t\to 0_-}\left(-5(-x)^{1/5}\Big|_{-1}^t\right)$$

$$= \lim_{t\to 0_-}\left(-5(-x)^{1/5} + 5\cdot 1^{1/5}\right) = 5$$

$$\int_0^1 |x|^{-4/5}dx = \int_0^1 x^{-4/5}dx = \lim_{t\to 0_+}\left(\int_t^1 x^{-4/5}dx\right) = \lim_{t\to 0_+}\left(5x^{1/5}\Big|_t^1\right)$$

$$= \lim_{t\to 0+}\left(5 - 5\cdot t^{1/5}\right) = 5$$

$$\Rightarrow \int_{-1}^1 \frac{1}{|x|^{4/5}}dx = \boxed{10}$$ (Convergent).

3. $\int_0^1 \frac{1}{x^5}dx = \lim_{t\to 0_+}\left(\int_t^1 \frac{1}{x^5}dx\right) = \lim_{t\to 0_+}\left(-\frac{1}{4}\frac{1}{x^4}\Big|_t^1\right)$

$$= \lim_{t\to 0_+}\left(-\frac{1}{4} + \frac{1}{4}\frac{1}{t^4}\right) = \boxed{\infty}$$ <u>Divergent</u>

4. $\int_0^2 x^2 \ln x\, dx = \lim_{t\to 0+}\left(\int_t^2 x^2\ln x\,dx\right) = \lim_{t\to 0+}\left(\frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{3}\underbrace{t^3\ln t}_{\to 0} + \frac{1}{9}\underbrace{t^3}_{\to 0}\right) = \boxed{\frac{8}{3}\ln 2 - \frac{8}{9}}$

$$\int_t^2 x^2\ln x\,dx = \frac{1}{3}x^3\ln x\Big|_t^2 - \int_t^2 \frac{1}{3}x^2 dx = \frac{8}{3}\ln 2 - \frac{1}{3}t^3\ln t - \frac{1}{9}x^3\Big|_t^2$$

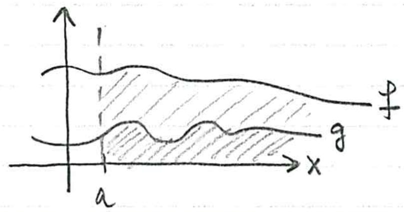
$u=\ln x$; $dv = x^2dx$

$du = \frac{1}{x}dx$; $v=\frac{1}{3}x^3$

$$= \frac{8}{3}\ln 2 - \frac{1}{3}t^3\ln t - \frac{8}{9} + \frac{1}{9}t^3$$

$\boxed{\lim_{t\to 0+} t^3\ln t = 0}$ ($0\cdot\infty$): $\lim_{t\to 0_+} t^3\ln t \;(0\cdot\infty) = \lim_{t\to 0_+}\frac{\ln t}{\frac{1}{t^3}} \overset{\text{l'H}}{\underset{\frac{\infty}{\infty}}{=}} \lim_{t\to 0+}\frac{\frac{1}{t}}{-\frac{3}{t^4}} = \lim_{t\to 0+}\left(-\frac{t^3}{3}\right) = 0.$

Comparison Theorem: If f, g are continuous, and $f(x) \geq g(x) \geq 0$ for all $x \geq a$:



- If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
- If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

1. $\int_1^\infty \frac{2+e^{-x}}{x}dx$ $\qquad f(x) = \frac{2+e^{-x}}{x} \geq \frac{2}{x} = g(x)$ for all x

$$\int_1^\infty g(x)dx = \int_1^\infty \frac{2}{x}dx = 2\ln x\Big|_1^\infty = \infty$$

$\Rightarrow \int_1^\infty \frac{2+e^{-x}}{x}dx$ is divergent by the Comparison Test

2. $\int_0^\infty \frac{\arctan x}{2+e^x}dx$

$f(x) = \frac{\arctan x}{2+e^x} < \frac{\pi/2}{2+e^x} < \frac{2}{2+e^x} < \frac{2}{e^x} = g(x)$ $\qquad \int_0^\infty \frac{2}{e^x}dx = -2e^{-x}\Big|_0^\infty = 2$

$f(x) \leq g(x)$ for all $x \geq 0$, $\int_0^\infty g(x)dx$ converges $\Rightarrow \int_0^\infty \frac{\arctan x}{2+e^x}dx$ converges by the Comparison Test.

3. $\int_1^\infty \frac{5}{\sqrt{x^6+4}}dx$ $\qquad$ Choose a function to compare: $\int_1^\infty g(x)dx$

A. $g(x) = 5\sqrt{x}$
B. $g(x) = \frac{5}{\sqrt{x}}$
C. $g(x) = \frac{5}{x^3}$
D. $g(x) = \frac{5}{4x}$

C. $g(x) = \frac{5}{x^3} \geq \frac{5}{\sqrt{x^6+4}} = f(x)$

$\int_1^\infty g(x)dx = \int_1^\infty \frac{5}{x^3}dx = -\frac{5}{2}\frac{1}{x^2}\Big|_1^\infty = \frac{5}{2} < \infty$ Convergent! $\Rightarrow \int_1^\infty f(x)dx$ converges by Comparison Test.

4. $\int_0^\infty \frac{4}{4x+e^{4x}}dx$ $\qquad$ Choose a function to compare: $\int_1^\infty g(x)dx$

A. $g(x) = \frac{4}{4x}$
B. $g(x) = \frac{4}{e^{4x}}$
C. $g(x) = \frac{4}{e^{8x}}$
D. $g(x) = 4e^{4x}$

A.? $\frac{4}{4x+e^{4x}} \leq \frac{4}{4x}$; $\int_0^\infty \frac{1}{x}dx = \ln x\Big|_{0_+}^\infty = \infty - (-\infty) = \infty$

So $f(x) \leq g(x)$ and $\int_0^\infty g(x)dx = \infty$... Tells us nothing about f!

B.? $\frac{4}{4x+e^{4x}} \leq \frac{4}{e^{4x}}$; $\int_0^\infty \frac{4}{e^{4x}}dx = -e^{-4x}\Big|_0^\infty = 1 < \infty$!

$f(x) = \frac{4}{4x+e^{4x}} \leq g(x) = \frac{4}{e^{4x}}$, $\int_0^\infty g(x)dx = 1 < \infty$ $\Rightarrow \int_0^\infty f(x)dx$ converges by Comparison Test

Limit Comparison Test: If f, g are positive continuous functions on $[a, \infty)$ and if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$ exists and is finite $(0 < L < \infty)$

then: $\boxed{\int_a^\infty f(x)dx}$ & $\boxed{\int_a^\infty g(x)dx}$

either both converge or both diverge.

1. $\int_{10}^{\infty} \frac{8}{\sqrt{x-9}}dx$ Converge or diverge? Use Limit Comparison. $f(x) = \frac{8}{\sqrt{x-9}}$

Example of $g(x)$: $g(x) = \frac{1}{\sqrt{x}}$ $\quad \lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{\frac{8}{\sqrt{x-9}}}{\frac{1}{\sqrt{x}}} = \lim\limits_{x\to\infty} 8\frac{\sqrt{x}}{\sqrt{x-9}} = 8 < \infty$

$$\int_{10}^{\infty} g(x)dx = \int_{10}^{\infty} \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_{10}^{\infty} = \infty \quad \text{Divergent}$$

$\Rightarrow \int_{10}^{\infty} \frac{8}{\sqrt{x-9}}dx$ is divergent by the Limit Comparison Test.

2. $\int_5^\infty \frac{\sqrt{x+4}}{2x^5}dx$ Same question.

$$f(x) = \frac{\sqrt{x+4}}{2x^5}$$

Suggestion: $g(x) = \frac{1}{x^{5-\frac{1}{2}}} = \frac{1}{x^{9/2}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{x+4}}{2x^5} \cdot x^{9/2} = \lim_{x\to\infty} \frac{\sqrt{x+4}}{2\sqrt{x}} = \frac{1}{2} < \infty$$

$$\int_5^\infty \frac{1}{x^{9/2}}dx = -\frac{2}{7}x^{-7/2}\Big|_5^\infty = -\frac{2}{7}\frac{1}{x^{7/2}}\Big|_5^\infty = \frac{2}{7}\frac{1}{5^{7/2}} < \infty \quad \text{Convergent!}$$

$\Rightarrow \int_5^\infty f(x)dx$ also converges by Limit Comparison Test.