M133 - Series: Extra Problems (3)
Sections 11.2 — 11.7 (Solutions)

1). $\frac{7}{4} - \frac{7}{6} + \frac{7}{8} - \frac{7}{10} + \frac{7}{12} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{7}{2(n+1)}$ converges by AST

$$b_n = \frac{7}{2(n+1)} \geq \frac{7}{2(n+2)} = b_{n+1} \Rightarrow \{b_n\} \text{ is decreasing}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{7}{2(n+1)} = 0$$

2). $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot e^{3/n}$ divergent by the Test for Divergence

$$\lim_{n\to\infty} e^{3/n} = e^0 = 1 \Rightarrow \lim_{n\to\infty} (-1)^{n-1} \cdot 1 \text{ does not exist}$$

3). $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{4n-1} = \sum_{n=1}^{\infty} a_n$ conditionally convergent

- $b_n = \frac{1}{4n-1} \geq \frac{1}{4(n+1)-1} = b_{n+1} \Rightarrow \{b_n\}$ is decreasing

  $\lim_{n\to\infty} b_n = 0$

  $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test

- $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{4n-1}$ diverges by the Comparison Test

  $\frac{1}{4n-1} \geq \frac{1}{4n}$ and $\sum_{n=1}^{\infty} \frac{1}{4n}$ diverges (harmonic series)

4). $\sum_{n=1}^{\infty} \frac{n}{6^n}$ convergent by the Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)}{6^{n+1}} \cdot \frac{6^n}{n} = \lim_{n\to\infty} \frac{n+1}{6n} = \frac{1}{6} < 1$$

5). $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ divergent by Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n+1}{10} = \boxed{\infty}$$

6). $\sum_{n=1}^{\infty} \frac{\sin(7n)}{5^n}$ convergent (because it is absolutely convergent)

$$|a_n| = \frac{|\sin(7n)|}{5^n} \leq \frac{1}{5^n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{5^n} \text{ is a convergent geometric series } \left(r = \frac{1}{5} < 1\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| \text{ is convergent (Comparison Test)} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}$$

We cannot apply the Comparison Test to the original series because the Comparison Test only applies to sequences with positive terms.

7). $\sum_{n=2}^{\infty} \left(\frac{-4n}{n+1}\right)^{3n}$ diverges by the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{+4n}{n+1}\right)^3 = 4^3 > 1$$

8). $\sum_{n=1}^{\infty} \left(\frac{n^3+3}{9n^3+1}\right)^n$ converges by the Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^3+3}{9n^3+1} = \frac{1}{9} < 1$$

9). $\sum_{n=1}^{\infty} 3\left(1+\frac{1}{n}\right)^{n^2} = 3 \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ divergent by the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

10). $\sum_{n=1}^{\infty} \left(\frac{3n+10}{3n+3}\right)^{4n+2}$ diverges by the Test for Divergence

The Root Test is inconclusive: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{3n+10}{3n+3}\right)^{\frac{4n+2}{n}} = 1^4 = 1$

$$\lim_{n\to\infty} \left(\frac{3n+10}{3n+3}\right)^{4n+2} = \lim_{n\to\infty} \left(\frac{3n+3+7}{3n+3}\right)^{4n+2} = \lim_{n\to\infty} \left(1+\frac{7}{3n+3}\right)^{4n+2}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{\frac{3n+3}{7}}\right)^{4n+2}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{\frac{3n+3}{7}}\right)^{\frac{3n+3}{7}}\right]^{\frac{7}{3n+3}(4n+2)} = e^{\frac{28}{3}} \neq 0$$

11). $\sum_{n=1}^{\infty} \frac{3\cdot 6\cdot 9\cdot 12\cdot \ldots \cdot (3n)}{n!} = \sum_{n=1}^{\infty} \frac{(3\cdot 1)\cdot(3\cdot 2)\cdot(3\cdot 3)\cdot(3\cdot 4)\cdot\ldots\cdot(3\cdot n)}{n!}$

$= \sum_{n=1}^{\infty} \frac{3^n\cdot n!}{n!} = \sum_{n=1}^{\infty} 3^n$ diverges by the Test for Divergence

$\lim_{n\to\infty} 3^n = \infty$

12). $a_1 = 3;\ a_{n+1} = \frac{6n^2+3}{5n^2+n} a_n$ diverges by the Ratio Test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{6n^2+3}{5n^2+n} = \frac{6}{5} > 1$

13). For what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$ for all x

The series converges for all real numbers x. Therefore:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x$$

(because if $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$).

14). $\sum_{n=1}^{\infty} \frac{9^n}{4^n+3^n}$ diverges by the Test for Divergence

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(\frac{9}{4}\right)^n}{1+\left(\frac{3}{4}\right)^n} = \infty$

15). $\sum_{n=1}^{\infty} \frac{n^{10}+1}{8n^{11}+7n^6+4}$ diverges by the Limit Comparison Test

$a_n = \frac{n^{10}+1}{8n^{11}+7n^6+4}$; $b_n = \frac{1}{n}$ (divergent - harmonic series)

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n^{10}+1)}{8n^{11}+7n^6+4} = \frac{1}{8}$

16). $\sum_{n=1}^{\infty} \frac{4n!}{e^{n^2}}$ converges by the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{4n!}\right| = \lim_{n\to\infty}\frac{n+1}{e^{n^2+2n+1-n^2}}$$

$$= \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = 0 < 1$$

17). $\sum_{n=1}^{\infty} \frac{3^n \cdot n^6}{n!}$ converges by the Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)^6}{(n+1)!}\cdot\frac{n!}{3^n\cdot n^6} = \lim_{n\to\infty}\frac{3(n+1)^6}{(n+1)\cdot n^6} = 0 < 1$$

18). $\sum_{n=1}^{\infty} \sin(8n)$ diverges by the Test for Divergence $\left(\lim_{n\to\infty}\sin(8n) \text{ DNE}\right)$

19). $\sum_{n=1}^{\infty} \frac{(7n+1)^n}{n^{6n}}$ converges by the Root Test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{7n+1}{n^6} = 0 < 1$$

20). $\sum_{n=1}^{\infty} (\sqrt[n]{6}-1)^n$ converges by the Root Test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}(\sqrt[n]{6}-1) = \lim_{n\to\infty}(6^{1/n}-1) = 6^0-1 = 1-1 = 0 < 1$$

21). $\sum_{n=1}^{\infty} \frac{\sin(8n)}{1+6^n}$ convergent because it is absolutely convergent

$|a_n| = \frac{|\sin(8n)|}{1+6^n} \le \frac{1}{1+6^n} \le \frac{1}{6^n}$ and $\sum_{n=1}^{\infty}\frac{1}{6^n}$ is a convergent geometric series $(r = \frac{1}{6} < 1)$

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

22). $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{e} < 1$$